Closure for $\{K_{1,4}, K_{1,4} + e\}$-free graphs

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Joint work with Petr Vrána, Plzeň, and Shipeng Wang, Beijing

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Graph: simple finite undirected

$\mathcal{X}$ - a family of graphs: a graph G is $\mathcal{X}$-free if G does not contain a copy of any graph from $\mathcal{X}$ as an induced subgraph.

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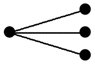

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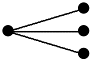

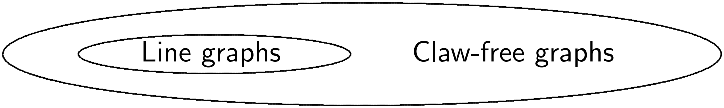

Closure for hamiltonicity in claw-free graphs

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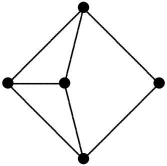

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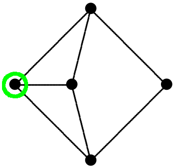

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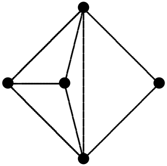

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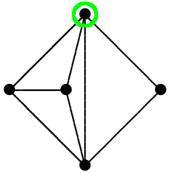

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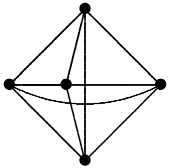

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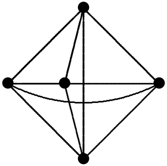

cl(G) is complete

Theorem [ZR 1997]. *Let G be a claw-free graph. Then*

(i) $cl(G)$ *is uniquely determined,*

(ii) $cl(G)$ *is the line graph of a triangle-free graph,*

(iii) $c(G) = c(cl(G))$,

(iv) G *is hamiltonian if and only if* $cl(G)$ *is hamiltonian.*

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The closure operation $\mathrm{cl}(G)$:

- turns a claw-free graph into the line graph of a triangle-free graph,
- preserves the value of circumference,
- preserves hamiltonicity or non-hamiltonicity.

Applications.

Conjecture 1 [Matthews, Sumner 1984]. *Every* 4*-connected claw-free graph is hamiltonian.*

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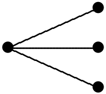
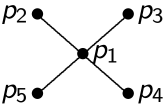
"Strongest" known, still equivalent:

Conjecture 5.
Every 4*-connected claw-free graph is 1-Hamilton-connected.*

Extending the closure to $\{K_{1,4}, K_{1,4}+e\}$-free graphs?



The claw $K_{1,3}$



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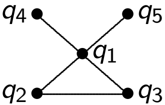

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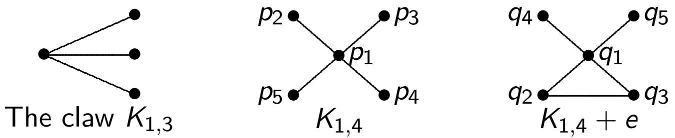

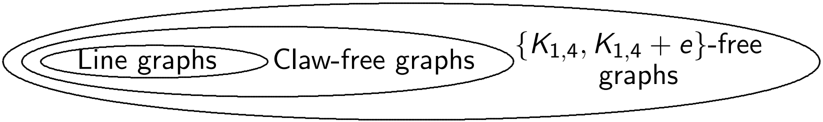

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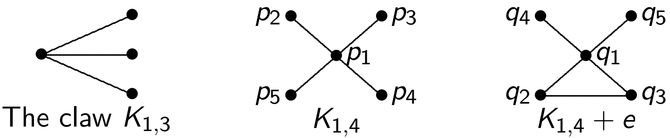

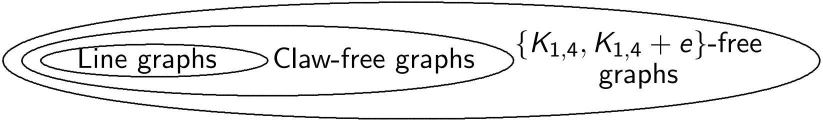

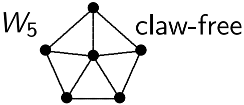

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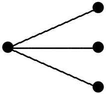

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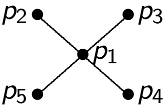

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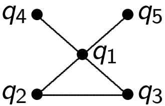

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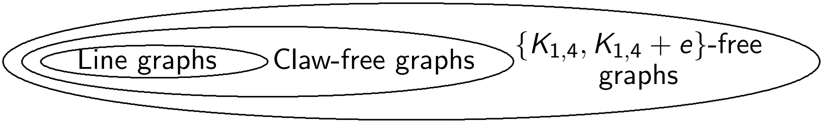
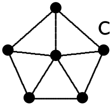
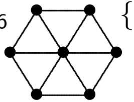


W_5



claw-free

W_6



$\{K_{1,4}, K_{1,4}+e\}$-free (not claw-free)

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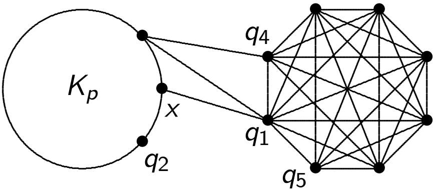
For $x \in V_{EL}(G)$, the *local completion of G at x*:
the graph G_x^*, obtained from G by adding all missing edges to the noncomplete component of $\langle N_G(x) \rangle_G$
(i.e., by replacing the noncomplete component of $\langle N_G(x) \rangle_G$ with a clique).

The edges in $E(G_x^*) \setminus E(G)$: *new edges*.

A problem: G_x^* is not necessarily $\{K_{1,4}, K_{1,4}+e\}$-free !!!



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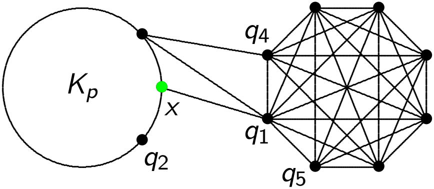

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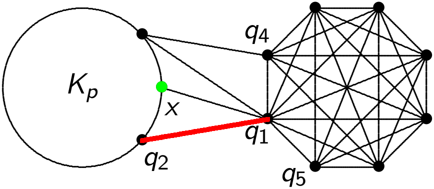

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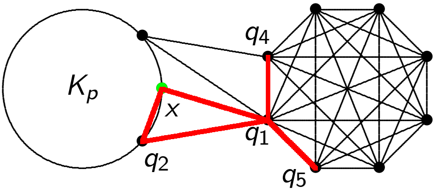

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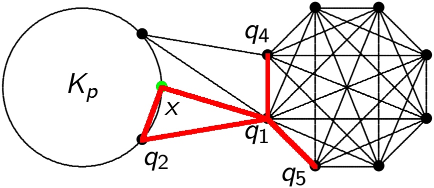

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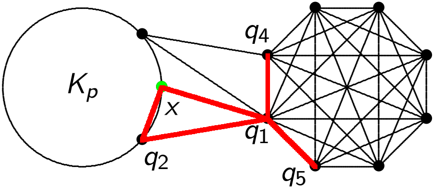

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Can we choose another eligible vertex???

No!

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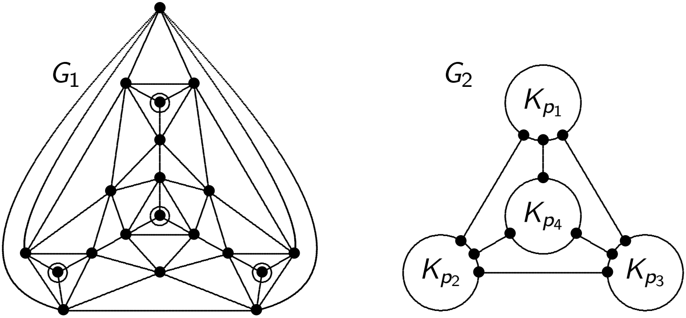

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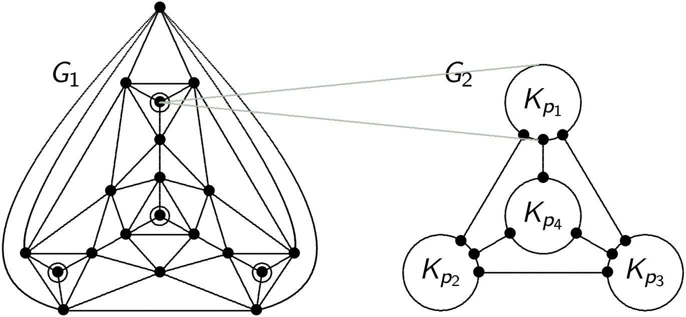

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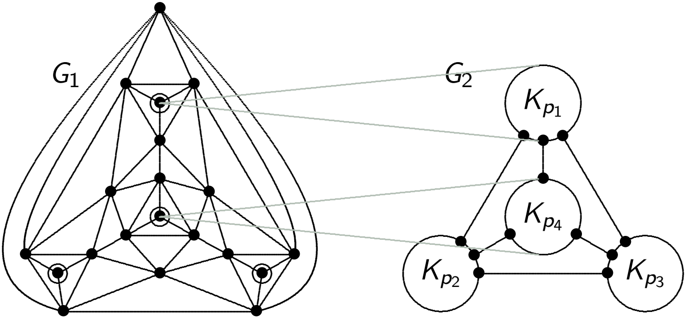

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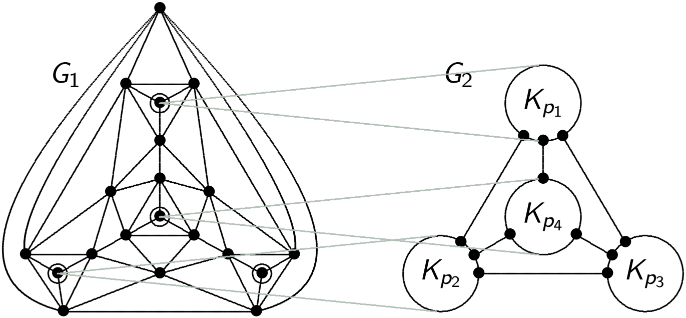

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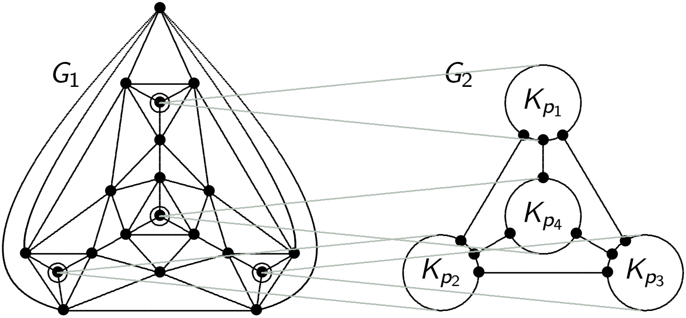

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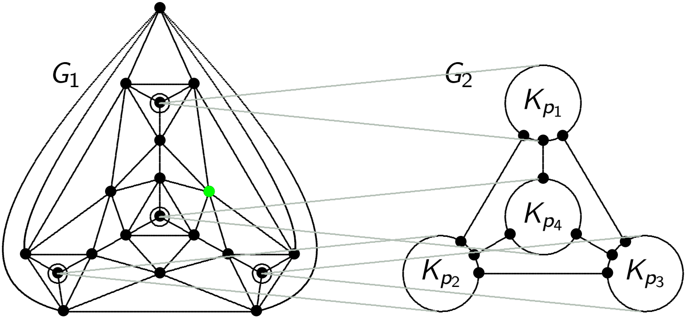

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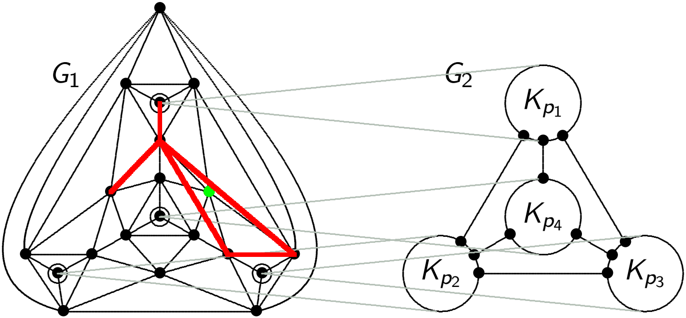

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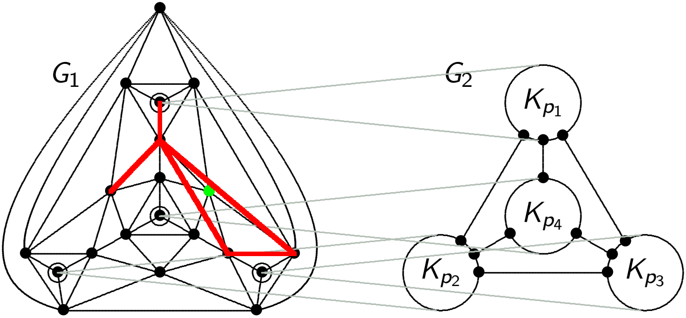

G is $\{K_{1,4}, K_{1,4}+e\}$-free, but G^*_x contains an induced $K_{1,4}+e$ **for any** $x \in V_{EL}(G)$!!!

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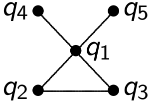
  (i) $\{q_1, q_2, q_3\} \subset V(\mathcal{K}_G)$,



$K_{1,4}+e$

Class $\mathcal{F}$

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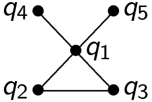

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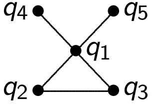

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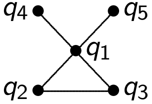
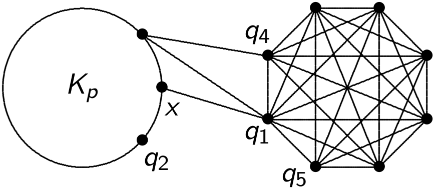


$K_{1,4}+e$

Clearly: $\mathcal{F}$ contains all $\{K_{1,4}, K_{1,4}+e\}$-free graphs.



K_p
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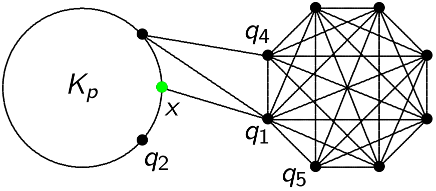
q_1
q_2
q_5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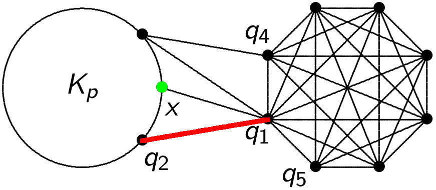
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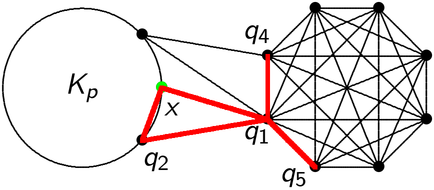
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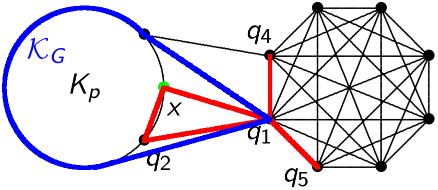
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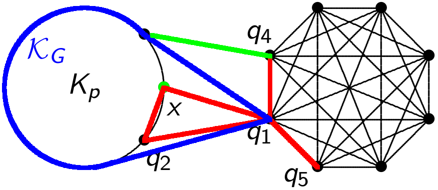
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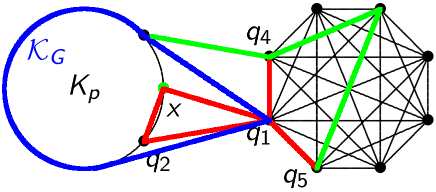
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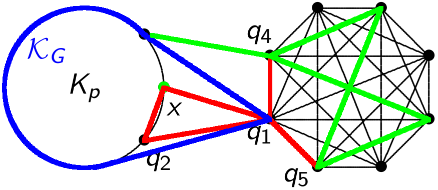

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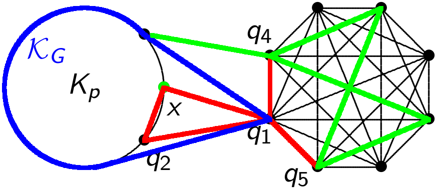

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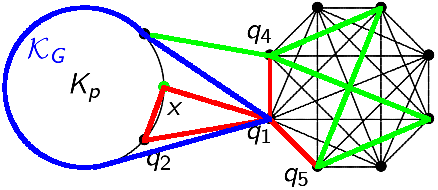

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Closure. Let $G \in \mathcal{F}$. The *h-closure of* G, denoted $\mathrm{cl}^h(G)$, is the graph obtained from G by recursively performing the local completion operation at vertices $x \in V_{EL}(G) \cap V(\mathcal{K}_G)$, or $x \in V_{EL}(G)$ if $V(\mathcal{K}_G) = \emptyset$, as long as this is possible.

(More precisely, there is a sequence of graphs $G_1, \ldots, G_k$ such that

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Theorem. *Let* $G \in \mathcal{F}$. *Then*

(i) $cl^h(G)$ *is well-defined (i.e., uniquely determined),*

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Applications of the closure.

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Thomassen's, Matthews-Sumner's and Bondy's conjectures
Recall equivalent conjectures:

Conjecture A. *Every 4-connected line graph is hamiltonian.*
Conjecture B. *Every 4-connected claw-free graph is hamiltonian.*
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As a weaker version of Conjecture C, Bondy [1989] suggested:

Conjecture E. *There is a constant* c_0 *with* $0 < c_0 \leq 1$ *such that every cyclically 4-edge-connected cubic graph* H *of order* n *has a cycle of length at least* $c_0 n$.

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Theorem. *Conjecture F and Conjecture G are equivalent.*

Hamiltonicity of graphs with high connectivity

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Degree and neighborhood conditions for hamiltonicity

[Matthews, Sumner 1985] *Every 2-connected claw-free graph G with* $\delta(G) \geq \frac{n-2}{3}$ *is hamiltonian.*
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- each $\mathcal{F}_i$ is generated by a single graph, and
- every "sufficiently large" claw-free graph G satisfying $\sigma_k(G) \geq n + k^2 - 4k + 7$ is either hamiltonian, or $\mathrm{cl}(G) \in \cup_{i=1}^{r_k} \mathcal{F}_i$.

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Performed for $k = 6$ (manually), and for $k = 8$, with a computer.

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Easy: can be directly extended to $\{K_{1,4}, K_{1,4} + e\}$-free graphs with minimum degree at least 6 using the h-closure operation.

We formulate this fact in the form of the following "metatheorem".

Theorem. *Let k an κ be positive integers, and let $f_k(n)$ be a function and $\mathcal{F}_k$ a family of line graphs such that every κ-connected claw-free graph G of order n satisfying $\sigma_k(G) \geq f_k(n)$ is either hamiltonian, or $cl(G) \in \mathcal{F}_k$. Then every κ-connected $\{K_{1,4}, K_{1,4} + e\}$-free graph G of order n satisfying $\delta(G) \geq 6$ and $\sigma_k(G) \geq f_k(n)$ is either hamiltonian, or $cl^h(G) \in \mathcal{F}_k$.*

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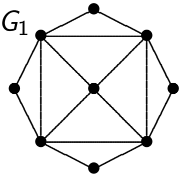
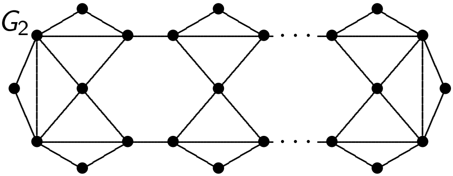
Similar: sufficient conditions in terms of the neighborhood union $|N_G(x_1) \cup \ldots \cup N_G(x_k)|$ taken over all independent sets $\{x_1, \ldots, x_k\} \subset V(G)$ can be also directly extended to $\{K_{1,4}, K_{1,4}+e\}$-free graphs with $\delta(G) \geq 6$ using the h-closure.

Examples.

1.





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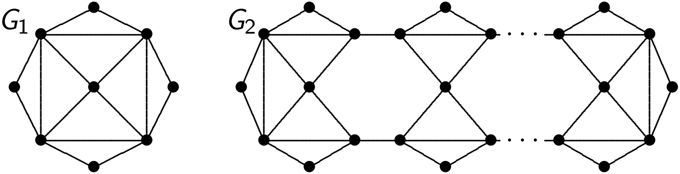

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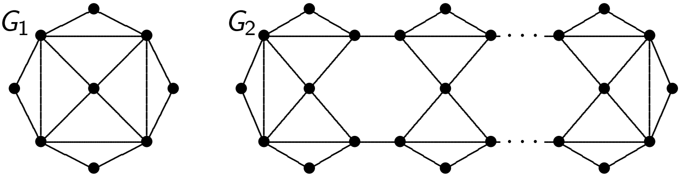

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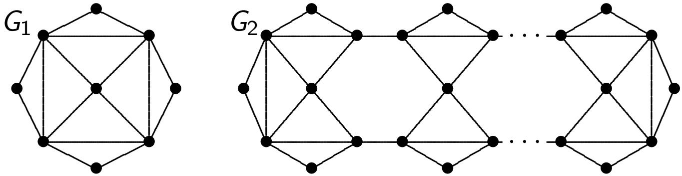

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Thus: the results cannot be true without an assumption on $\delta(G)$.

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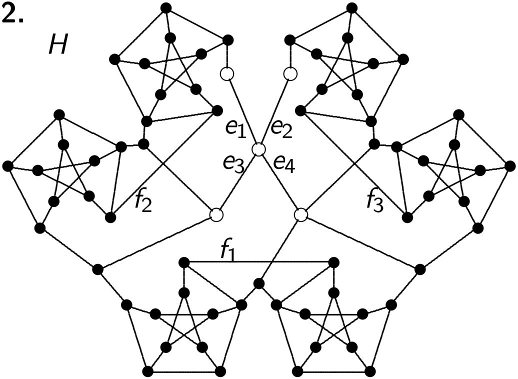

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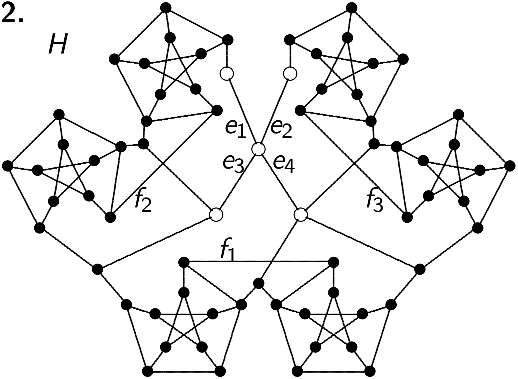

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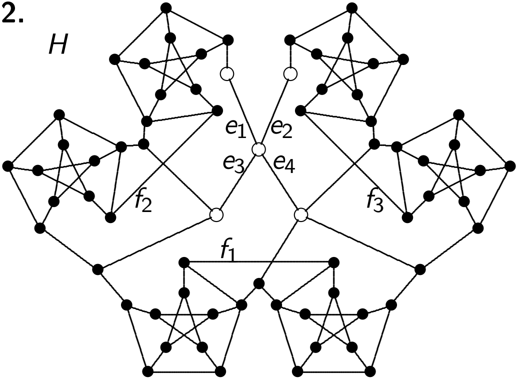

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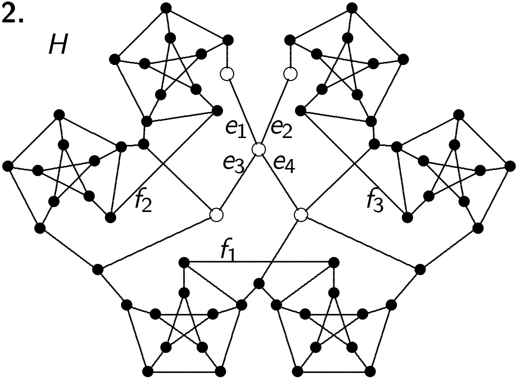

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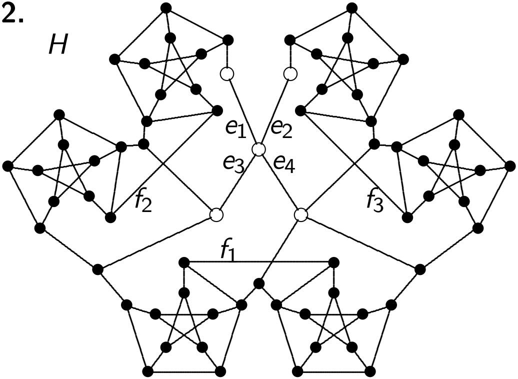

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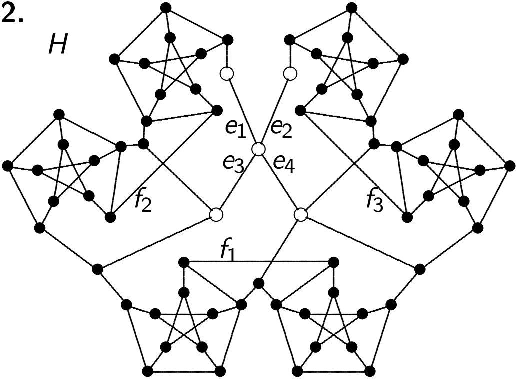

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Since $\delta(G) = 4$ (and, moreover, G is 3-connected), the results cannot be true even for $\delta(G) \geq 4$.

Open question.

We admit that the results could be true for $\delta(G) \geq 5$, but, since our proof heavily relies on the condition $\delta(G) \geq 6$, the proof of such an improvement would require a new idea, and we leave this as an open question.

Thank you