# Applications of adjacency properties from the countable random graph to graph-theoretic problems

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# Adjacency properties

(Erdős-Rényi 1963, Blass-Harary 1979)

$G$ satisfies $\boldsymbol{\mathcal{P}(l, m)} \underset{\text{def}}{\Leftrightarrow}$ $\forall A, B \subset V(G)$ $(|A| = l, |B| = m, A \cap B = \emptyset)$,
$\exists z_{A,B} \notin A \cup B$ satisfying the following property $(*)_{A,B}$:

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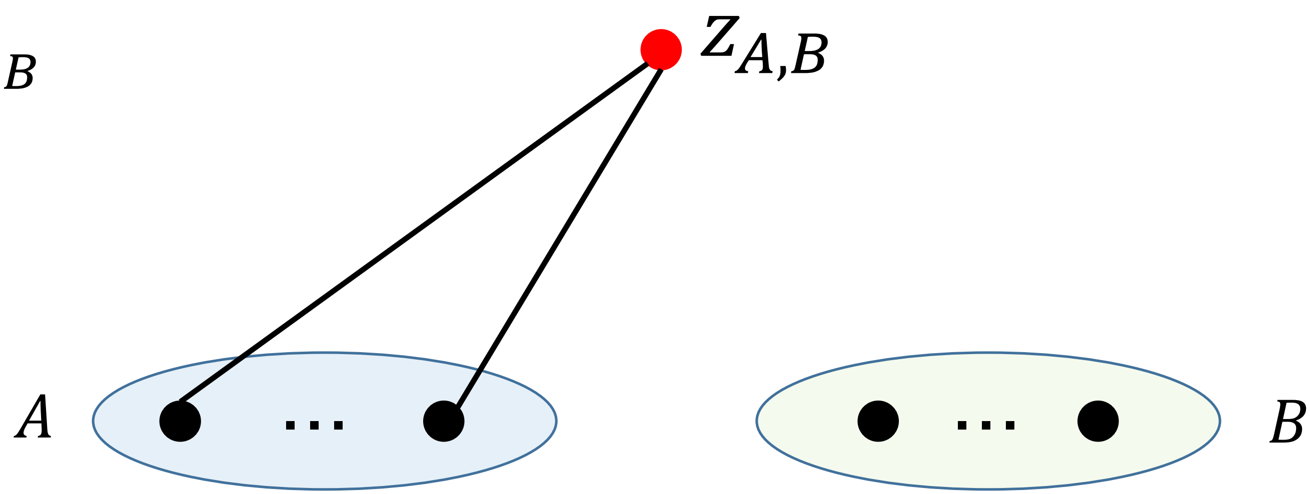
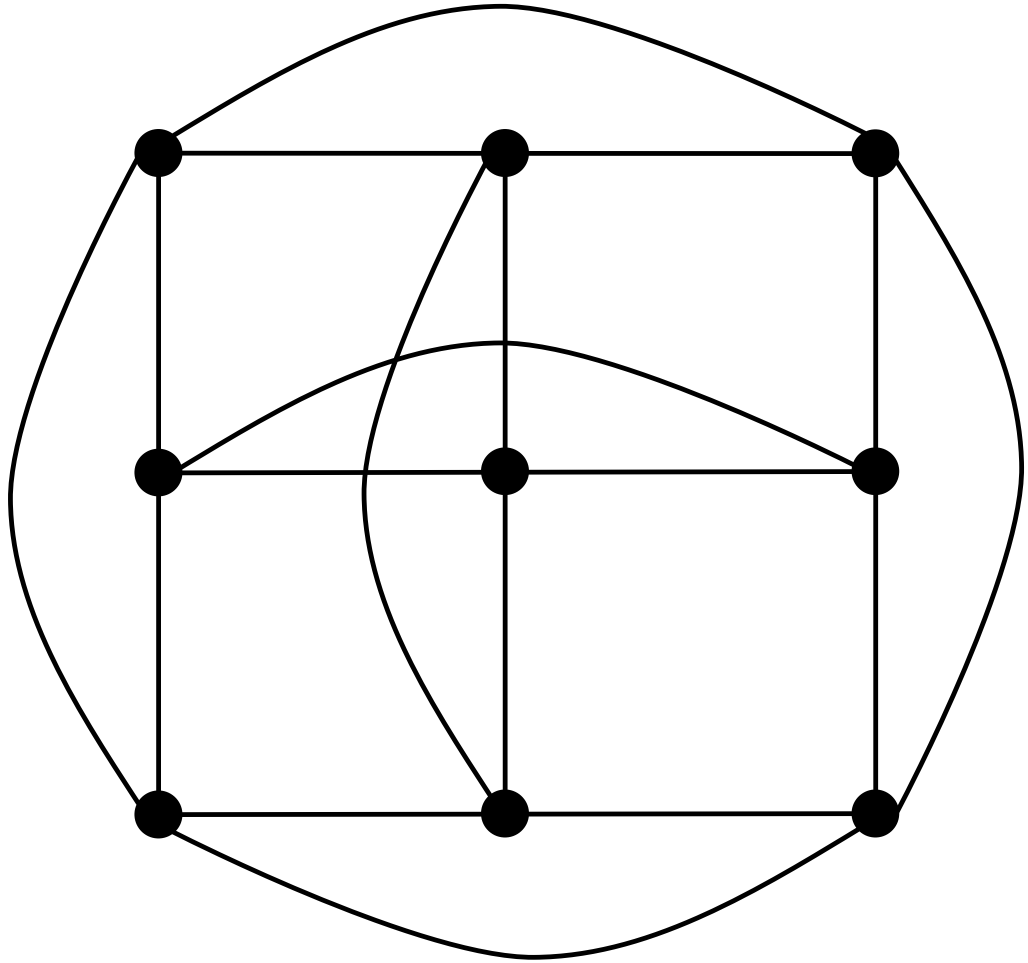



If $G$ satisfies $\mathcal{P}(l, m)$ for all $(l, m)$ s.t. $l + m = n$, $G$ is called **$\boldsymbol{n}$-e.c. ($\boldsymbol{n}$-existentially closed).**

# Adjacency properties



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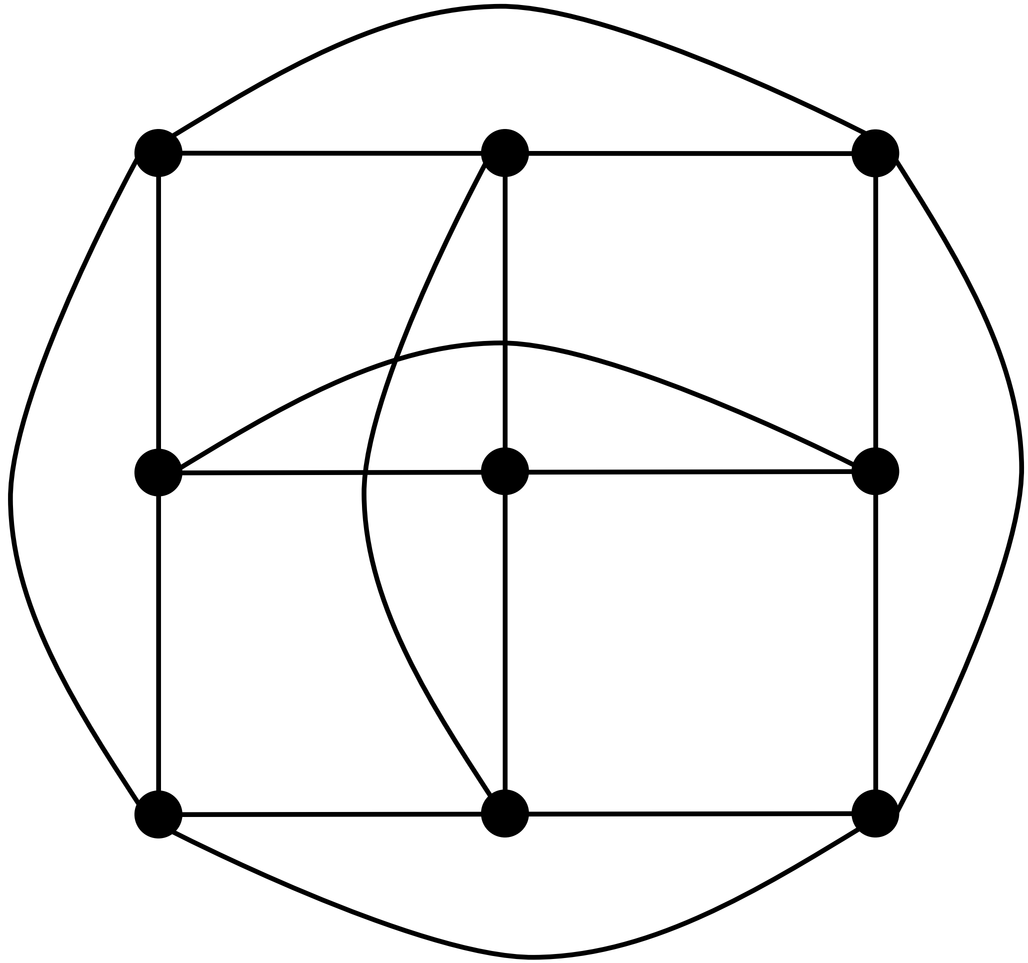

$\mathcal{P}(2,0)$ (moreover, 2-e.c.)

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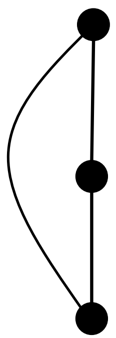

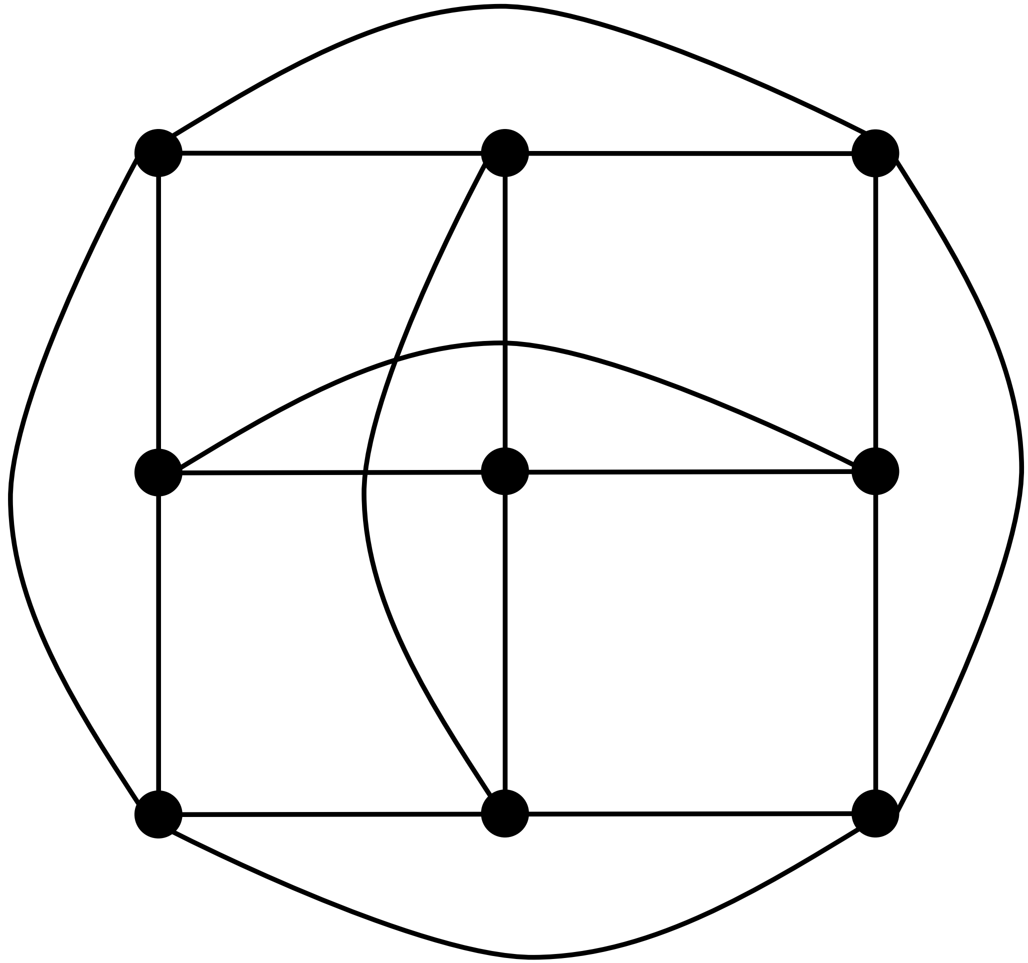

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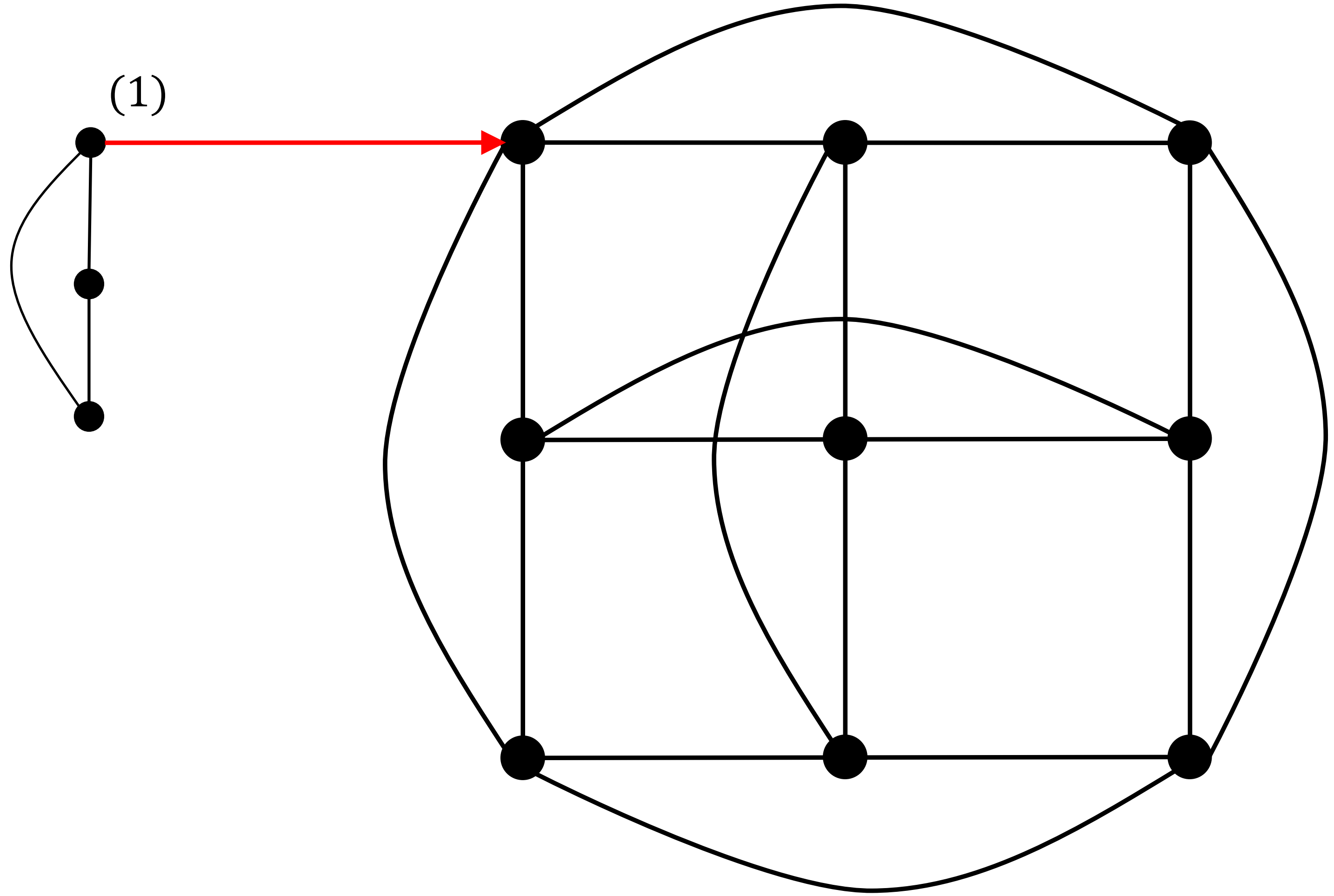

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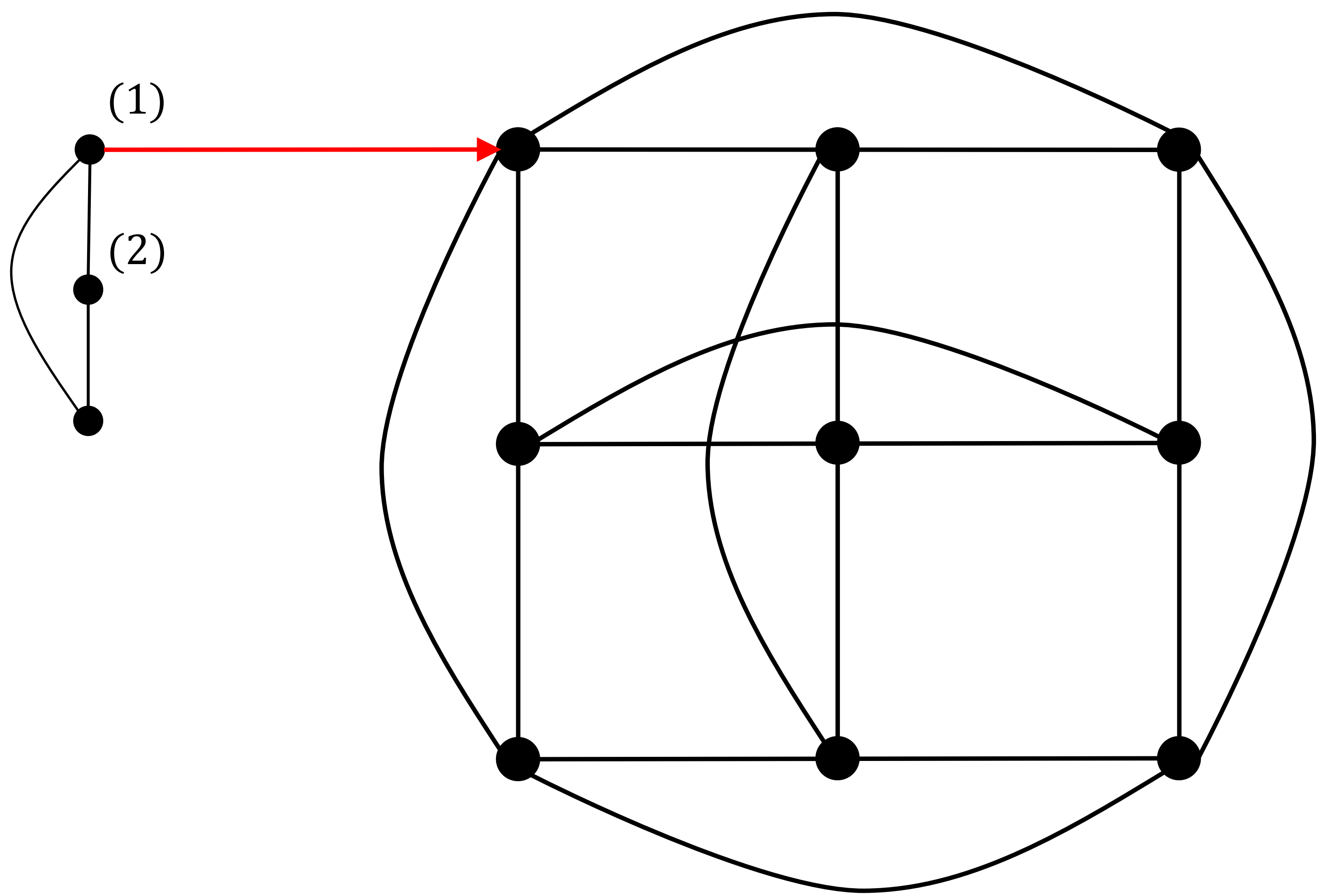

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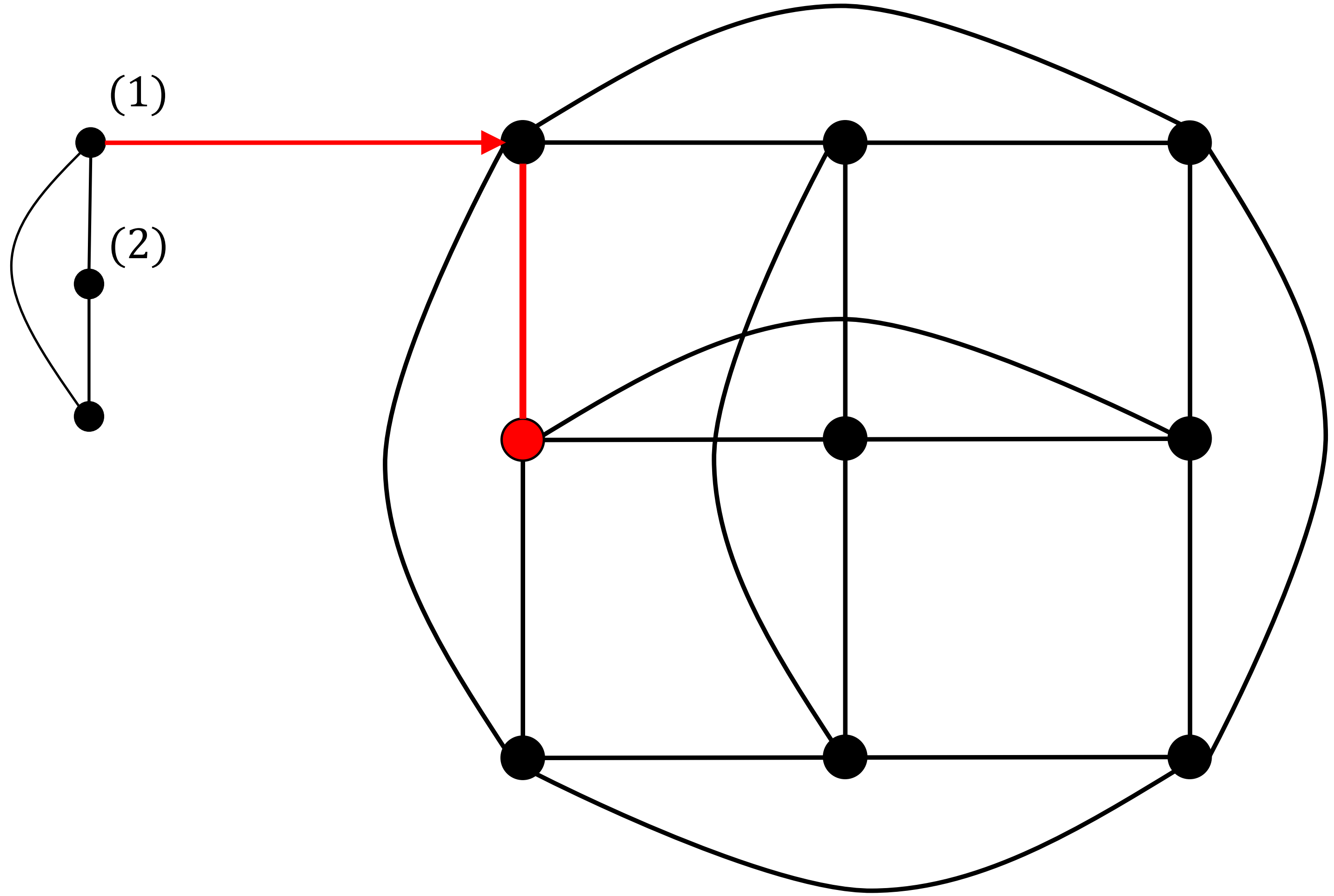

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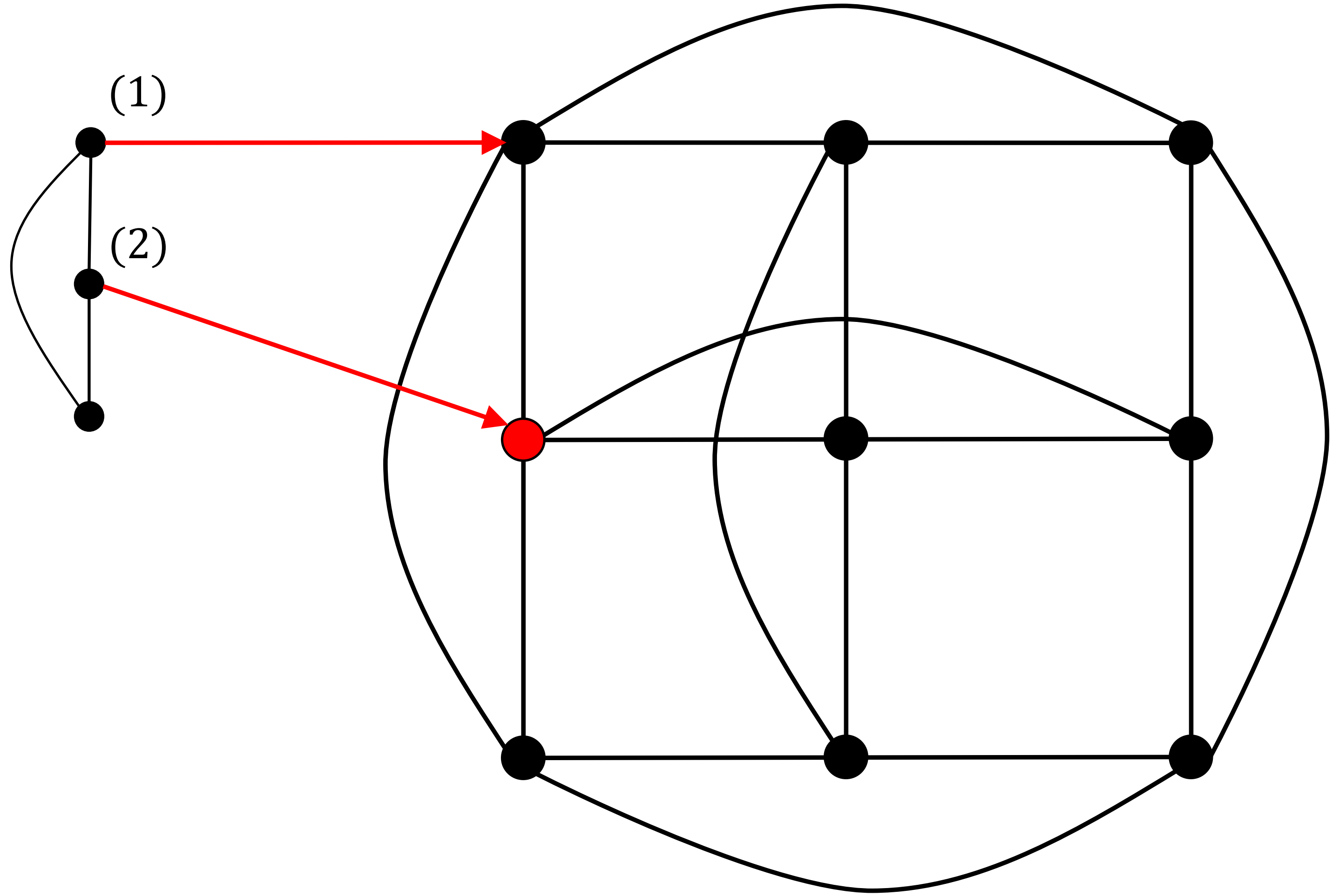

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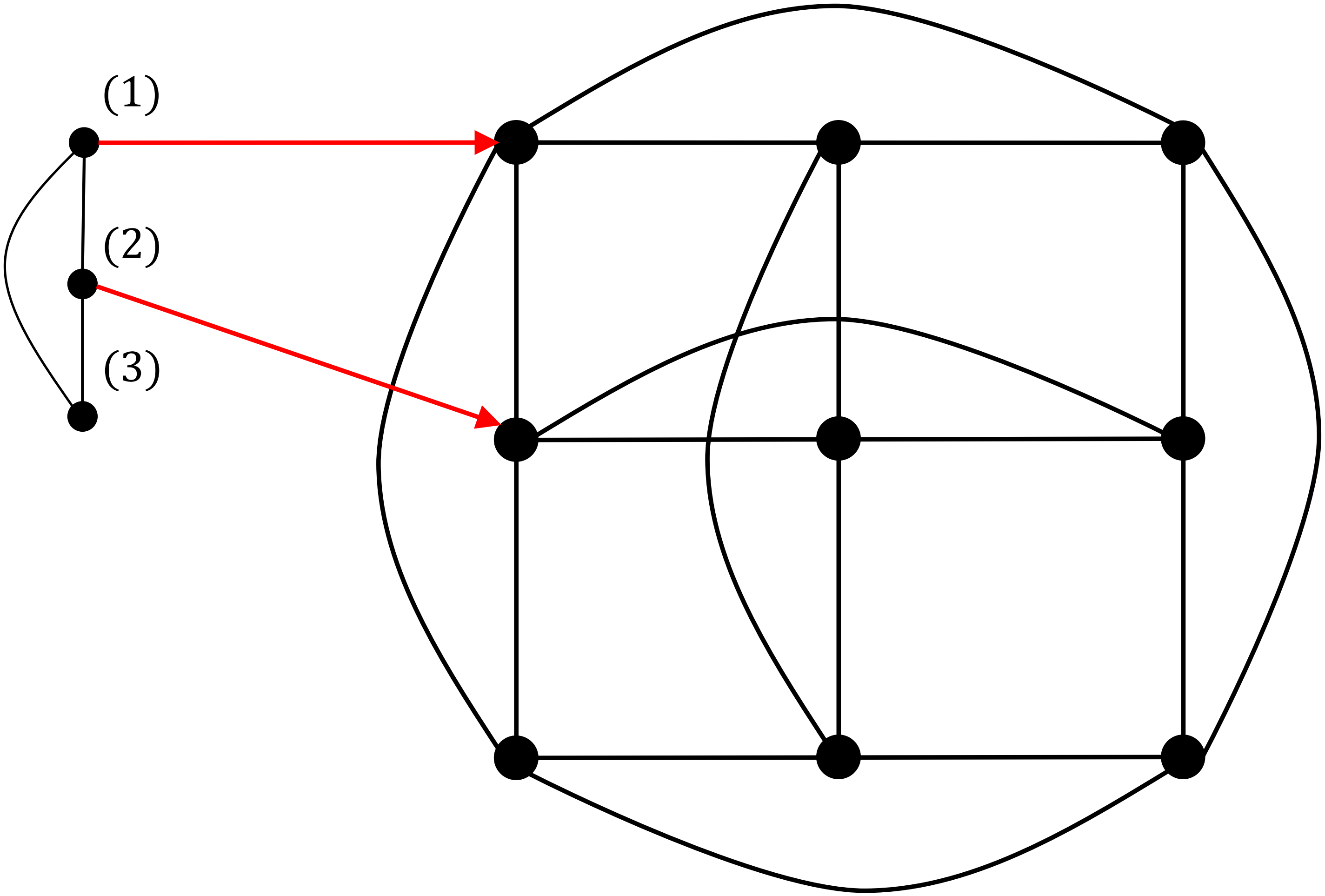

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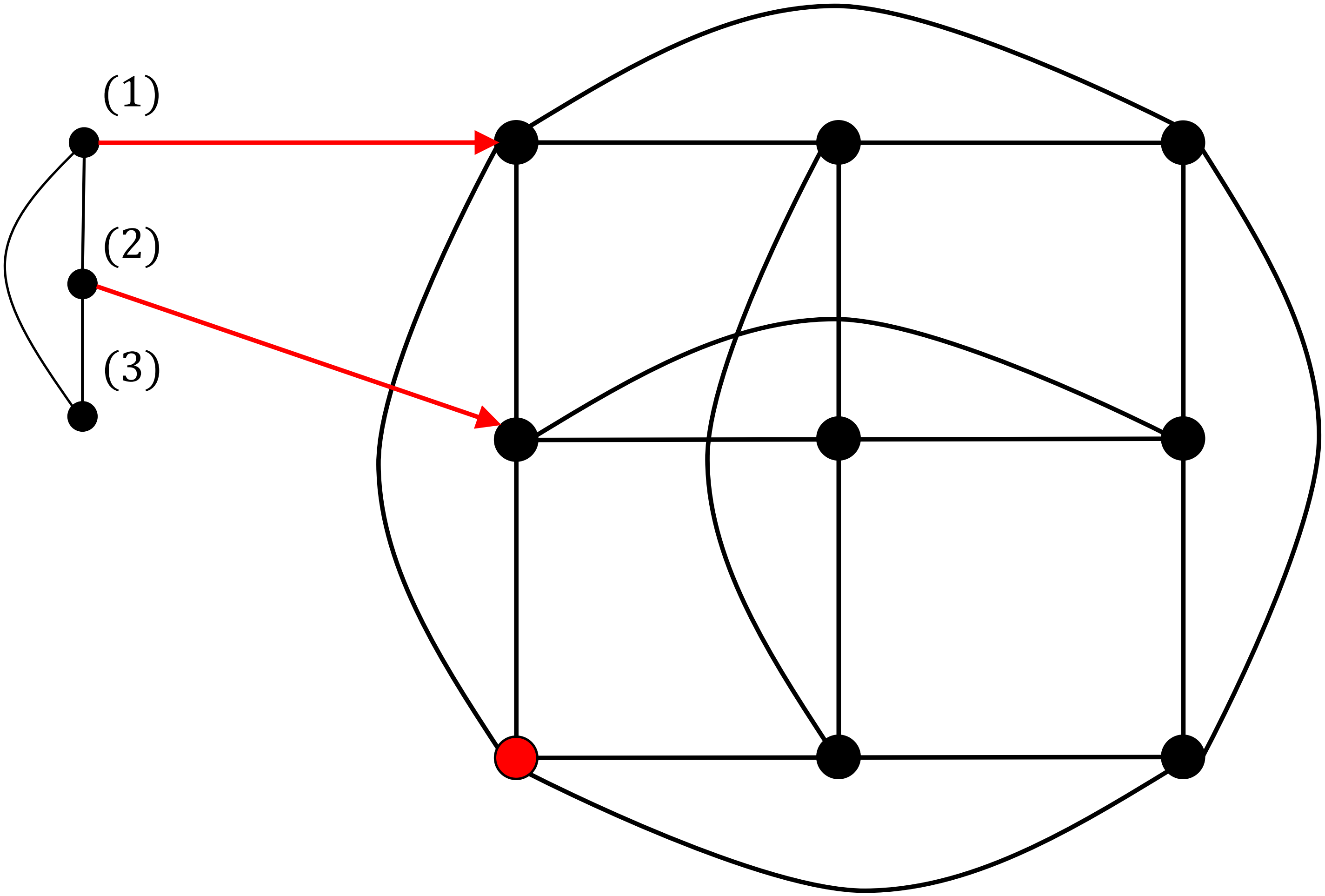

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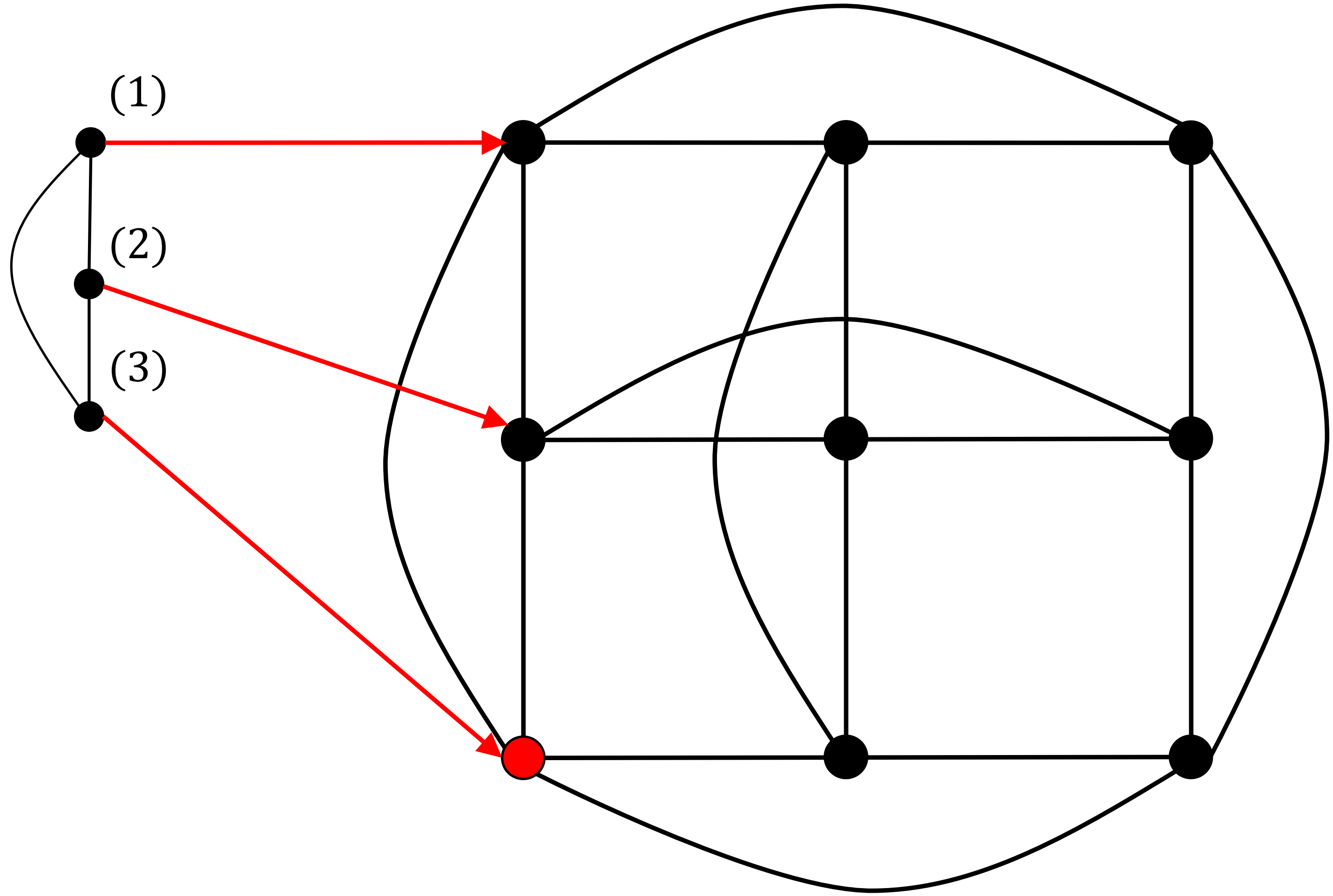

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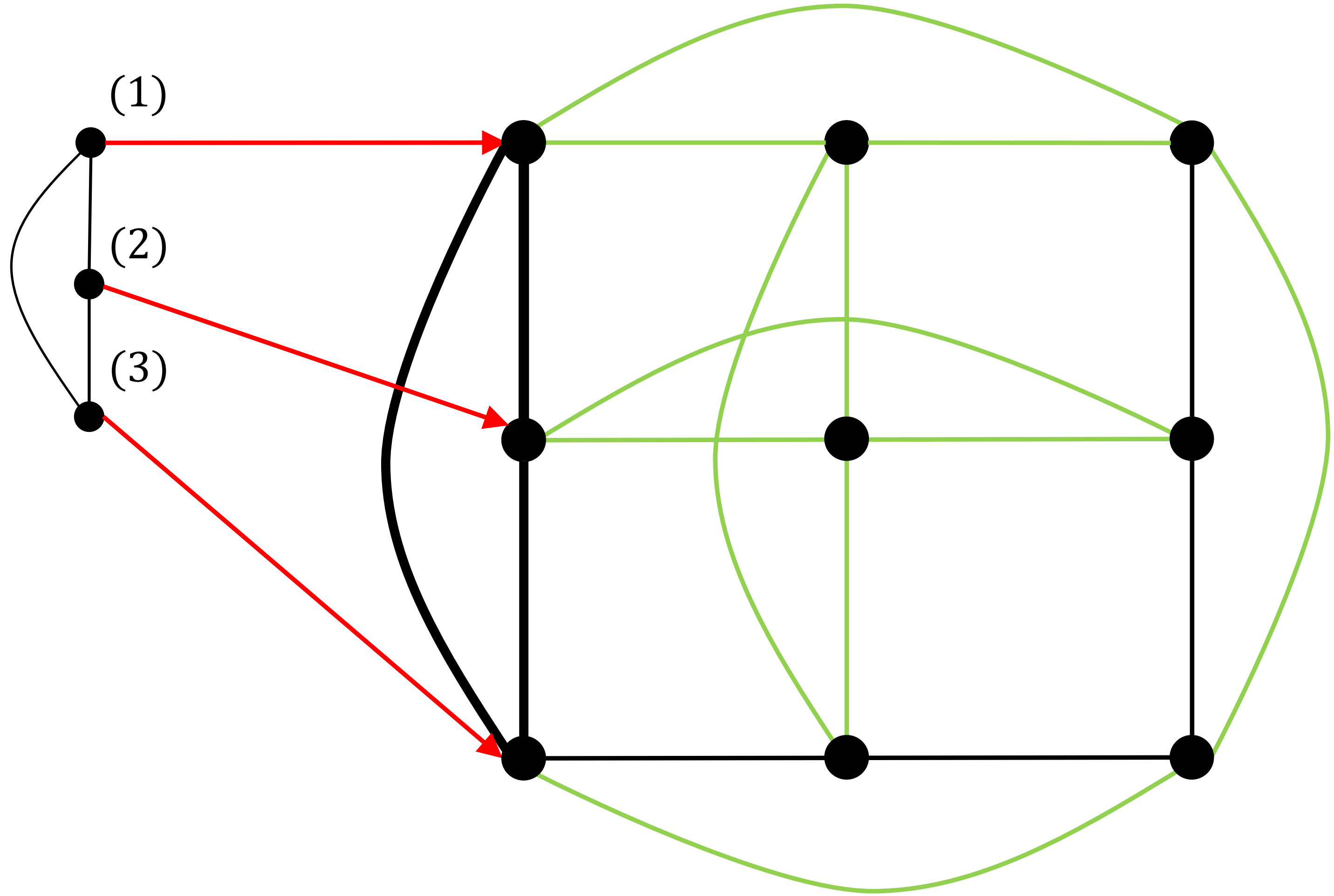


$G$: $\mathcal{P}(l, 0) \Rightarrow G$ contains all cliques with $l+1$ vertices

($G$: $n$-e.c. graphs $\Rightarrow G$ contains all graphs with $n+1$ vertices)

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- By Weil's theorem, $G_p(k)$ satisfies $\mathcal{P}(l, 0)$ if $p - l(l-1)k^l\sqrt{p} - lk^{l-1} > 0$.

# A problem in number theory

Problem (Ebert 1981, originally for $k = 2$)
Find the maximum size $R_k(p)$ of subsets of $k$-th power residues of $\mathbb{F}_p$ s.t. difference of any two distinct elements is a $k$-th power residue.

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The key observation is that

$$G \text{ satisfies } \mathcal{P}(l,0) \Rightarrow G|_{N_G(v)} \text{ satisfies } \mathcal{P}(l-1,0) \text{ for all } v \in V(G)$$

where $N_G(v) = \{u \in V(G) \mid \{u,v\} \in E(G)\}$.

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  - ◊ It is conjectured (cf. Croot-Lev 2007) that

$$\omega\left(G_p(2)\right) \leq O(p^{\varepsilon}) \text{ for all } \varepsilon \in (0,1).$$

Thank you for your attentions!!