On the Recognition of Unit Grid Intersection Graphs

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We are preparing a paper:

**On Unit Grid Intersection Graphs**

by

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Grid Intersection Graph (GIG)

- \( V(G) \): set of straight line segments such that
  - line segments are parallel to the \( x \)- and \( y \)-axes, and
  - no two parallel line segments intersect

- \( E(G) = \{(u, v) \mid u \text{ and } v \text{ intersect.}\} \).

\{ GIGs \} \subset \{ \text{bigraphs} \}

bipartite graphs
Grid Intersection Graph (GIG)

grid intersection representation

grid intersection graph
Unit Grid Intersection Graph (UGIG)

grid intersection graph such that every line segment (vertex) has the same length.

\{ \text{UGIGs} \} \subseteq \{ \text{GIGs} \}
Unit Grid Intersection Graph (UGIG)

unit grid intersection representation

unit grid intersection graph \( (C_{14}) \)
Unit Grid Intersection Graph (UGIG)

$G$: not UGIG

$G'$: planar
Unit Grid Intersection Graph (UGIG)

$G$: not UGIG

$G'$: planar

planar bipartite $\Rightarrow$ grid intersection
Unit Grid Intersection Graph (UGIG)

$G$: not UGIG

$G$: GIG

$\Downarrow$

$\{\text{UGIGs}\} \subset \{\text{GIGs}\}$

planar bipartite $\Rightarrow$ grid intersection
Orthogonal Ray Graph (ORG)

- \( V(G) \): set of axis parallel rays such that no two parallel rays intersect
- \( E(G) = \{ (u, v) \mid u \text{ and } v \text{ intersect.} \} \).
Orthogonal Ray Graph (ORG)

- orthogonal ray representation
- orthogonal ray graph
Orthogonal Ray Graph (ORG)

orthogonal ray representation

orthogonal ray graph
Orthogonal Ray Graph (ORG)

- orthogonal ray representation
- orthogonal ray graph
Orthogonal Ray Graph (ORG)

Every orthogonal ray graph has a unit grid intersection representation.
Orthogonal Ray Graph (ORG)

\{ \text{ORGs} \} \subseteq \{ \text{UGIGs} \}
Orthogonal Ray Graph (ORG)

{ ORGs } ⊆ { UGIGs } 

- Cycle $C_{2n}$ of length $2n$: orthogonal ray gray
  $\Leftrightarrow 2 \leq n \leq 6$
  ($C_{14}$: not ORG)
  [A.M.S. Shrestha, et.al., '10]

- even length cycle:
  unit grid intersection graph
  ($C_{14}$: UGIG)
Orthogonal Ray Graph (ORG)

\{ \text{ORGs} \} \subset \{ \text{UGIGs} \}

- Cycle $C_{2n}$ of length $2n$: orthogonal ray gray
  $\Leftrightarrow 2 \leq n \leq 6$
  \cite{A.M.S. Shrestha, et.al., '10}
- even length cycle: unit grid intersection graph
  \cite{C_{14}: \text{UGIG}}
Relation between Graph Classes

\{ Bipartite Graphs \}

\cup

\{ Grid Intersection Graphs \}

\cup

\{ Unit Grid Intersection Graphs \}

\cup

\{ Orthogonal Ray Graphs \}
<table>
<thead>
<tr>
<th>Graph Class</th>
<th>Complexity</th>
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<tbody>
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<td>Bigraphs</td>
<td>$O(</td>
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<tr>
<td>GlGs</td>
<td>NP-complete</td>
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<tr>
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[ Kratochivíl, et.al. ’94]
### RECOGNITION PROBLEM

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[ Kratochivíl, et.al. ’94]
Theorem  The recognition of unit grid intersection graphs is NP-complete.

Proof:

- The recognition of unit grid intersection graph is in NP.

- A polynomial time reduction from CONSTRAINT GRAPH SATISFIABILITY (NP-C) to our problem.
Characterization of ORGs

**Theorem**

$G$: ORG $\iff$

$G$ has a UGIR lying inside an open square of side length $1 + \epsilon$ for any $\epsilon > 0$. 
Theorem
\( G: \text{ORG} \iff G \text{ has a UGIR lying inside an open square of side length } 1 + \epsilon \text{ for any } \epsilon > 0. \)
Theorem

\[ G : \text{ORG} \iff G \text{ has a UGIR lying inside an open square of side length } 1 + \varepsilon \text{ for any } \varepsilon > 0. \]
Theorem
$G$: ORG $\iff$
$G$ has a UGIR lying inside an open square of side length $1 + \epsilon$ for any $\epsilon > 0$. 

Characterization of ORGs
The complexity of the recognition of orthogonal ray graphs is open.

[Mustață, et.al., '16]
References

