Results related to structures in Gallai colorings

Roman Čada

Department of Mathematics & NTIS
University of West Bohemia
Czech Republic

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edge-colored graph – a triple $G = (V(G), E(G), c)$, where $(V(G), E(G))$ is a (simple finite undirected) graph and $c : E(G) \to \mathbb{Z}^+$ (edge coloring of G)

an edge set $F \subset E(G)$ is called rainbow if no two distinct edges in F receive the same color

a graph is called rainbow if its edge set is rainbow (heterochromatic, multicolored)

$c(e)$ the color of e
$d_G^c(u)$ color degree of u
$\delta^c(G)$ minimum color degree of G

study: colorings without a rainbow subgraph and corresponding Ramsey type problems

Theorem (Gallai, 1967)

In any rainbow triangle-free coloring of a complete graph, there exists a non-trivial partition of the vertices such that between the parts there is a total of at most two colors, and between each pair of parts there is only one color on the edges.

$\Rightarrow$ Gallai partition

reduced graph of a Gallai partition with parts $A_1, A_2, \ldots, A_t$ – a 2-colored complete K_t graph with vertices $a_1, a_2, \ldots, a_t$ and color of $a_i a_j$ as the color of edges between A_i and A_j

Given two graphs G and H and an integer k, the Gallai-Ramsey number $\mathrm{gr}_k(G : H)$ is the minimum integer n such that every k-coloring of K_n contains either a rainbow copy of G or a monochromatic copy of H as a subgraph.

general behaviour of $\mathrm{gr}_k(G : H)$

Theorem (Gyárfás, Sárközy, Sebó, Selkow)

Let H be a fixed graph with no isolated vertices. If H is not bipartite, then $gr_k(G : H)$ is exponential in k. If H is bipartite, then $gr_k(G : H)$ is linear in k.

Ramsey number $r_k(H)$ of a graph H is the least positive integer n such that every k-coloring of K_n contains a monochromatic copy of H

clearly, $\mathrm{gr}_k(H) \leq r_k(H)$

H is a clique

Conjecture (Fox, Grinshpun, Pach)

For integers $k \geq 1$ and $t \geq 3$

$$gr_k(K_3 : K_t) = \begin{cases} (r_2(K_t) - 1)^{k/2} + 1 & \text{if } k \text{ is even} \\ (t-1)(r_2(K_t) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd} \end{cases}$$

Constructions:

For $n = 1$

Theorem (Chung, Graham, 1983)

$$gr_k(K_3 : K_3) = \begin{cases} 5^{k/2} & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} & \text{if } k \text{ is odd} \end{cases}$$

For $n = 2$

Theorem (Liu, Magnant, Saito, Schiermeyer, Shi)

$$gr_k(K_3 : K_4) = \begin{cases} (r_2(K_4) - 1)^{k/2} + 1 & \text{if } k \text{ is even} \\ (t - 1)(r_2(K_4) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd} \end{cases}$$

H is and odd/even cycle (non bipartite vs. bipartite graph)

Theorem (Fujita, Magnant, 2011)

For any positive integer $k \geq 2$ *and* $n \geq 1$,

$$n2^k + 1 \leq gr_k(K_3 : C_{2n+1}) \leq k(n-1) + n(4n+1)3^{k-1}.$$

Theorem (Hall, Magnant, Ozeki, Tsugaki)

For any positive integer $k \geq 1$ and $n \geq 2$,

$$n2^k + 1 \leq gr_k(K_3 : C_{2n+1}) \leq (2^{k+3} - 3)n \log n.$$

classical Ramsey numbers of cycles

Theorem (Faudree, Schelp, 1974; Rosta, 1973)

For all $n \geq 2$,

$$r(C_{2n}, C_{2n}) = 3n - 1.$$

Theorem (Bondy, Erdös, 1973)

For all $n \geq 2$,

$$r(C_{2n+1}, C_{2n+1}) = 4n + 1.$$

so we have $r_2(C_{2n})$ and $r_2(C_{2n+1})$

classical Ramsey numbers of paths

Theorem (Gerencsér, Gyárfás, 1967)

For all $n \geq 2$,

$$r(P_n, P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - 1.$$

Theorem (Fujita, Magnant, 2011)

$gr_k(K_3 : C_5) = 2 \cdot 2^k + 1$ *for all* k.

Theorem (Bruce, Song)

$gr_k(K_3 : C_7) = 3 \cdot 2^k + 1$ *for all* k.

Theorem (Bosse, Song)

$gr_k(K_3 : C_9) = 4 \cdot 2^k + 1$ *for all* k.

Theorem (Bosse, Song)

$gr_k(K_3 : C_{11}) = 5 \cdot 2^k + 1$ *for all* k.

Theorem (Bosse, Song, Zhang)

$gr_k(K_3 : C_{13}) = 6 \cdot 2^k + 1$ *for all* k.

Theorem (Bosse, Song, Zhang)

$gr_k(K_3 : C_{15}) = 7 \cdot 2^k + 1$ *for all* k.

back to classical Ramsey numbers

Conjecture (Bondy, Erdös, 1973)

$r_k(C_n) = 2^{k-1}(n-1)+1$ *for odd* $n > 3$
equivalently
$r_k(C_{2\ell+1}) = \ell \cdot 2^k + 1$ *for* $\ell > 1$

Theorem (Bondy, Erdös, 1973)

$r_2(C_n) = 2n-1$
$r_k(C_n) \leq (k+2)!n$ *(n odd)*

Bondy-Erdös conjecture proved if $k = 3$ for large n

Theorem (Luczak, 1999)

$r_3(C_n) = 4n + o(n)$, *n odd*

Theorem (Kohayakawa, Simonovits, Skokan, 2005)

$r_3(C_n) = 4n - 3$, *n odd, sufficiently large*

for $k \geq 4$

Theorem (Luczak, Simonovits, Skokan, 2011)

For every $k \geq 4$ and odd n

$$r_k(C_n) \leq k2^k n + o(n) \quad as\ n \to +\infty.$$

it is a k vs. n problem
an optimization problem on $r(k, n)$
what is going on if $k \to +\infty$ or $n \to +\infty$

Theorem (Day, Johnson, 2017)

For all odd n and all k sufficiently large, there exists a constant $\epsilon = \epsilon(n) > 0$ such that

$$r_k(C_n) > (n-1)(2+\epsilon)^{k-1}.$$

so Bondy-Erdös conjecture disproved in case if k much larger than n

but on the other hand...

Theorem (Jenssen, Skokan)

For any fixed $k \geq 2$ and odd n sufficiently large,

$$r_k(C_n) = 2^{k-1}(n-1) + 1.$$

n must be large with respect to k
so Bondy-Erdös conjecture proved in case if n much larger than k

let $G = (V(G), E(G))$, denote $e(G) = |E(G)|$
let A, B be two disjoint subsets of $V(G)$
then $e(A, B)$ denotes the number of edges vw with $v \in A$ and $w \in B$
a pair (A, B) is (ϵ, G)-regular for some $\epsilon > 0$ if for every $A' \subseteq A$ and $B' \subseteq B$ with
$|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$, we have

$$\left| \frac{e(A', B')}{|A'||B'|} - \frac{e(A, B)}{|A||B|} \right| < \epsilon$$

a partition $\Pi = (V_i)_{i=0}^{k}$ of the vertex set $V(G)$ of G is (ϵ, k)-equitable if $|V_0| \leq \epsilon |V(G)|$ and $|V_1| = |V_2| = \ldots = |V_k|$
an (ϵ, k)-equitable partition Π is (k, ϵ, G)-regular if at most $\epsilon \binom{k}{2}$ pairs (V_i, V_j) with $1 \leq i < j \leq k$ are not (ϵ, G)-regular

Lemma (special case of Szemerédi's Regularity lemma)

For every $\epsilon > 0$ and k_0 there exists $K_0 = K_0(\epsilon, k_0) \geq k_0$ such that the following holds.
For all graphs, G_1, G_2, G_3, G_4, where $V(G_1) = V(G_2) = V(G_3) = V(G_4) = V$ and $|V| \geq k_0$, there exists a partition $\Pi = (V_0, V_1, V_2, \ldots, V_k)$ of V such that $k_0 \leq k \leq K_0$ and Π is (k, ϵ, G_s)-regular for $s = 1, 2, 3, 4$.

Theorem (Erdös, Gallai)

Each graph with n vertices and at least $(m-1)(n-1)/2+1$ *edges* ($3 \leq m \leq n$), *contains a cycle of length at least m.*

Dirac's and Ore's degree (sum) conditions for hamiltonian-connectivity

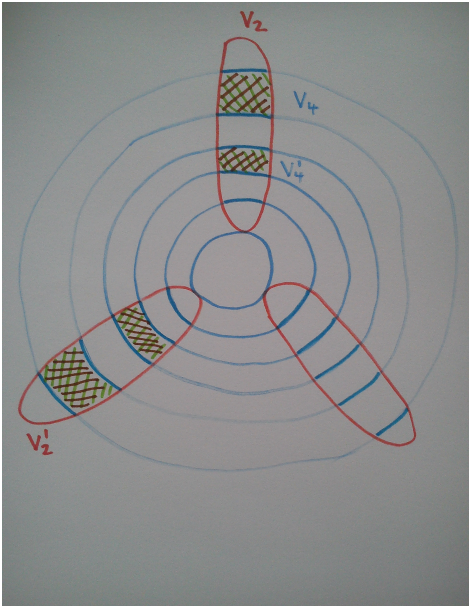
Lemma (Luczak)

For every small δ, $\alpha > 2\delta$ and $n = n(\delta/\alpha)$ sufficiently large the following holds. Each graph G on n vertices which contains no odd cycles longer than αn contains subgraphs G' and G'' such that:

- *$V(G')$ and $V(G'')$ form a partition of $V(G)$ and each of the sets $V(G')$ and $V(G'')$ is either empty or contains at least $\alpha\delta n/2$ vertices,*
- *G' is bipartite,*
- *G'' contains not more than $\alpha n|V(G'')|/2$ edges,*
- *all except no more than δn^2 edges of G belong to either G' or G''.*

Lemma (Luczak)

Let η be small and $n = n(\eta)$ sufficiently large. Furthermore, let G be a graph with $\approx 2n$ vertices and at least $\binom{V(G)|}{2} - f(\eta, n)$ vertices. Then every 2-coloring of the edges of G leads to a monochromatic odd cycle of length at least $(1 + \eta/const)n$



V2
V4
V4'
V2'

Let G and H be k-colored graphs with $V(H) \subseteq V(G)$. Let $\epsilon > 0$, then we say that G is ϵ-close to H if $|G_i \Delta H_i| \leq \epsilon v(G)^2$ for all $i \in [k]$.
Let F be a connected graph whose largest matching saturates m vertices, then F is called a connected matching (Luczak).

Jenssen, Skokan's proof:
transformation to a nonlinear optimization problem
$\max f(\|x\|)$ (linear)
subject to
$F(x) = w^T x + x^T Hx$
H relates to matchings in a hypercube Q_k

in Gallai coloring the two colors joining particular sets of partitions are from orthogonal subspaces

Theorem

$gr_k(K_3 : C_{2n+1}) = n2^k + 1$

structure of the set $X(\gamma)$ (defined by Jenssen and Skokan):
For $\gamma \geq 0$, let $X(\gamma)$ denote the set of elements $x \in R^\ell$ satisfying:

- $F(x) \leq \gamma$
- $x_\tau \leq 1 + \gamma$ whenever $w(\tau) = 1$ ("dimension")
- $x_\tau x_\sigma \leq \gamma$ whenever τ and σ are incompatible
- $x_\tau \geq 0$ for all τ

Kuhn+Karush+Tucker conditions

Let $f, g_1, g_2, \dots, g_r : R^m \to R$ be convex, differentiable functions and let

$$S = \{x \in R^m : g_i(x) \leq 0 \text{ for } i = 1, 2, \dots, r\}.$$

Suppose that there exists an $x_0 \in R^m$ such that $g_i(x_0) < 0$ for $i = 1, 2, \dots, r$. Then if $x^* \in S$ is such that

$$f(x^*) = sup_{x \in S} f(x),$$

then there exist $\lambda_1, \lambda_2, \dots, \lambda_r \in R$ such that

- $\nabla f(x^*) = \sum_{i=1}^{r} \lambda_i \nabla g_i(x^*)$
- $\lambda_i \geq 0$, $i = 1, 2, \dots, r$,
- $\lambda_i g_i(x^*) = 0$, $i = 1, 2, \dots, r$.

even n remains. . .

Theorem (Sung, Young, Xu, Li, 2006)

$r_k(C_n) \geq (k-1)n - 2k + 4$ for n even

Theorem (Luczak, Simonovits, Skokan, 2011)

For every $k \geq 2$ and even n

$$r_k(C_n) \leq kn + o(n) \quad as\ n \to +\infty.$$

Theorem (Sárközy, 2016)

For every $k \geq 2$ and even n

$$r_k(C_n) \leq \left(k - \frac{1}{16k^3 + 1}\right) n + o(n) \quad as\ n \to +\infty.$$

Theorem (Davies, Jenssen, Roberts, 2017)

For every $k \geq 2$ and even n

$$r_k(C_n) \leq \left(k - \frac{1}{4}\right) n + o(n) \quad as\ n \to +\infty.$$

Theorem (Knierim, Su)

For every $k \geq 2$ and even n

$$r_k(C_n) \leq \left(k - \frac{1}{2}\right) n + o(n) \quad as\ n \to +\infty.$$

Thank you for your attention.