Forbidden subgraphs for constant domination number

Michitaka Furuya (Kitasato University)

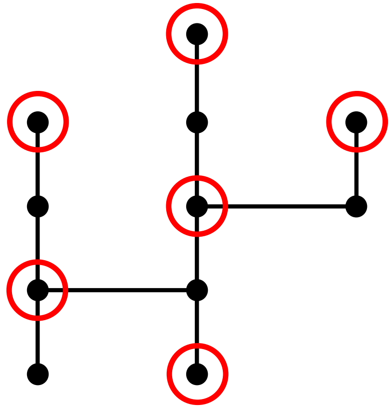
Domination

Let G be a graph.

A set $S \subseteq V(G)$ is a dominating set of G if

$$\text{for } \forall\, x \in V(G) - S,\ \exists\, y \in S \ \text{ s.t. } \ xy \in E(G).$$

The minimum cardinality of a dominating set of G is called the domination number of G, and is denoted by $\gamma(G)$.



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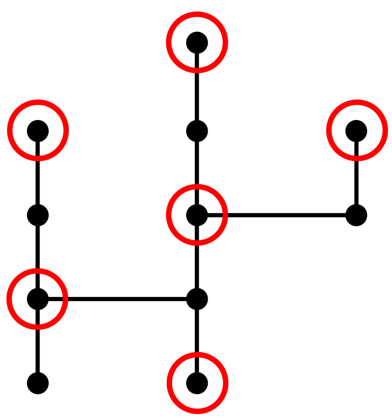

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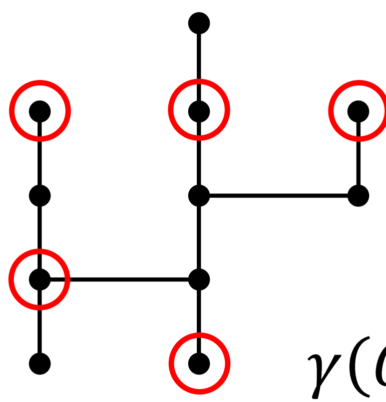

$\gamma(G) = 5$

Domination

Theorem 1 (Ore, 1962)
Let G be a conn. graph of order $n \geq 2$.
Then $\gamma(G) \leq n/2$.

Theorem 2 (Fink et al., 1985; Payan and Xuong, 1982)
A conn. graph G of order n satisfies $\gamma(G) = n/2$ if and only if $G = C_4$ or G is the corona of a conn. graph.

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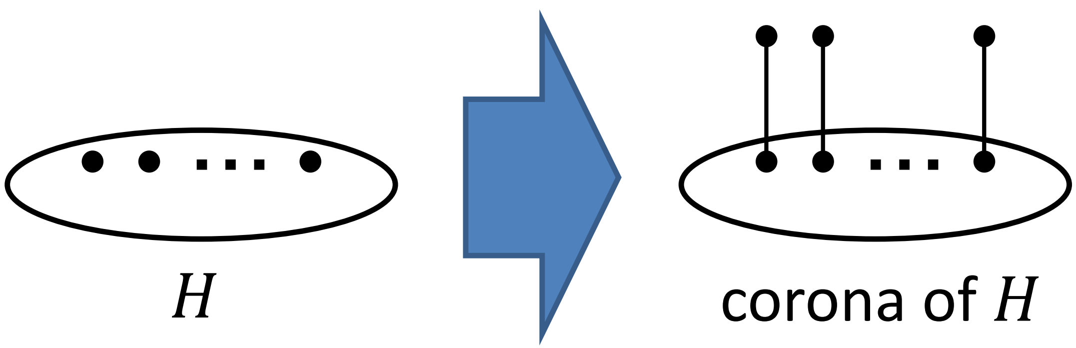
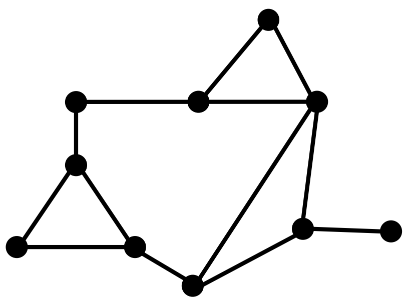
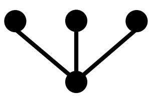


Forbidden subgraph

Let $\mathcal{H}$ be a set of conn. graphs.

A graph G is $\mathcal{H}$-free if G has no graph in $\mathcal{H}$ as an induced subgraph.
(If G is $\{H\}$-free, then G is simply said to be H-free.)
In this context, graphs in $\mathcal{H}$ are called forbidden subgraphs.



$K_{1,3}$-free graph
($K_{1,3}$: )

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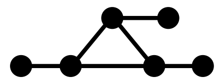
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be sets of conn. graphs.

We write $\mathcal{H}_1 \leq \mathcal{H}_2$ if

for $\forall H_2 \in \mathcal{H}_2$, $\exists H_1 \in \mathcal{H}_1$ s.t. H_1 is an induced subgraph of H_2.

<u>Remark</u>
If $\mathcal{H}_1 \leq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is $\mathcal{H}_2$-free.

Domination and forbidden subgraph

Theorem 3 (Cockayne et al., 1985)
Let G be a conn. $\{K_{1,3}$,  $\}$-free graph of order n.
Then $\gamma(G) \leq \lceil n/3 \rceil$.

Let $\gamma_{\text{pr}}(G)$ be the minimum cardinality of a dominating set S of G s.t. $G[S]$ has a perfect matching.

Theorem 4 (Dorbec et al., 2006)
Let G be a conn. $K_{1,m}$-free graph of order $n \geq 2$.
Then $\gamma_{\text{pr}}(G) \leq 2(mn+1)/(2m+1)$.

Theorem 5 (Dorbec and Gravier, 2008)
Let G be a conn. P_5-free graph of order $n \geq 2$.
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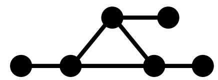
Let G be a conn. $\{K_{1,3},$  $\}$-free graph of order n.
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Domination and forbidden subgraph

We focus on the most effective case, that is, sets $\mathcal{H}$ of connected graphs satisfying the following:

$\exists$ const. $c = c(\mathcal{H})$ s.t. for $\forall$ conn. $\mathcal{H}$-free graph G, $\gamma(G) \leq c$.

What graphs do belong to $\mathcal{H}$??

Let K_{c+1}^* be the corona of K_{c+1}.

Since $\gamma(K_{c+1}^*) = c + 1$, K_{c+1}^* is *not* $\mathcal{H}$-free.

➡ $\exists H \in \mathcal{H}$ s.t. H is an induced subgraph of K_{c+1}^*.

➡ $\mathcal{H} \leq \{K_{c+1}^*\}$.

By similar argument, $\mathcal{H} \leq \{$ $, P_{3c+1}\}$.

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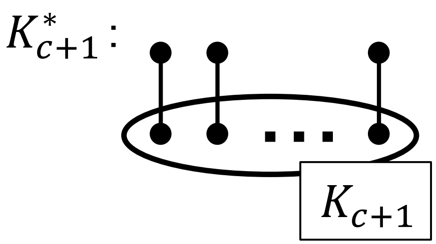

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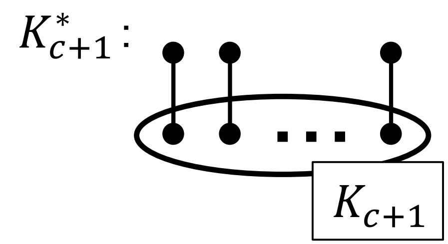

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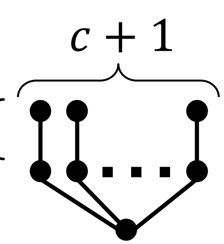
By similar argument, $\mathcal{H} \leq \{$  $, P_{3c+1}\}$.

Main result

<u>Theorem</u>

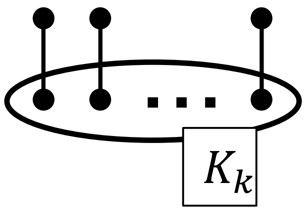
Let $\mathcal{H}$ be a set of conn. graphs.

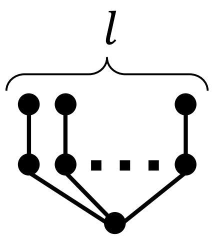
Then

$\exists$ const. $c = c(\mathcal{H})$ s.t. for $\forall$ conn. $\mathcal{H}$-free graph G, $\gamma(G) \leq c$

if and only if

$\mathcal{H} \leq \{K_k^*,\ S_l^*,\ P_m\}$ for some positive integers k, l and m

where $K_k^* =$ 

and $S_l^* =$ 

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Outline of proof of main result

We show that

$$\gamma(G) \leq 1 + \sum_{2 \leq i \leq m-2} f_{k,l}(i) R(k,l)$$

for $\forall$conn. $\{K_k^*,\ S_l^*,\ P_m\}$-free graph G, where

$$f_{k,l}(i) := \begin{cases} 1 & (i = 1) \\ R\big(k, (l-1) f_{k,l}(i-1) + 1\big) - 1 & (i \geq 2) \end{cases}.$$

Let $x \in V(G)$, and let $X_i = \{y \in V(G) : \mathrm{dist}(x,y) = i\}$.
Then $V(G) = \bigcup_{0 \leq i \leq m-2} X_i$.

<u>Key Lemma</u>
For $i \geq 2$, the set X_i is dominated by at most $f_{k,l}(i) R(k,l)$ vertices.

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Let $U \subseteq X_{i-1}$ be a smallest set dominating X_i.

If U is "large"...

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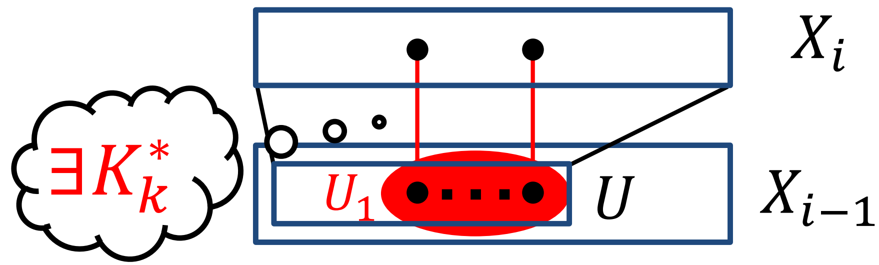
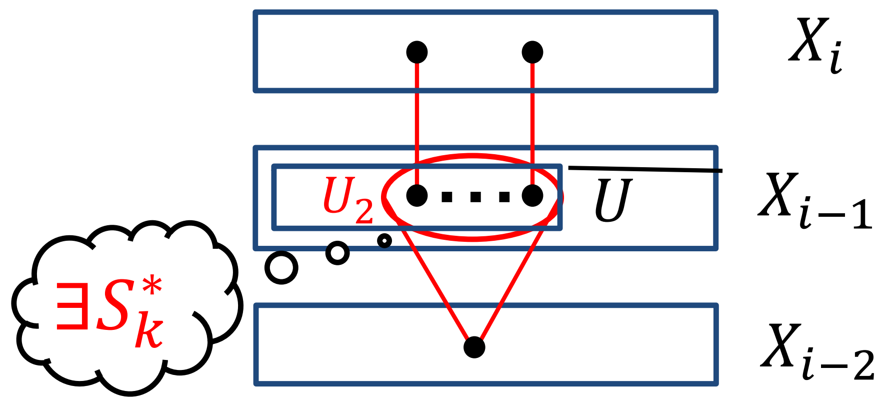


- If ∃large indep. set $U_2 \subseteq U$ …



Extension of main result

Corollary

Let μ be an invariant of graphs s.t.

$c_1\gamma(G) \leq \mu(G) \leq c_2\gamma(G)$ for $\forall$ conn. graph G of suff. large order.

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Thank you for your attention!