On extremal mixed graphs

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Joint work with Dr. Grahame Erskine

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Terminology

Definition

Let G be a mixed graph. The undirected subgraph is G^U and the directed subgraph G^Z.

The directed out-degree of a vertex v is $d^+(v)$.
The directed in-degree of v is $d^-(v)$.
The undirected degree of v is $d(v)$.
G is out-regular if G^U is regular and G^Z out-regular.
G is totally regular if G^U is regular and G^Z is diregular.

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Degree/diameter problem

What is the largest possible order of a mixed graph with maximum undirected degree r, maximum directed out-degree z and diameter k?

The mixed Moore bound

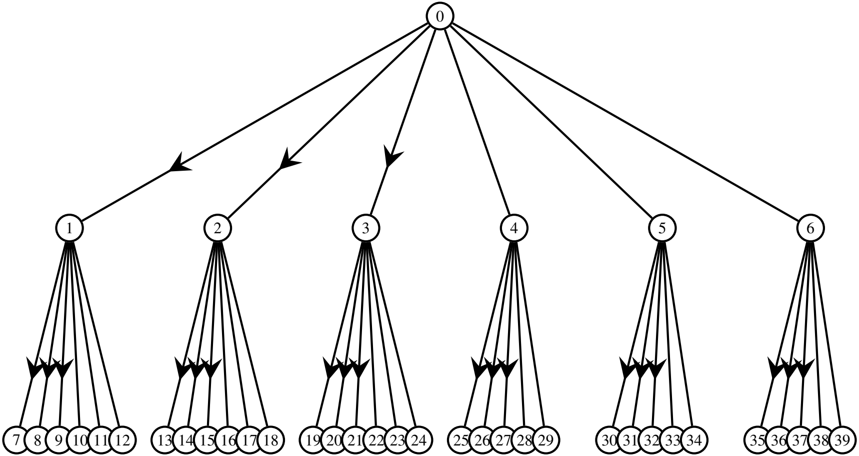

Figure: The labelled Moore tree for $z = 3, r = 3, k = 2$

The mixed Moore bound

For $1 \leq i \leq k$, let L_i be the maximum possible number of vertices at level i of the Moore tree. Then $L_0 = 1, L_1, r + z$ and

$$L_{i+2} = (r + z - 1)L_{i+1} + zL_i.$$

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Solving the resulting recurrence relations yields

$$|G| \leq M(r, z, k) = A\frac{u_1^{k+1} - 1}{u_1 - 1} + B\frac{u_2^{k+1} - 1}{u_2 - 1},$$

where $v = (z + r)^2 + 2(z - r) + 1$ and

$$u_1 = \frac{z + r - 1 - \sqrt{v}}{2}, u_2 = \frac{z + r - 1 + \sqrt{v}}{2}$$

$$A = \frac{\sqrt{v} - (z + r + 1)}{2\sqrt{v}}, B = \frac{\sqrt{v} + (z + r + 1)}{2\sqrt{v}}$$

Mixed Moore graphs

A mixed graph is k-geodetic if between any ordered pair of vertices u, v there is at most one non-backtracking walk of length $\leq k$.

Theorem

A mixed graph is Moore if and only if it is out-regular, has diameter k and is k-geodetic.

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Theorem Nguyen, Miller, Gimbert, 2007

There are no mixed Moore graphs with $k \geq 3$.

Mixed Moore graphs with $k = 2$

The mixed Moore bound for $k = 2$ is $(r + z)^2 + z + 1$.

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Theorem Bosák 1979

If G is a non-trivial mixed Moore graphs with diameter $k = 2$, then there exists an odd integer c such that $c | (4z-3)(4z+5)$ and $r = \frac{1}{4}(c^2+3)$.

Open cases

Undirected degree r	Directed degree z	Order n
1	any	$r^2 + 2r + 3$
3	1	18
	3	40
	4	54
	6	88
	7	108
	...	...
7	2	84
	5	150
	7	204
	...	...
13	4	294
	6	368
	...	...
21	1	486
	...	...
...	...	...

Kautz graphs

Take an alphabet Ω of size $z+2$. The vertices of $Kautz(z)$ are words ab, where $a \neq b$. Adjacencies are given by

$$ab \to bc, c \neq a$$

$$ab \sim ba.$$

$Kautz(z)$ has undirected degree $r=1$, directed out-degree z, order z^2+3z+2 and diameter $k=2$.

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Theorem Gimbert

There is a unique Moore graph with diameter $k=2$, undirected degree $r=1$ and any value of z.

Kautz graphs: example

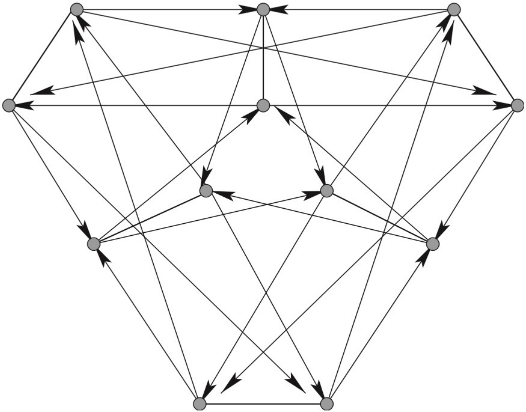

Figure: A mixed Moore graph with $r = 1, z = 2$

Cayley mixed Moore graphs

Take a group G and subsets A and B such that

$$e \notin A \cup B$$

$$A = A^{-1}, B \cap B^{-1} = \varnothing$$

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$$V(Cay(G, A, B)) = G,$$

$$u \sim ua, a \in A, u \to ub, b \in B.$$

Cayley mixed Moore graphs

$Kautz(z)$ is Cayley iff $z + 2$ is a prime power.

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Theorem Bosák, Jørgenson

There is a Cayley mixed Moore graph with $r = 3, z = 1, n = 18$ and two Cayley mixed Moore graphs with $r = 3, z = 7, n = 108$.

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Theorem Erskine

There are no further Cayley mixed Moore graphs with order ≤ 485.

Theorem López, Miret and Fernández

There are no mixed Moore graphs with orders 40, 54 or 84.

Degree/diameter for mixed graphs

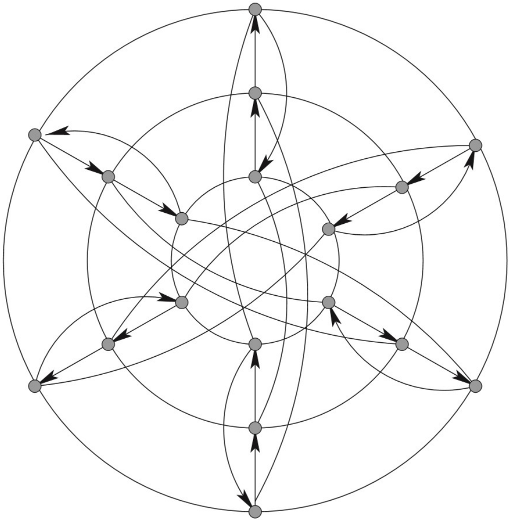

Figure: The Bosák graph

Totally regular almost mixed Moore graphs

A mixed graph with defect one is an almost mixed Moore graph.

Theorem López, Miret

If G is an almost mixed Moore graph, then r is even. If G is a totally regular almost mixed Moore graph with $r > 2, z \geq 1$ and diameter $k = 2$, then either
There exists odd $c \in \mathbb{Z}$ such that $4r + 1 = c^2$ and $c|(4z+1)(4z-7)$ or there exists odd $c \in \mathbb{Z}$ such that $4r - 7 = c^2$ and $c|(16z^2 + 40z - 23)$.

An almost mixed Moore graph

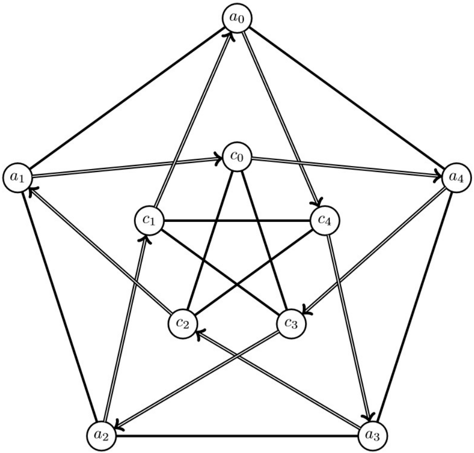

Figure: A mixed graph with defect $\delta = 1$, diameter $k = 2$, $r = 2, z = 1$.

Total regularity of almost mixed Moore graphs

Let G be an almost mixed Moore mixed graph with undirected degree r, directed out-degree z and diameter $k = 2$ that is not totally regular. We write

$$S = \{v \in V(G) : d^-(v) < z\}, S' = \{v' \in V(G) : d^+(v') > z\}$$

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$$U(u) = \{v \in V(G) : u \sim v\}$$

$$Z^+(u) = \{v \in V(G) : u \to v\}, Z^-(u) = \{v \in V(G) : v \to u\}$$

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$$Z^+(u) = \{v \in V(G) : u \to v\}, Z^-(u) = \{v \in V(G) : v \to u\}$$

$$N^+(u) = U(u) \cup Z^+(u), N^-(u) = U(u) \cup Z^-(u).$$

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If $v \in S$, then $d^-(v) = z - 1$ and for all $u \in V(G)$ we have $S \subseteq N^+(r(u))$.

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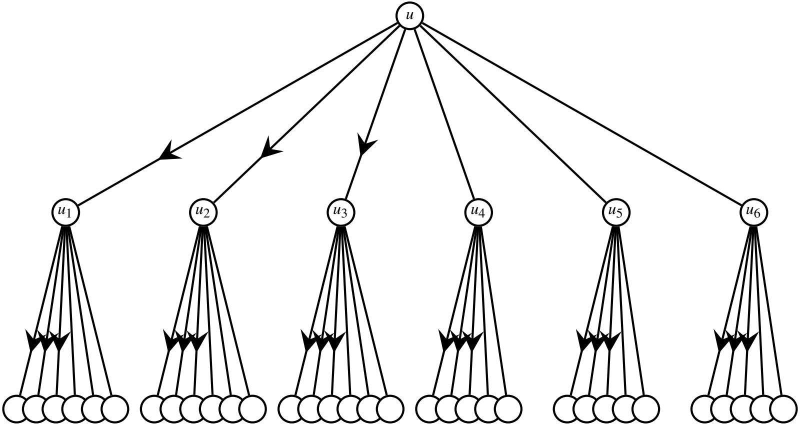

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Lemma

For all $u \in V(G)$ we have $S' \subseteq r(N^+(u))$.

Total regularity of almost mixed Moore graphs

Observe that $\sum_{v' \in S'} (d^-(v') - z) = \sum_{v \in S} (z - d^-(v)) = |S|$.

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Theorem

$|S| = r + z$.

Proof

Let $v \in S$. We have $d^-(v) = z - 1$. Every vertex can reach v in G by a path of length ≤ 2, so by drawing a Moore tree for the converse graph of G we see that we obtain an upper bound for the order of G by assuming that $S' \in N^-(u)$, yielding

$$|V(G)| \leq 1 + r + (z - 1) + r(r + z - 1) + (z - 1)(r + z) + |S|.$$

Rearranging, $|S| \geq r + z$.

Total regularity of almost mixed Moore graphs

Observe that $\sum_{v' \in S'}(d^-(v') - z) = \sum_{v \in S}(z - d^-(v)) = |S|$.

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Rearranging, $|S| \geq r + z$.

It follows that $S = N^+(r(u))$ for all $u \in V(G)$.

Total regularity of almost mixed Moore graphs

Lemma

$r \neq 1$

Proof

If $r = 1$, then G contains a perfect matching, so that G must have even order, whereas $|V(G)| = z^2 + 3z + 1$ is odd.

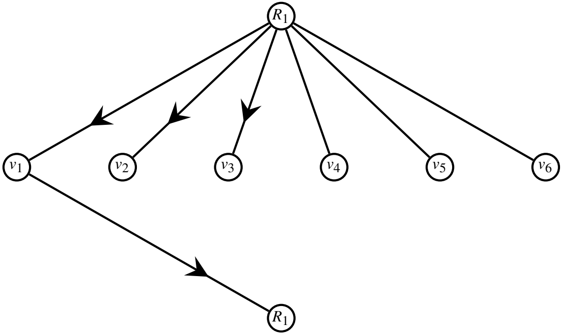
Total regularity of almost mixed Moore graphs

Lemma

$r \neq 1$

Lemma

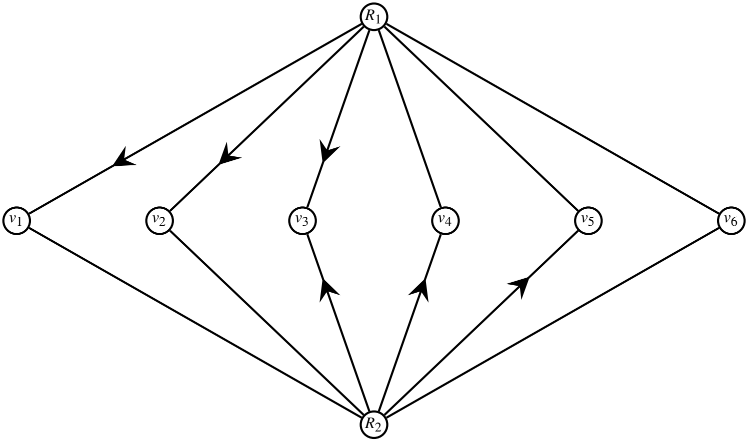
G has at least two repeats.



Total regularity of almost mixed Moore graphs

Theorem

$r = 2$ and there are exactly two repeats, R_1 and R_2.



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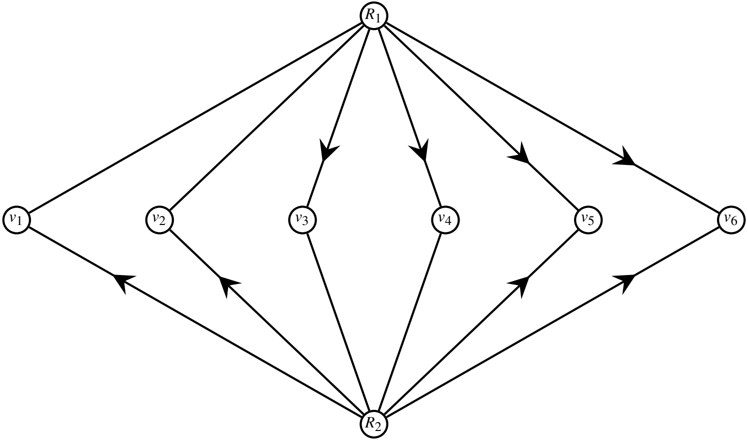

Total regularity of almost mixed Moore graphs

Suppose that there are m_1 vertices with repeat R_1 and m_2 vertices with repeat R_2. Label the vertices of G $u_1, u_2, \ldots, u_n$, so that $u_1 = R_1, u_2 = R_2$, $r(u_{2+j}) = R_2$ for $1 \leq j \leq m_2 - 1$ and $r(u_{1+m_2+s}) = R_1$ for $1 \leq s \leq m_1 - 1$. If A is the adjacency matrix of G, then

$$I + A + A^2 = J + 2I + P,$$

where I is the $n \times n$ identity matrix, J is the $n \times n$ all-one matrix and P is the matrix with entries given by

$$P_{ij} = \begin{cases} 1, & \text{if } r(u_i) = u_j. \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

All non-zero entries of P occur in the first two columns.

Total regularity of almost mixed Moore graphs

Inspection shows that the matrix $J+P$ has the following eigenvalues:
i) eigenvalue $n+1$ with multiplicity one.
Corresponding eigenvector: all-one vector of length n
ii) eigenvalue -1 with multiplicity one.
Corresponding eigenvector:

$$f(u) = \begin{cases} 1, & \text{if } r(u) = R_1. \\ -(m_2+1)/(m_1+1), & \text{if } r(u) = R_2. \end{cases} \tag{2}$$

Total regularity of almost mixed Moore graphs

iii) eigenvalue 0 with multiplicity $n-2$.
Corresponding eigenvectors:

$$f_i(u_j) = \begin{cases} 1, & \text{if } j = i+2, \\ -1, & \text{if } j = n, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

for $1 \leq i \leq n-3$ and

$$g(u_i) = \begin{cases} 1, & \text{if } i = 1 \text{ or } 2, \\ -3/(n-2), & \text{otherwise.} \end{cases} \tag{4}$$

Total regularity of almost mixed Moore graphs

It follows that $I + A + A^2$ has spectrum $\{n+3, 1, 2^{(n-2)}\}$, so that G must have spectrum $\sigma(G) = \{\lambda_1, \ldots, \lambda_n\}$, where

λ_1 is a solution of $\lambda^2 + \lambda - (n+2) = 0$
λ_2 is a solution of $\lambda^2 + \lambda = 0$
λ_i is a solution of $\lambda^2 + \lambda - 1 = 0$ for $3 \le i \le n$.

The solutions of the first equation are $\lambda_1 = \frac{-1 \pm \sqrt{4n+9}}{2}$. As $r = 2$, the order of G is $n = z^2 + 5z + 4$, so $4n + 9 = 4z^2 + 20z + 25 = (2z+5)^2$, yielding $\lambda_1 = z + 2$ or $-z - 3$.

Trivially $\lambda_2 = 0$ or -1.

The third equation has solutions $\frac{-1 \pm \sqrt{5}}{2}$.

Total regularity of almost mixed Moore graphs

G has no loops, so $Tr(A) = \sum_{\lambda \in \sigma(G)} \lambda = 0$.

The trace is rational $\implies$ half of the eigenvalues $\lambda_2, \ldots, \lambda_n$ take the $+$ sign and half the $-$ sign.

Thus $Tr(A) = -\frac{n}{2} + 1 + \lambda_1 + \lambda_2$.

The order $n = (r+z)^2 + z$ of G is equal to one of the following:

$$2z+6, 2z+4, -2z-4 \text{ or } -2z-6$$

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Theorem Tuite

Almost mixed Moore graphs with diameter two are totally regular.

The ‘Fiol bound’ on totally regular mixed graphs

Theorem Dalfó, M. A. Fiol, N. López

A totally regular mixed graph with undirected degree r and diameter k has defect $\delta \geq r$.

The ‘Fiol bound’ on totally regular mixed graphs

Theorem Dalfó, M. A. Fiol, N. López

A totally regular mixed graph with undirected degree r and diameter k has defect $\delta \geq r$.

Theorem

A totally regular mixed graph with undirected degree r, directed out-degree $z = 1$ and diameter k has defect

$$\delta \geq \frac{1}{2}\Big[(\sum_{i=1}^{k-3} Z_i) - Down(k-4) - Down(k-3)\Big] + Z_{k-2} + Z_{k-1},$$

where Z_t is the number of vertices in undirected branches of a Moore tree that arise as end-points of arcs and $Down$ is the solution of recurrence relations.....

The 'Fiol bound' on totally regular mixed graphs

The numbers $Up(t)$ and $Down(t)$ satisfy the relations

$$Up(t) + Down(t) = Z_t$$

and

$$Up(t+2) = Down(t)$$

for $1 \leq t \leq k-3$.

For $k \equiv 0 \pmod 4$, we have

$$Down(k-3) = \frac{r}{\sqrt{r^2+4}}\left[\frac{\lambda_1^2(\lambda_1^{k-4}-1)}{\lambda_1^2+1} - \frac{\lambda_2^2(\lambda_2^{k-4}-1)}{\lambda_2^2+1}\right],$$

where

$$\lambda_1 = \frac{1}{2}(r + \sqrt{r^2+4})$$

and

$$\lambda_2 = \frac{1}{2}(r - \sqrt{r^2+4}).$$

The 'Fiol bound' is tight

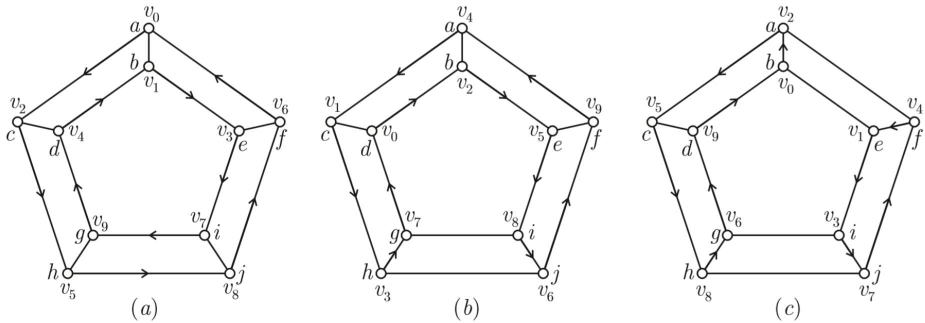

Figure: Three $(1, 1, k; -1)$-graphs

Mixed graphs with small excess

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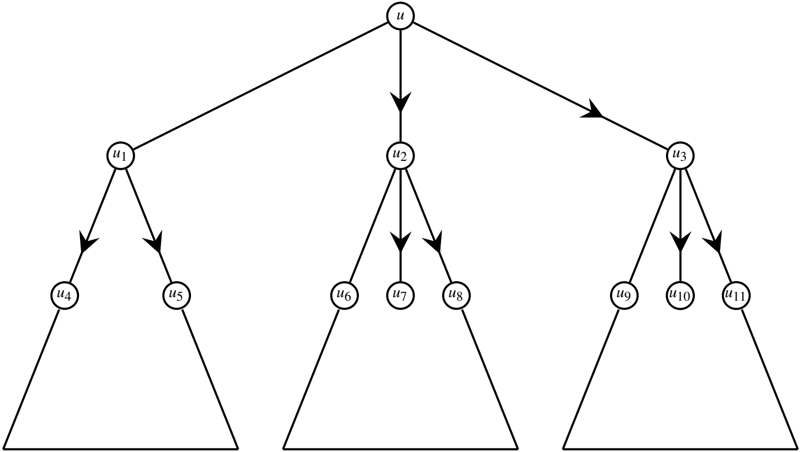
Theorem Tuite

Mixed graphs with excess $\epsilon = 1$ are out-regular and are totally regular if:

- $k = 2$
- $k = 3$ and $z = 1$
- $k \geq 4$ and $z = 1$.

A mixed graph with $\epsilon = 1$ is totally regular if and only if o is an automorphism.

Bounds on totally regular graphs



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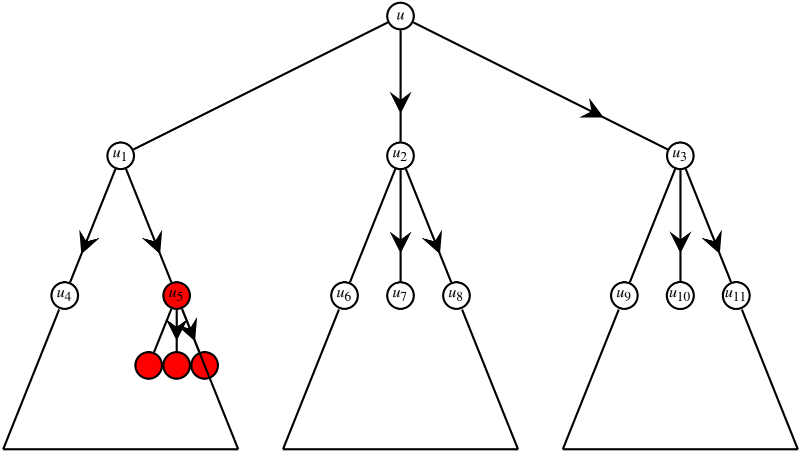

Bounds on totally regular graphs

Z_t (respectively U_t) is the number of vertices in the undirected branches at level t in the Moore tree based at u that are end-points of arcs (edges) emanating from level $t-1$.
$U_1 = r, Z_1 = 0$ and $Z_2 = rz$. These numbers satisfy the recurrence relations

$$U_{t+1} = (r-1)U_t + rZ_t, Z_{t+1} = zU_t + zZ_t$$

for $t \geq 1$. It follows that

$$Z_{t+2} = zU_{t+1} + zZ_{t+1} = z((r-1)U_t + rZ_t) + zZ_{t+1}.$$

Substituting using the second relation,

$$Z_{t+2} = zZ_{t+1} + rzZ_t + z(r-1)(1/z)(Z_{t+1} - zZ_t) = (r+z-1)Z_{t+1} + zZ_t.$$

Bounds on totally regular graphs

This second-order recurrence relation has characteristic equation

$$\lambda^2 - (r + z - 1)\lambda - z = 0,$$

with solutions

$$\lambda_1 = \frac{1}{2}(r + z - 1 + \phi)$$

and

$$\lambda_2 = \frac{1}{2}(r + z - 1 - \phi),$$

where

$$\phi = \sqrt{(r + z - 1)^2 + 4z}.$$

The discriminant $\phi^2 = (r + z - 1)^2 + 4z$ is strictly positive, so λ_1, λ_2 are real and distinct. It follows that

$$Z_t = A\lambda_1^t + B\lambda_2^t$$

for $t \geq 1$ and some constants A and B.

Bounds on totally regular graphs

Substituting $Z_1 = 0, Z_2 = rz$, we obtain

$$Z_t = \frac{rz}{\phi}(\lambda_1^{t-1} - \lambda_2^{t-1})$$

for $t \geq 1$. Summing, we find that there are

$$\sum_{i=0}^{k-2} \frac{rz}{\phi}(\lambda_1^i - \lambda_2^i) = \frac{rz}{\phi}\Big[\frac{\lambda_1^{k-1} - 1}{\lambda_1 - 1} - \frac{\lambda_2^{k-1} - 1}{\lambda_2 - 1}\Big]$$

such vertices. As the union of the outlier sets of the vertices in $Z(u)$ contain a maximum of $z\epsilon$ distinct vertices between them, it follows that

$$z\epsilon \geq \frac{rz}{\phi}\Big[\frac{\lambda_1^{k-1} - 1}{\lambda_1 - 1} - \frac{\lambda_2^{k-1} - 1}{\lambda_2 - 1}\Big].$$

Bounds on totally regular graphs

Theorem Tuite

For $k \geq 3$, the excess ϵ of a totally regular k-geodetic mixed graph with undirected degree r, directed degree z satisfies

$$\epsilon \geq \frac{r}{\phi}\left[\frac{\lambda_1^{k-1}-1}{\lambda_1 - 1} - \frac{\lambda_2^{k-1}-1}{\lambda_2 - 1}\right],$$

where

$$\phi = \sqrt{(r+z-1)^2 + 4z},$$

$$\lambda_1 = \frac{1}{2}(r+z-1+\phi)$$

and

$$\lambda_2 = \frac{1}{2}(r+z-1-\phi).$$

Bounds for $k = 4$

r/z	1	2	3	4	5	6
1	2	3	4	5	6	7
2	6	8	10	12	14	16
3	12	15	18	21	24	27
4	20	24	28	32	36	40
5	30	35	40	45	50	55
6	42	48	54	60	66	72
7	56	63	70	77	84	91
8	72	80	88	96	104	112
9	90	99	108	117	126	135
10	110	120	130	140	150	160
11	132	143	154	165	176	187
12	156	168	180	192	204	216
13	182	195	208	221	234	247
14	210	224	238	252	266	280
15	240	255	270	285	300	315

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6	42	48	54	60	66	72
7	56	63	70	77	84	91
8	72	80	88	96	104	112
9	90	99	108	117	126	135
10	110	120	130	140	150	160
11	132	143	154	165	176	187
12	156	168	180	192	204	216
13	182	195	208	221	234	247
14	210	224	238	252	266	280
15	240	255	270	285	300	315

Challenge: find a tight bound!

A graph with excess one

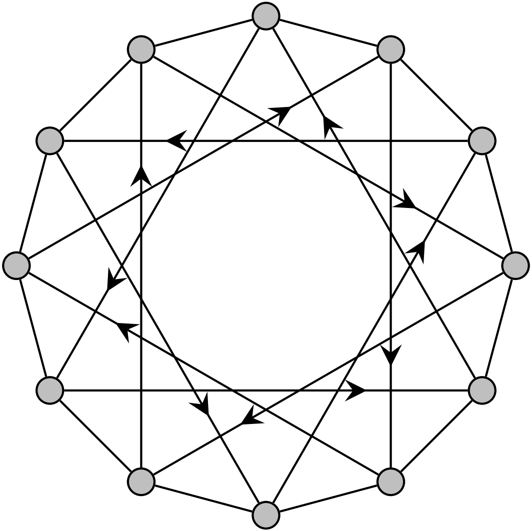

Figure: The unique mixed graph with $k = 2, r = 2, z = 1$ and excess $\epsilon = 1$.

Theorem Tuite

Let G be a totally regular $(r, z, 2, +1)$-graph. Then either

$r = 2$

$4r + 1$ is square and $c|(16z^2 - 24z + 25)$ or

$4r - 7$ is square and $c|(16z^2 + 40z + 9)$.

Spectral theory of mixed graphs with excess one

Let G be a totally regular $(r, z, 2, +1)$-graph with adjacency matrix A and outlier function o, where $r > 2$. We have

$$I + A + A^2 = J + rI - P,$$

where P is a permutation matrix. P is similar to a matrix in block diagonal form, where each block P_n is a cyclic permutation matrix. The characteristic polynomial of $J - P_n$ is

$$(x - (n-1))(x+1)^{-1}(x^n - (-1)^n),$$

Suppose that o has permutation cycle structure $(m_1, m_2, \ldots, m_n)$. Let $m'(n) = \sum_{n|(2i+1)} m_{2i+1}$, $m''(n) = \sum_{n|2i} m_{2i}$ and $m(n) = m'(n) + m''(n)$. The characteristic polynomial of $J - P$ is

$$(x-(n-1))(x-1)^{m''(1)}(x+1)^{m(2)+m'(1)-1} \prod_{i=3,\text{i odd}}^{n} \Phi_i(x)^{m''(i)} \Phi_{2i}(x)^{m'(i)} \prod_{i=4,i \text{ even}}^{n} \Phi_i(x)^{m(i)}$$

Hence $det((x-r)I-(J-P))$ is

$$(x-r-(n-1))(x-r-1)^{m''(1)}(x+1-r)^{m(2)+m'(1)-1}$$

$$\times \prod_{i=3,\text{i odd}}^{n} \Phi_i(x-r)^{m''(i)}\Phi_{2i}(x-r)^{m'(i)} \prod_{i=4,i\text{ even}}^{n} \Phi_i(x-r)^{m(i)}$$

Spectral theory of mixed graphs with excess one

Therefore $\sigma(A)$ contains:

1 solution of $1 + x + x^2 = n + r - 1$
$m''(1)$ solutions of $1 + x + x^2 = r + 1$
$m(2) + m'(1) - 1$ solutions of $1 + x + x^2 = r - 1$
$m''(i)$ solutions of $\Phi_i(1 + x + x^2 - r) = 0$, $3 \le i \le n$, i odd
$m'(i)$ solutions of $\Phi_{2i}(1 + x + x^2 - r) = 0$, $3 \le i \le n$, i odd
$m(i)$ solutions of $\Phi_i(1 + x + x^2 - r) = 0$, $4 \le i \le n$, i even.

Theorem López, Miret

The polynomials $\Phi_i(1 + x + x^2 - r)$ are irreducible for $r \ge 1$.

We have $n + r - 1 = (r + z)^2 + (r + z) + 1$, so the first eigenvalue is $d = r + z$. The discriminant of $x^2 + x - r$ is $\Delta = 4r + 1$ and the discriminant of $x^2 + x - r + 2$ is $\Delta = 4r - 7$.

If neither $4r+1$ nor $4r-7$ is square

In this case x^2+x-r and $x^2+x-r+2$ are irreducible, so

$$\phi_G(x) = (x-d)(x^2+x-r)^{m''(1)/2}(x^2+x+2-r)^{\frac{m(2)+m'(1)-1}{2}}$$

$$\times \prod_{i=3,\text{i odd}}^{n} \Phi_i(1+x+x^2-r)^{m''(i)/2}\Phi_{2i}(1+x+x^2-r)^{m'(i)/2}$$

$$\times \prod_{i=4,i \text{ even}}^{n} \Phi_i(1+x+x^2-r)^{m(i)/2}$$

If neither $4r + 1$ nor $4r - 7$ is square

G has no loops, so $Tr(A) = 0$. Summing the eigenvalues, we obtain

$$0 = d + (-1)[\frac{m''(1) + m(2) + m'(1) - 1}{2}] - \frac{1}{2} \sum_{i=3,\text{i odd}}^{n} m''(i)\psi(i)$$

$$-\frac{1}{2} \sum_{i=3,\text{i odd}}^{n} m'(i)\psi(2i) - \frac{1}{2} \sum_{i=4,\text{i even}}^{n} m(i)\psi(i)$$

$$= d + \frac{1}{2} - \frac{1}{2} \sum_{i=1}^{n} m(i)\psi(i)$$

If neither $4r+1$ nor $4r-7$ is square

Lemma

$\sum_{i=1}^{n} m(i)\psi(i) = n$

Thus

$$n = 2d + 1,$$

yielding $d = r = 1$.

If $4r+1$ is square, but $4r-7$ not

$x^2+x-r+2$ is irreducible. Let $4r+1=c^2$, c odd. Then $x^2+x-r=0$ has roots α,β with respective multiplicities a,b, where $a+b=m''(1)$. We have

$$\alpha=\frac{-1+c}{2},\beta=\frac{-1-c}{2}.$$

Now

$$Tr(A)=0=d+a\alpha+b\beta-\frac{m(2)+m'(1)-1}{2}-\frac{n}{2}+\frac{m(1)+m(2)}{2}$$

$$=d+a\frac{(-1+c)}{2}+b\frac{(-1-c)}{2}+\frac{1}{2}+\frac{m''(1)}{2}.$$

Hence $n=2d+1+c(a-b)$. Substituting for n and c, we obtain

$$c^4+(8z-10)c^2-16(a-b)c+16z^2-24z+25=0.$$

Automorphisms of mixed graphs with excess one

Theorem Tuite

Let G be a $(r, z, 2; +1)$-graph and $\psi \in Aut(G)$. Then the subgraph induced by the fixed points of ψ is isomorphic to one of:

K_1

$2K_2$

A $(r', z', +1)$-graph with $r' \leq r, z' \leq z$.

Mixed geodetic cages exist

Definition

Mixed geodetic cages are k-geodetic mixed graphs with minimum undirected degree r, minimum directed out-degree z and smallest possible order.

Mixed geodetic cages exist

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Mixed geodetic cages are k-geodetic mixed graphs with minimum undirected degree r, minimum directed out-degree z and smallest possible order.

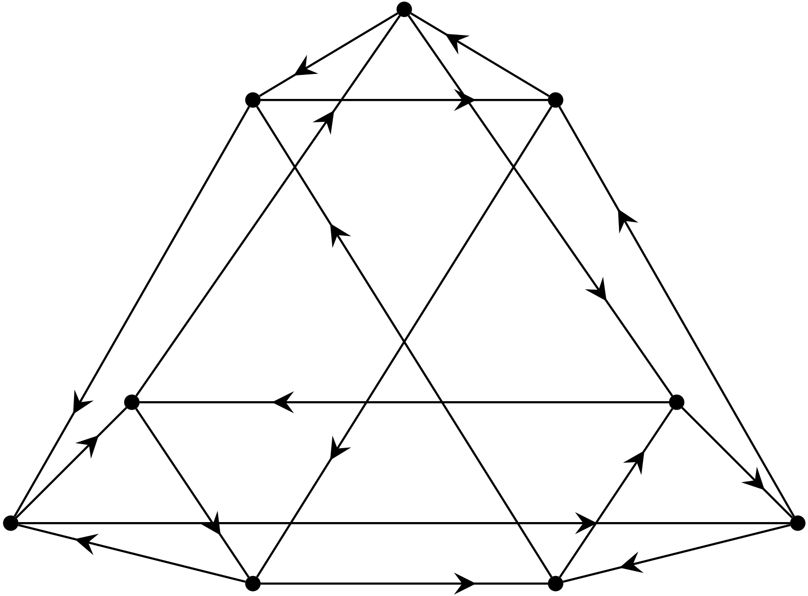
Theorem

Mixed geodetic cages exist for all r, z and k.

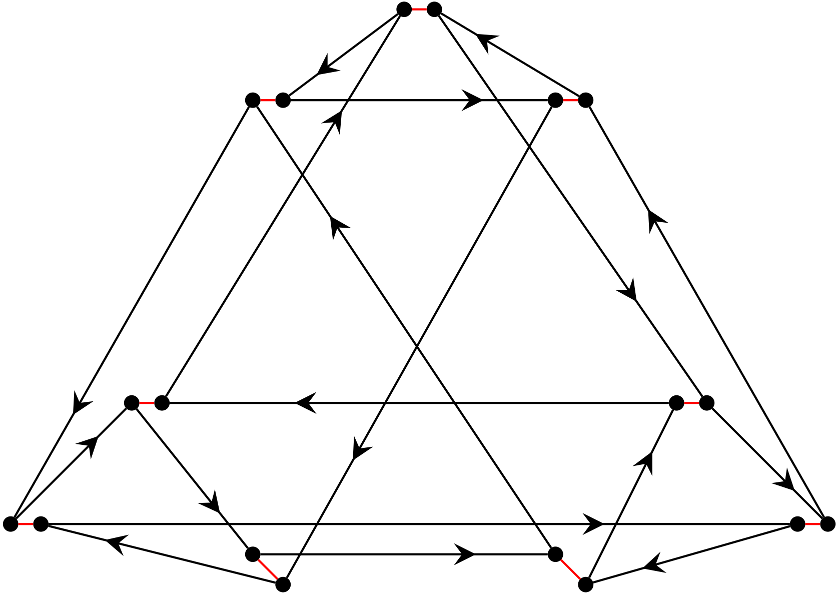
Mixed geodetic cages exist

Let G be a cage with degree r, girth $g = 2k + 1$ and order n. Let G' be a directed geodetic cage with geodetic girth k and directed out-degree nz. Identify each vertex of G' with a copy of G and connect the vertices of copies of G together in accordance with the topology of G'.

Example



Example



Monotonicity

Definition

$f(r, z; k)$ is the order of a (r, z, k)-cage.

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Theorem Tuite

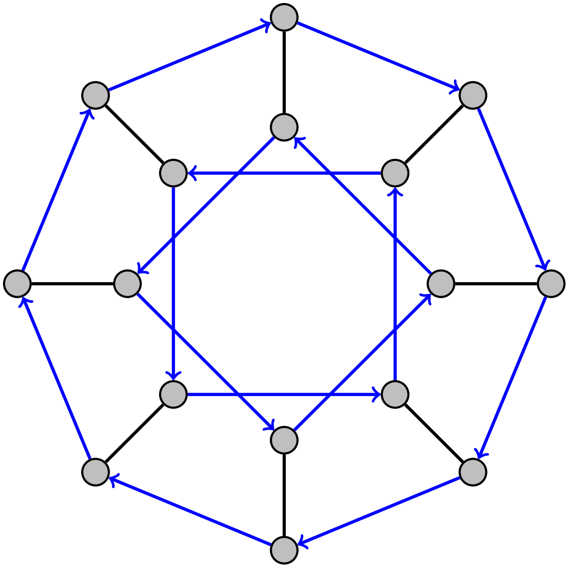
$f(r, z; k) < f(r, z; k+1)$

A regular cage for $r = z = 1$, $k = 3$

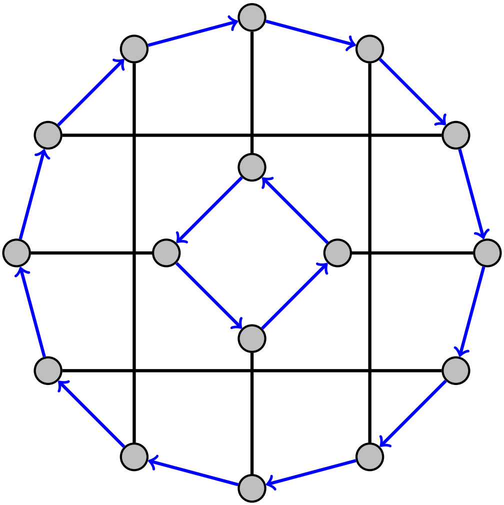
Theorem Tuite

Totally regular $(1, 1, 3; +\epsilon)$-graphs have excess at least 5.

A regular cage for $r = z = 1, k = 3$



Another regular cage for $r = z = 1$, $k = 3$



Cayley graph search

k	z	r	n	ϵ	G
1	1	2	6	0	S_3
1	1	3	20	9	$AGL(1,5)$
1	1	4	32	13	$(\mathbb{Z}_8 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
1	1	5	54	22	$(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
1	1	6	120	67	$S_3 \times AGL(1,5)$
1	2	2	12	1	D_{12}
1	2	3	48	20	$\mathbb{Z}_2 \times S_4$
1	3	2	18	0	$\mathbb{Z}_3 \times S_3$
1	3	3	128	67	$(\mathbb{Z}_8 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4$
1	4	2	30	3	$\mathbb{Z}_5 \times S_3$
1	5	2	48	10	D_{48}
1	6	2	70	19	$\mathbb{Z}_5 \times D_{14}$
1	7	2	96	30	$\mathbb{Z}_{48} \rtimes \mathbb{Z}_2$

Cayley graph search

k	z	r	n	ϵ	G
2	1	2	12	0	A_4
2	1	3	64	30	$((\mathbb{Z}_8 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
2	2	2	24	5	$SL(2,3)$
2	3	2	42	14	$\mathbb{Z}_7 \times S_3$
2	4	2	48	9	$D_8 \times S_3$
2	5	2	80	28	$D_8 \times D_{10}$
2	6	2	108	41	$S_3 \times D_{18}$
3	1	2	20	0	$AGL(1,5)$
3	2	2	39	10	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$
3	3	2	52	12	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_4$
3	4	2	80	27	$\mathbb{Z}_5 \rtimes \mathbb{Z}_{16}$
3	5	2	100	32	$\mathbb{Z}_5 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$

Cayley graph search

k	z	r	n	ϵ	G
4	1	2	42	12	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
4	2	2	54	13	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
4	3	2	72	18	$\mathbb{Z}_3 \times S_4$
4	4	2	96	27	$(SL(2,3) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
5	1	2	42	0	$AGL(1,7)$
5	2	2	75	20	$(\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_3$
5	3	2	96	26	$((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
5	4	2	120	33	$\mathbb{Z}_2 \times A_5$
6	1	2	56	0	$AGL(1,8)$
6	2	2	96	25	$((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
6	3	2	108	20	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_4$
7	1	2	72	0	$AGL(1,9)$
7	2	2	108	19	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_4$
7	3	2	108	0	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_4$
8	1	2	108	18	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_4$
9	1	2	110	0	$AGL(1,11)$

Further research

- Are all mixed graphs with order one away from the Moore bound totally regular?
- Are mixed cages totally regular? What are their connectivity properties?
- Find a tight bound on totally regular mixed graphs with small excess
- Identify cages and largest possible graphs for small r, z and k
- Search for Cayley and voltage graphs with order close to the Moore bound

Thank you!

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