Degree powers in graphs with a forbidden forest

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Joint work with Yongxin Lan[2], Zhongmei Qin[3], and Yongtang Shi[2]

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Introduction and overview

For a fixed graph H, let $\mathrm{ex}(n, H)$ denote the classical *Turán function*. That is

$$\mathrm{ex}(n, H) = \max\{e(G) : |V(G)| = n \text{ and } G \not\supset H\}.$$

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- Important results include the Erdős-Stone Theorem (1946) and "stability" theorems of Erdős and Simonovits (1960s).
- A well known conjecture: $\mathrm{ex}(n, C_{2k}) = (c_k + o(1))n^{1+1/k}$. Only known to be true for $k = 2, 3, 5$.

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For $\ell \geq 2$, we have $\mathrm{ex}(n, P_\ell) \leq (\frac{\ell}{2} - 1)n$. Moreover, if $\ell - 1$ divides n, then equality holds only for the graph with vertex-disjoint copies of $K_{\ell-1}$.

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Theorem 3 (Faudree and Schelp, 1975)

Let $\ell \geq 2$ and $n = a(\ell - 1) + b$, where $a \geq 0$ and $0 \leq b < \ell - 1$. We have $\mathrm{ex}(n, P_\ell) = a\binom{\ell-1}{2} + \binom{b}{2}$. Moreover, an extremal graph (among others) is $aK_{\ell-1} \cup K_b$.

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Bushaw and Kettle (2011) had proved Theorem 5 for $n \geq 7k$.

For $F = \bigcup_{i=1}^{k} P_{\ell_i}$ a linear forest, let $b = \sum \lfloor \frac{\ell_i}{2} \rfloor - 1$. Let $H(n, F)$ be $K_b + E_{n-b}$ with a single edge added to E_{n-b} if all ℓ_i are odd, and $H(n, F) = K_b + E_{n-b}$ otherwise. Note: $H(n, F)$ is F-free.

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Theorem 6 (Lidický, Liu, Palmer, 2013)

Let $k \geq 2$, and $F = \bigcup_{i=1}^{k} P_{\ell_i}$ be a linear forest, where $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k \geq 2$ and $\ell_i \neq 3$ for some i. Let $n \geq n_0(F)$ be sufficiently large. We have $\mathrm{ex}(n, F) = e(H(n, F))$. Moreover, $H(n, F)$ is the unique extremal graph.

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Bushaw and Kettle (2011) had proved Theorem 6 for $F = kP_\ell$ with $\ell \geq 4$. Note that the extremal graph $H(n, F)$ for $\mathrm{ex}(n, F)$ in Theorem 6 is very different from those for $\mathrm{ex}(n, P_\ell)$ in Theorem 3.

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Let $F = \bigcup_{i=1}^{k} S_{r_i}$ be a star forest where $r_1 \geq \cdots \geq r_k \geq 1$. Let $G(n, i, r_i)$ be $K_{i-1} + L$ where L is a near $(r_i - 1)$-regular graph on $n - i + 1$ vertices. Let $G(n, F)$ be any graph where $e(G(n, i, r_i))$ is maximised over $1 \leq i \leq k$. Note: Any such $G(n, F)$ is F-free.

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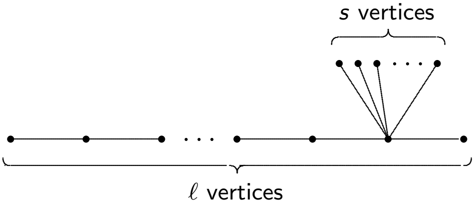

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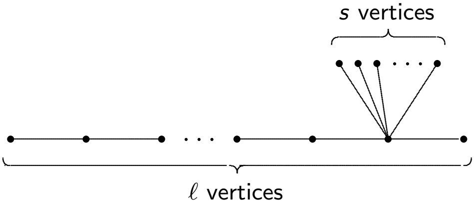

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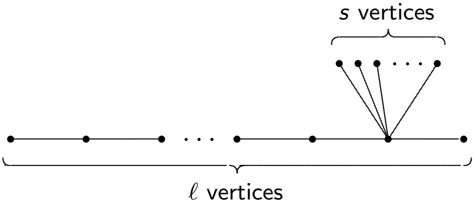

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Sun and Wang (2011) determined the functions $\mathrm{ex}(n, B_{4,s})$ and $\mathrm{ex}(n, B_{5,s})$ exactly. Their paper suggests that the exact determination of $\mathrm{ex}(n, B_{\ell,s})$ may be complicated.

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Let $p \geq 2$.

(a) $\mathrm{ex}_p(n, P_2) = 0$. *Unique extremal graph is E_n.*

(b) $\mathrm{ex}_p(n, P_3) = 2\lfloor \frac{n}{2} \rfloor$. *Unique extremal graph is M_n.*

(c) *For $\ell \geq 4$, and $n \geq n_0(\ell)$ sufficiently large, we have* $\mathrm{ex}_p(n, P_\ell) = e_p(H(n, P_\ell))$. *Unique extremal graph is $H(n, P_\ell)$.*

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New results and open problems

Theorem 11 (Lan, L., Qin, Shi, 2018+)

Let $k, p \geq 2$, and $F = \bigcup_{i=1}^{k} S_{r_i}$ be a star forest, where $r_1 \geq \cdots \geq r_k \geq 1$. Let $n \geq n_0(F)$ be sufficiently large. We have $\mathrm{ex}_p(n, F) = e_p(G(n, k, r_k))$. Extremal graphs are the $G(n, k, r_k)$.

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Enough to show: If $G = G(n)$ is F-free and $G \neq G(n, k, r_k)$, then $e_p(G) < e_p(G(n, k, r_k))$.

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- If $\leq k - 2$ vertices of G have degree $\geq \sum r_i + k$, then $e_p(G) < (k-1)n^p + o(n^p) = e_p(G(n, k, r_k))$.

New results and open problems

Theorem 11 (Lan, L., Qin, Shi, 2018+)

Let $k, p \geq 2$, and $F = \bigcup_{i=1}^{k} S_{r_i}$ be a star forest, where $r_1 \geq \cdots \geq r_k \geq 1$. Let $n \geq n_0(F)$ be sufficiently large. We have $\mathrm{ex}_p(n, F) = e_p(G(n, k, r_k))$. Extremal graphs are the $G(n, k, r_k)$.

Sketch proof.

Enough to show: If $G = G(n)$ is F-free and $G \neq G(n, k, r_k)$, then $e_p(G) < e_p(G(n, k, r_k))$.

- If $\leq k - 2$ vertices of G have degree $\geq \sum r_i + k$, then $e_p(G) < (k-1)n^p + o(n^p) = e_p(G(n, k, r_k))$.
- Otherwise, $\exists U \subset V(G)$, $|U| = k - 1$, and each vertex of U has degree $\geq \sum r_i + k$. Identifying U with K_{k-1} in $G(n, k, r_k)$, easy to show that $G \subset G(n, k, r_k)$. □

Corollary 12

Let $k, p \geq 2$ and $n \geq n_0(k)$ be sufficiently large. We have $\mathrm{ex}_p(n, kP_3) = e_p(K_{k-1} + M_{n-k+1})$. Moreover, $K_{k-1} + M_{n-k+1}$ is the unique extremal graph.

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Theorem 13 (Lan, L., Qin, Shi, 2018+)

Let $k, p \geq 2$, and $F = \bigcup_{i=1}^{k} P_{\ell_i}$ be a linear forest, where $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k \geq 2$ and $\ell_i \neq 3$ for some i. Let $n \geq n_0(F)$ be sufficiently large. We have $\mathrm{ex}_p(n, F) = e_p(H(n, F))$. Moreover, $H(n, F)$ is the unique extremal graph.

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Theorem 13 extends all previous results involving paths and linear forests.

Sketch proof of Theorem 13.

- Let $G = G(n)$ be F-free with $G \neq H(n, F)$ and $e_p(G)$ maximum. Enough to show $e_p(G) < e_p(H(n, F))$.

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- Lemma of Caro and Yuster (using Theorem 1 of Erdős and Gallai) $\Rightarrow \exists X \subset V(G)$, $|X| = b$, whose vertices have degrees $> 0.65n$. Note that any two vertices of X have at least $0.29n$ common neighbours in $Y = V(G) \setminus X$.

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- Can show that every vertex of Y has a neighbour in X.
- If some ℓ_i is even, then $G[Y]$ cannot contain an edge $\Rightarrow G \subset H(n, F)$.
- If all ℓ_i are odd, then $G[Y]$ consists of independent edges and isolated vertices. If there are at least two such edges in $G[Y]$, then either $e_p(G) < e_p(H(n, F))$, or $\exists F$-free $G' = G'(n)$ with $e_p(G) < e_p(G')$, a contradiction. □

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(a) *For $n > (2s+10)^2$, we have*

$$\mathrm{ex}_p(n, B_{5,s}) = \begin{cases} e_p(H(n, P_5)) & \text{if } s = 0, \\ e_p(K_1 + M_{n-1}) & \text{if } s \geq 1. \end{cases}$$

Unique extremal graph is $H(n, P_5)$ if $s = 0$, and $K_1 + M_{n-1}$ if $s \geq 1$.

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(b) *For $n > (2s+12)^2$, we have $\mathrm{ex}_p(n, B_{6,s}) = e_p(H(n, P_6))$. Unique extremal graph is $H(n, P_6)$.*

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(b) *For $n > (2s + 12)^2$, we have $\mathrm{ex}_p(n, B_{6,s}) = e_p(H(n, P_6))$. Unique extremal graph is $H(n, P_6)$.*

(c) *For $n > (3s + 31)^2$, we have $\mathrm{ex}_p(n, B_{7,s}) = e_p(H(n, P_7))$. Unique extremal graph is $H(n, P_7)$.*

Sketch proof of Theorem 14(c).

- Let $G = G(n)$ be $B_{7,s}$-free with $G \neq H(n, P_7)$ and $e_p(G)$ maximum. Enough to show $e_p(G) < e_p(H(n, P_7))$.

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- Suffices to consider G is connected. Otherwise, if G has many components of the form $H(t, P_7)$ and a remaining graph with small maximum degree $d = d(s)$, then the sum of e_p for these subgraphs is $< e_p(H(n, P_7))$.

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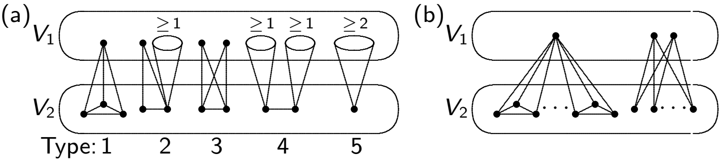
- We may then assume that $V_4 = V_3 = \emptyset$.

Sketch proof of Theorem 14(c) (ctd.)

- Structure between V_1 and V_2 looks like:



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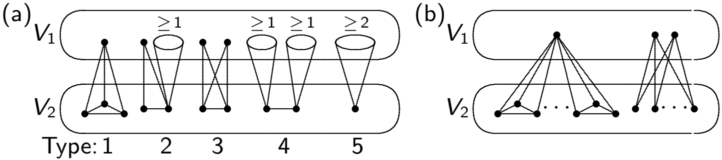

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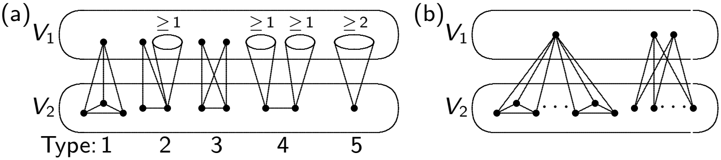

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- Connect v to all vertices of V_2 to obtain G^*, so that $e_p(G) \leq e_p(G^*)$, and $G^* - v$ consists of components of types 1 to 5 as shown in (a). Then G^* is $B_{7,s}$-free.

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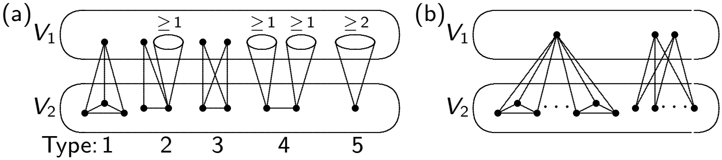

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- Replace $G^* - v$ with $H(n-1, P_5)$. Then can show that $e_p(G) \leq e_p(G^*) < e_p(H(n, P_7))$. □

Conjecture 15 (Lan, L., Qin, Shi, 2018+)

Let $p \geq 2$, $\ell \geq 6$, $s \geq 0$, and $n \geq n_0(\ell, s)$ be sufficiently large. We have

$$\mathrm{ex}_p(n, B_{\ell,s}) = e_p(H(n, P_\ell)).$$

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Problem 16 (Lan, L., Qin, Shi, 2018+)

For $p \geq 2$, a fixed broom forest F, and $n \geq n_0(F)$ sufficiently large, determine $\mathrm{ex}_p(n, F)$ and the extremal graphs.

Thank you!