Classification problem of certain spherical embeddings of strongly regular graphs

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May 24, 2018

at JCCA2018 (Sendai, Japan)

Joint work with
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$$S^{n-1} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$$

$X \subset S^{n-1}$, $0 < |X| < \infty$. and let $t \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Def. **(Spherical t-designs)** (Delsarte-Goethals-Seidel, 1977)

X is called a spherical design on S^{n-1}, if

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x) = \frac{1}{|X|} \sum_{x \in X} f(x),$$

for $^\forall f(x) = f(x_1, x_2, \ldots, x_n)$, polynomials of degree $\leq t$,

$$\iff \sum_{x \in X} f(x) = 0 \text{ for } ^\forall f(x) \in \mathrm{Harm}_i(\mathbb{R}^n),\ 1 \leq i \leq t,$$

$$\iff \sum_{(x,y) \in X \times X} Q_i(x \cdot y) = 0,\ 1 \leq i \leq t,$$

($Q_i(x)$ = Gegenbauer polynumial of degree i)

($\iff$ many other equivalent conditions.)

Let $T \subset \mathbb{N} = \{1, 2, 3, \ldots\}$.

Def. (Spherical T-designs)

X is called a spherical T-design on S^{n-1}, if

$$\iff \sum_{x \in X} f(x) = 0 \text{ for } {}^{\forall} f(x) \in \mathrm{Harm}_i(\mathbb{R}^n), \text{ for all } i \in T,$$

$$\iff \sum_{(x,y) \in X \times X} Q_i(x \cdot y) = 0, \text{ for all } i \in T.$$

Remark. X is a spherical t-design on S^{n-1}, if and only if X is a spherical T-design with $T = \{1, 2, \ldots, t\}$.

Tight spherical t-designs.

If X is a spherical t-design on S^{n-1}, then

$$|X| \geq \binom{n-1+e}{e} + \binom{n-1+e-1}{e-1}, \text{ if } t = 2e,$$

$$|X| \geq 2\binom{n-1+e}{e}, \qquad \text{if } t = 2e+1,$$

" $=$ " holds $\Longleftrightarrow$ X is called a tight spherical t-design.

Let X be a t-design and s-distance set, i.e., $s = |A(X)|$, where $A(X) = \{x \cdot y \mid x, y \in X, x \neq y\}$. Then

(i) $t \leq 2s$.

(ii) $t = 2s \Longleftrightarrow X$ is a tight $2s$-design.

(iii) $t = 2s-1$ and X is antipodal $\Longleftrightarrow X$ is a tight $(2s-1)$-design.

(iv) $t \geq 2s-2 \Longrightarrow X$ has the structure of a Q-poly. A.S.

Classification of tight t-designs on S^{n-1}

Tight t-designs on S^n are classified for all $t \neq 4, 5, 7$. See
Delsarte-Goethals-Seidel(1977), Bannai-Damerell(1979,1980), Bannai-Sloane (1981)
For further non-existence results for $t = 4, 5, 7$, see
Makhnev (2002), Bannai-Munemasa-Venkov (2004), Nebe-Venkov (2013).

Remark. Lower bounds for T-designs are known. If $2e \in T$, then $|X| \geq c_e n^e$. (Bannai-Okuda-Tagami, 2015)

Remark. There is a concept that X is a tight frame. It is known that $X \subset S^{n-1}$ is a tight frame, if and only if X is a spherical $\{2\}$-design, i.e. T-design with $T = \{2\}$.

Main Theorem (Eiichi Bannai, Etsuko Bannai, Ziqing Xiang, Wei-Hsuan Yu, and Yan Zhu, in preparation)
Let $Y \subset S^{n-1}$ be a 2-distance $\{4, 2, 1\}$-design. (Then Y is a SRG, since $t \geq 2s - 2$.) Then we can determine the possible parameters of the SRG. Moreover, Y must be either a tight spherical 4-design on S^{n-1} with $|Y| = \frac{n(n+3)}{2}$ or a half of a tight spherical 5-design on S^{n-1} with $|Y| = \frac{n(n+1)}{2}$.

(Here, a half of an antipodal design means take one point from each pair of antipodal two points. It is not known which of such half of an antipodal tight 5-design becomes a $\{4, 2, 1\}$-design.)
(For a related work, see "Half of an antipodal spherical design" (Arkiv der Math., 2018) by Bannai, Da Zhao, Lin Zhu, Yan Zhu, and Yinfeng Zhu.)

Rough sketch of the idea and some speculations.

Let Y has the structure of a SRG of type (v, k, λ, μ). Let k, x, y be the eigenvalues of the SRG. Then we have:
$k = \mu - xy,$
$v = \frac{1}{\mu}(k - x)(k - y),$
$\lambda = x + y + \mu,$
$n = m_x = \frac{(\mu - xy)(\mu - xy - y)(y+1)}{\mu(y-x)}.$
Then we can show that using the condition that Y is a $\{4, 2, 1\}$-design, we get

$$F_3(x, y, \mu) = 0$$

must be satisfied, where

$$F_3(x, y, \mu) = (y + 1)(x - y(y^2 + 3y + 3))\mu^3$$
$$+(x^2(3y^2 + 8y + 3) + yx(3y^4 + 10y^3 + 6y^2 - 7y - 2)$$
$$+y^3(y + 3)(y + 2))\mu^2$$
$$-x(y + 1)(x^2(3y^2 - 2y - 2) + yx(3y^4 + 5y^3 - 4y^2 + y + 1)$$
$$+y^4(2y + 5))\mu - x^2y^2(y + 1)^2(x + 1)(x - y^3).$$

As an experiment, we determined small solutions of the equation $F_3(x, y, \mu) = 0$ with the property $xy < 0$.

x	y	n	v	k	λ	μ
−81	3	47	1128	567	246	324
−81	3	46	1127	486	165	243
−28	2	23	276	140	58	84
−28	2	22	275	112	30	56
−5	1	7	28	15	6	10
−5	1	6	27	10	1	5
−1	1	2	3	2	1	1
4	−2	6	27	16	10	8
4	−2	7	28	12	6	4
27	−3	22	275	162	105	81
27	−3	23	276	135	78	54
80	−4	46	1127	640	396	320
80	−4	47	1128	560	316	240

From this data, we were able to conjecture that all the integer solutions of $F_3(x, y, \mu)$ must be in the form stated below. Then, we could solve this diophantine equation also using some additional conditions such as $|Y| \leq n(n+3)/2$.
(It seems that the problem "whether all the integer solutions of this diophantine equation could be determined" remains as an interesting purely number theoretical open problem.)
If $v \leq \frac{n(n+3)}{2}$ and $y \neq -1$, then all integer solutions of $F_3(x, y, \mu) = 0$ so that $xy < 0$ and n, v are integers are

1. $-x = y = \mu = 1$,
2. $x = -y^2(2y+3)$ and $\mu = -xy$,
3. $x = -y^2(2y+3)$ and $\mu = -x(y+1)$.

This completes the proof of our Theorem. (This proof uses computer extensively.)

Why I think this work is interesting ?

(i) If we try to describe all possibilities of $Y \subset S^{n-1}$ with Y a 2-distance $\{3,2,1\}$-design, then there are too many possibilities (some infinite families and some sporadic feasible parameters), and determining the existence or the non-existence would be very interesting problems (for any of these remaining parameters.)

(ii) In the case of 2-distance $\{4,2,1\}$-design, our Theorem almost determines the possibilities.

(iii) If we consider $Y \subset S^{n-1}$ with Y a 3-distance $\{5,4,3,2,1\}$-design, then the possible feasible parameter sets (by numerical experiment) are only those coming from tight spherical 5-designs and those coming from a section of tight 7-design.
(Although this case is far more complicated than the case of 2-distance $\{4,2,1\}$-design, I optimistically think that our method may work in this case.)

(iv) I believe that the situation is similar for s-distance $(2s-1)$-designs (for larger s).

Thank You