Spherical embeddings of symmetric association schemes

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Outline

- **What is association scheme?**
  From groups, polytopes, and graphs.
- **Concept related to association scheme**
  Idempotent, spherical embedding, primitive,
  metic (≈distance-regular graphs), and cometric (?)
- **Some conjectures and results revisited**
- **New results**
  Classification of A.S. with faithful spherical embedding in $\mathbb{R}^3$*,*
  and partial result on cometric A.S. with $m_1 = 4$*.*
- **Sketch of proof**

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- ▷ **Some conjectures and results revisited**
- ▷ **New results**
  Classification of A.S. with faithful spherical embedding in $\mathbb{R}^3$*, and partial result on cometric A.S. with* $m_1 = 4$.
- ▷ **Sketch of proof**

In case you get bored ...

4	8		5	2			9	
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	9		3			8		
	4			9	6		1	7

Time killer

4	8		5	2			9	
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			9		1		8	2
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	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Examples

4	8		5	2			9	
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	1						4	
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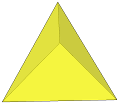
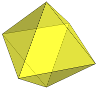
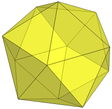
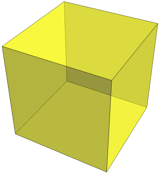
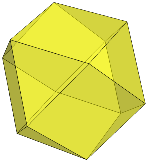
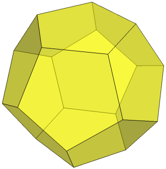
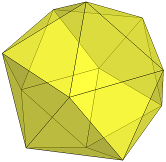
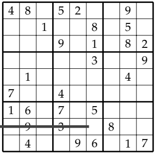
	9		3			8		
	4			9	6		1	7

Association scheme



Definition

- Let X be a finite set of size n.
- Let $\{R_i\}_{i=0}^{d}$ be a collection of binary relations $R_i \subseteq X \times X$.

  Let A_i be the corresponding adjacency matrix of R_i.

  They satisfy the following properties.

  1. $A_0 = I$.
  2. $A_0 + A_1 + \cdots + A_d = J$.
  3. $A_i^T \in \{A_0, A_1, \ldots, A_d\}$.
  4. $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$.

  Then we call $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ an *association scheme (A.S.)*.

  5. $A_i A_j = A_j A_i$. (commutative A.S.)
  6. $A_i^T = A_i$. (symmetric A.S. $\Longrightarrow$ commutative A.S.)

In this talk, we focus on symmetric A.S. .

More examples

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
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- Let (G, X) be a transitive permutation group. The orbits of G acting on $X \times X$ form an A.S..
- Let G be a finite group. The orbits of $G \times G$ acting on $G \times G$ by $(x, y)^{g,h} = (g^{-1}xh, g^{-1}yh)$ form an A.S..
- Let Γ be a distance-regular graph, then the distance relations form an A.S..

Idempotents

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
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Definition (primitive idempotent)

Given a commutative A.S., the adjacency matrices $\{A_i\}_{i=0}^{d}$ can be diagonalized simultaneously. The space $V = \mathbb{C}^{|X|}$ have the composition

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d$$

Let E_i be the projection $V \to V_i$, and we call it the i-th primitive idempotent.

1. $E_0 = \frac{1}{|X|}J$.
2. $E_0 + E_1 + \cdots + E_d = I$.
3. $E_i E_j = \delta_{ij} E_i$.
4. $E_i \circ E_j = \sum_{k=0}^{d} q_{ij}^{k} E_k$

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They forms two basis of the Bose-Mesner algebra.
We can take $V = \mathbb{R}^{|X|}$ if the A.S. is symmetric.

Spherical embedding

4	8		5	2			9	
		1			8		5	
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Let $k_i =$ the valency of A_i.
Let $m_i =$ the rank of E_i.

Definition (spherical embedding)

The spherical embedding of a symmetric A.S. $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ with respect to E_i is the mapping $X \to \mathbb{R}^{m_i}$ defined by

$$x \to \overline{x} = \sqrt{\frac{|X|}{m_i}} E_i \phi_x$$

where ϕ_x is the characteristic vector of x.

The image is on the unit sphere $S^{m_i - 1} \subset \mathbb{R}^{m_i}$.
We identify X and $\overline{X}$ when the embedding is faithful.

Primitive

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
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Definition (primitive)

A symmetric A.S. $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ is called **primitive** if each graph (X, R_i) is connected for $1 \leq i \leq d$.

metric and cometric

4	8		5	2			9	
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Definition (metric)

A symmetric association scheme is called metric (or P-polynomial) if there exists an ordering of relations such that $A_i = v_i(A_1)$, where v_i is a polynomial of degree i.

P-polynomial association scheme $\approx$ distance-regular graphs

Definition (cometric)

A symmetric association scheme is called cometric (or Q-polynomial) if there exists an ordering of primitive idempotents such that $E_i = v_i^*(E_1)$, where v_i^* is a polynomial of degree i, and the product is Schur product.

Conjectures and results revisited (I)

4	8		5	2			9	
		1			8		5	
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Conjecture (Bannai-Ito)

{primitive metric A.S.} = {primitive cometric A.S.} for large class number d.

Conjecture (Babai)

The maximum valency of a primitive association scheme is bounded by a function of the minimum (non-trivial) valency, i.e., $k_{\max} \leq f(k_{\min})$.

Conjectures and results revisited (II)

Theorem (Godsil 1988)

There are only finitely many connected co-connected distance-regular graphs with an eigenvalue multiplicity m for all $m \geq 3$.

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Theorem (Martin-Williford 2009)

There are finitely many cometric association schemes with multiplicity m_1 for all $m_1 \geq 3$.

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Theorem (Martin-Williford 2009)

There are finitely many cometric association schemes with multiplicity m_1 for all $m_1 \geq 3$.

Corollary

There are finitely many cometric association schemes with a relation of valency k for all $k \geq 3$.

Conjectures and results revisited (III)

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Theorem (Biggs-Boshier-ShaweTaylor 1986)

There are 13 distance-regular graphs of valency $k = 3$.

Theorem (Brouwer-Koolen 1999)

There are 17 possible parameters of distance-regular graphs of valency $k = 4$, each of which is determined and unique except possibly one parameter.

We aim to finish the classification dual to these two theorems.

New results (I)

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Theorem (Bannai-Zhao)

Let $\mathfrak{X}$ be a symmetric association scheme. If $\mathfrak{X}$ has a faithful spherical embeddings X with $m_1 = 3$ in $\mathbb{R}^3$, then it must be one of the followings:

1. the regular tetrahedron $(|X| = 4)$;
2. the regular octahedron $(|X| = 6)$;
3. the cube $(|X| = 8)$;
4. the regular icosahedron $(|X| = 12)$;
5. the quasi-regular polyhedron of type $[3, 4, 3, 4]$ $(|X| = 12)$;
6. the regular dodecahedron $(|X| = 20)$;

* The quasi-regular polyhedron of type $[3, 5, 3, 5]$ $(|X| = 30)$ is a non-commutative A.S. with a faithful spherical embeddings in $\mathbb{R}^3$.

Corollary

The Q-polynomial association schemes with $m_1 = 3$ are (1-4) in the above list.

New results (II)

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Lemma

Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a Q-polynomial association scheme. Suppose there exist $1 \leq i, j \leq d$ such that $p_{ij}^i = 1$, then the adjacency graph (X, R_i) has a cycle of length $3, 4, 5, 6, 8, 10$, or 12. So the girth of $\Gamma_i = (X, R_i)$ is at most 12. Moreover if $g = 3$, then $i = j$.

Corollary

Let $\mathfrak{X}$ be a primitive Q-polynomial association scheme with $m_1 = 4$. Then in the spherical embedding of $\mathfrak{X}$ on S^3, the nearest neighborhood $R_1(x)$ of a point $x \in X$ cannot be antipodal on the translated sphere S^2.

Remarks on previous result

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

The argument in Bannai-Bannai (2006) essentially proves:

Proposition

Let X be a faithful spherical embedding of a symmetric association scheme with $m_1 = 3$ in $\mathbb{R}^3$. Let $A(X) = \{\langle x, y\rangle \mid x, y \in X, x \neq y\}$ and $\alpha = \max A(X)$. Moreover, if we assume that $R_1 \subseteq \Gamma_\alpha = \{(x, y) \mid \langle x, y\rangle = \alpha\}$, then

1. The valency of the graph (X, Γ_α) is at most 5.
2. If we further assume $R_1 = \Gamma_\alpha$, we can show that $X \subset S^2$ is as follows.
   - 2.1 If $k_1 = 5$, then each connected component of (X, R_1) is the regular icosahedron ($|X| = 12$).
   - 2.2 If $k_1 = 4$, then each connected component of (X, R_1) is the regular octahedron ($|X| = 6$), the quasi-regular polyhedron of type $[3, 4, 3, 4]$ ($|X| = 12$), or the quasi-regular polyhedron of type $[3, 5, 3, 5]$ ($|X| = 30$).
   - 2.3 If $k_1 = 3$, then each connected component of (X, R_1) is the regular tetrahedron ($|X| = 4$), the cube ($|X| = 8$), or the regular dodecahedron ($|X| = 20$).

What shall we do to prove the first result?

4	8		5	2			9	
		1			8		5	
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The clean-up work

What shall we do to prove the first result?

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		1			8		5	
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					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

The clean-up work
If we can show that

1. $k_1 \neq 1$ and $k_1 \neq 2$.
2. $R_1 = \Gamma_\alpha$.
3. the graph (X, R_1) is connected.

Then we get a complete classification of symmetric A.S. with faithful spherical embedding in $\mathbb{R}^3$.

Proof of $k_1 \neq 1$

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Proof

- The eigenvalue of A_1 has to be -1.
- It is an antipodal relation in the spherical embedding.

Proof of $k_1 \neq 2$

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Proof

- (X, R_1) is a union of ℓ_j-cycles.
- All the ℓ_j's are equal.
- Geometrically they are regular ℓ-gons on great circles.
- Great circles intersect on S^2.

Proof of $R_1 = \Gamma_\alpha$

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Proof

$$3 + 3 > 5$$

Proof of connectedness

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Proof

- If there are two or more connected components.
- Then each connect component is in the list of Proposition.
- It is too crowded on the sphere S^2!

Remarks on previous result

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

In an informal paper in RIMS proceedings 1997, Attila Sali gives the following theorem.

Theorem

Let $\mathfrak{X}$ be a primitive Q-polynomial association scheme with $m_1 = 4$. Then in the spherical embedding of $\mathfrak{X}$ on S^3, the nearest neighborhood $R_1(x)$ of a point $x \in X$ is one of the following:

- icosahedron
- cube
- octahedron
- tetrahedron

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

Remarks on previous result

We can exclude some cases,
but we also find some missing cases.

Theorem

Let $\mathfrak{X}$ be a primitive Q-polynomial association scheme with $m_1 = 4$. Then in the spherical embedding of $\mathfrak{X}$ on S^3, the nearest neighborhood $R_1(x)$ of a point $x \in X$ is one of the following:

- ~~icosahedron~~
- ~~cube~~
- ~~octahedron~~
- tetrahedron
- triangle, pentagon, 3-prism, 5-prism, twisted 2-prism, twisted 4-prism, etc.

Proof of the second result

Lemma

Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ be a Q-polynomial association scheme. Suppose there exist $1 \leq i, j \leq d$ such that $p_{ij}^{i} = 1$, then the adjacency graph (X, R_i) has a cycle of length $3, 4, 5, 6, 8, 10,$ or 12. So the girth of $\Gamma_i = (X, R_i)$ is at most 12. Moreover if $g = 3$, then $i = j$.

Proof

- By Terwilliger's balanced set condition,
  $p_{ij}^{i} = 1$ implies that four points are co-plane.
- Inductively we get a regular N-gon.
- The splitting field is at most quadratic.
- $N = 3, 4, 5, 6, 8, 10$ or 12.

Further Discussion

4	8		5	2			9	
		1			8		5	
			9		1		8	2
					3			9
	1						4	
7			4					
1	6		7		5			
	9		3			8		
	4			9	6		1	7

- ▷ Partially metric association schemes with a multiplicity three. (van Dam-Koolen-Park 2017)
- ▷ Partially cometric association scheme (with $k_1 = 4$)?

Thank you for your attention!