Toric Fano varieties associated to graph cubeahedra

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Combinatorics on Convex Polytopes

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Table of Contents

1. Toric varieties and fans

Definition

An n-dimensional toric variety is a normal algebraic variety X over $\mathbb{C}$ containing $(\mathbb{C}^*)^n$ as an open dense subset, s.t. the natural action $(\mathbb{C}^*)^n \curvearrowright (\mathbb{C}^*)^n$ extends to an action on X.

Examples

$(\mathbb{C}^*)^n, \mathbb{C}^n, \mathbb{P}^n$ are toric varieties.

Definition

Two toric varieties X and X' are said to be isomorphic if there exists an isomorphism $f : X \to X'$ satisfying the following conditions:

- f induces an isomorphism of algebraic tori $f' : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$.
- f is equivariant with respect to f', i.e., $f(tx) = f'(t)f(x)$ for any $t \in (\mathbb{C}^*)^n$ and $x \in X$.

Definition

A rational strongly convex polyhedral cone is a cone $\sigma \subset \mathbb{R}^n$ generated by finitely many vectors in $\mathbb{Z}^n$ which does not contain any non-zero linear subspace of $\mathbb{R}^n$. A fan in $\mathbb{R}^n$ is a non-empty finite set Δ of such cones satisfying the following conditions:

- If $\sigma \in \Delta$, then each face of σ is in Δ.
- If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a face of each.

Definition

Two fans Δ and Δ' in $\mathbb{R}^n$ are said to be isomorphic if there exists an automorphism of $\mathbb{Z}^n$ that induces a bijection $\Delta \to \Delta'$.

Theorem

$$\{\text{fans in } \mathbb{R}^n\}/(\text{isom.}) \overset{1:1}{\longleftrightarrow} \{n\text{-dimensional toric varieties}\}/(\text{isom.}),$$
$$\Delta \mapsto X(\Delta).$$

We construct a toric variety $X(\Delta)$ from a fan Δ.

Step 1 (affine toric varieties)

For each $\sigma \in \Delta$, we construct an affine toric variety U_σ.

- $\sigma^\vee = \{u \in \mathbb{R}^n \mid \langle u, v\rangle \geq 0 \ \forall v \in \sigma\}$: the dual of σ.
- $\sigma^\vee \cap \mathbb{Z}^n$ is a commutative monoid.
- The monoid ring $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$ is a finitely generated integral domain over $\mathbb{C}$. So we put $U_\sigma = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$.

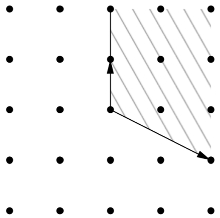
Step 2 (gluing)

Let τ be a face of σ and let $\tau \to \sigma$ be the inclusion.

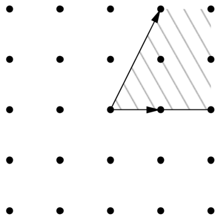
- $\rightsquigarrow$ a monoid homomorphism $\sigma^\vee \cap \mathbb{Z}^n \to \tau^\vee \cap \mathbb{Z}^n$.
- $\rightsquigarrow$ an open immersion $U_\tau \to U_\sigma$.
- Gluing $\{U_\sigma \mid \sigma \in \Delta\}$, we obtain the toric variety $X(\Delta)$.

Example

$\sigma = \mathbb{R}_{\geq 0}(2e_1 - e_2) + \mathbb{R}_{\geq 0}e_2 \subset \mathbb{R}^2$
$\rightsquigarrow \sigma^\vee = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(e_1 + 2e_2)$
$\rightsquigarrow \sigma^\vee \cap \mathbb{Z}^2 = \mathbb{Z}_{\geq 0}e_1 + \mathbb{Z}_{\geq 0}(e_1 + e_2) + \mathbb{Z}_{\geq 0}(e_1 + 2e_2)$.



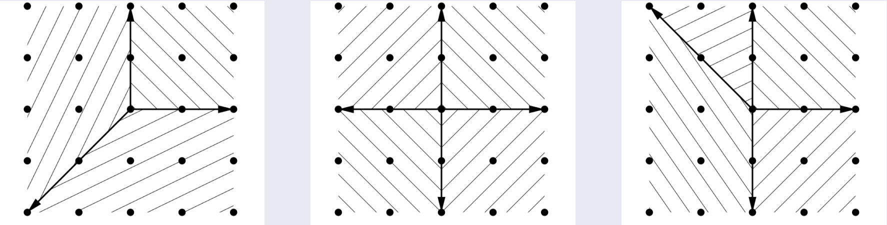
$\sigma = \mathbb{R}_{\geq 0}(2e_1 - e_2) + \mathbb{R}_{\geq 0}e_2$



$\sigma^\vee = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(e_1 + 2e_2)$

$\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2] = \mathbb{C}[X, XY, XY^2] = \mathbb{C}[U, V, W]/(UW - V^2)$. Therefore $U_\sigma = \mathrm{Spec}\mathbb{C}[U, V, W]/(UW - V^2)$.

Examples ($n = 2$)



$\mathbb{P}^2$ $\quad$ $\mathbb{P}^1 \times \mathbb{P}^1$ $\quad$ Hirzebruch surface F_1

Δ: a fan in $\mathbb{R}^n$.

Definition

- Δ is nonsingular $\Leftrightarrow$ every cone of Δ is generated by a part of a basis for $\mathbb{Z}^n$.
- Δ is complete $\Leftrightarrow$ $\bigcup_{\sigma \in \Delta} \sigma = \mathbb{R}^n$.

Fact

- $X(\Delta)$ is nonsingular $\Leftrightarrow$ Δ is nonsingular.
- $X(\Delta)$ is complete $\Leftrightarrow$ Δ is complete.

2. Graph cubeahedra

Let G be a finite simple graph, that is, a finite graph with no loops and no multiple edges.

- $V(G) = \{1, \ldots, n\}$: the node set.
- $E(G)$: the edge set.

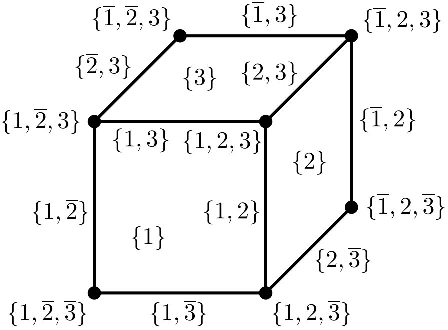
Definition

For $I \subset V(G)$, we define the induced subgraph $G|_I$ by

$$V(G|_I) = I, \quad E(G|_I) = \{\{v, w\} \in E(G) \mid v, w \in I\}.$$

Let $\square^n$ be the standard n-cube whose facets are labeled by $1, \ldots, n$ and $\overline{1}, \ldots, \overline{n}$, where the two facets labeled by i and $\overline{i}$ are on opposite sides.

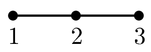
Then every face of $\square^n$ is labeled by a subset $I \subset \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$ such that $I \cap \{1, \ldots, n\}$ and $\{i \in \{1, \ldots, n\} \mid \overline{i} \in I\}$ are disjoint. The face corresponding to I is the intersection of the facets labeled by the elements of I.



Let $\mathcal{I}_G = \{I \subset V(G) \mid G|_I \text{ is connected}, I \neq \emptyset\}$. We call $\mathcal{I}_G$ the graphical building set of G.

Example

Let P_3 be a path with three nodes.



Then $\mathcal{I}_{P_3} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$.

Definition (Devadoss–Heath–Vipismakul, 2011)

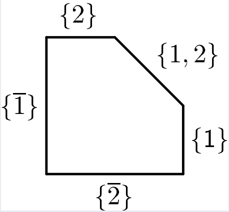
The graph cubeahedron $\square_G$ is obtained from $\square^n$ by truncating the faces labeled by the elements of $\mathcal{I}_G$ in increasing order of dimension.

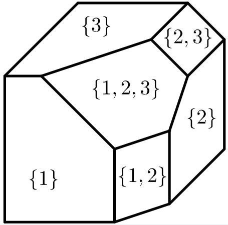
Example

Let P_n be a path with n nodes. Then

$$\mathcal{I}_{P_2} = \{\{1\},\{2\},\{1,2\}\},$$
$$\mathcal{I}_{P_3} = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,2,3\}\}$$

and we have the graph cubeahedra $\square_{P_2}$ and $\square_{P_3}$ in the following figures:

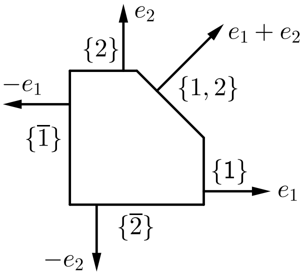




An n-dimensional simple convex polytope in $\mathbb{R}^n$ is called a Delzant polytope if for every vertex, the outward-pointing primitive normal vectors of the facets containing the vertex form a basis for $\mathbb{Z}^n$.

Example

The graph cubeahedron $\square_{P_2}$ can be realized as a Delzant polytope.



An n-dimensional Delzant polytope P defines a nonsingular complete fan Δ_P in $\mathbb{R}^n$.

By the following lemma, any graph cubeahedron can be realized as a Delzant polytope $\square_G$ in a canonical way. In particular, we obtain a nonsingular complete toric variety $X(\Delta_{\square_G})$.

Lemma

Let P be a Delzant polytope and let F be a face of codimension ≥ 2 of P. Then there exists a canonical truncation of P along F such that the result $\mathrm{Cut}_F(P)$ is also a Delzant polytope and the associated toric variety $X(\Delta_{\mathrm{Cut}_F(P)})$ is the blow-up of $X(\Delta_P)$ along the subvariety corresponding to F.

Remark

$X(\Delta_{\square_G})$ is in fact a nonsingular projective toric variety.

We have a one-to-one correspondence

$$\mathcal{I}_G \cup \{\{\bar{1}\}, \ldots, \{\bar{n}\}\} \overset{1:1}{\longleftrightarrow} \{\text{facets of } \square_G\}, I \mapsto F_I.$$

The outward-pointing primitive normal vector e_I of F_I is given by

$$e_I = \begin{cases} \sum_{i \in I} e_i & (I \in \mathcal{I}_G), \\ -e_i & (I = \{\bar{i}\}, i \in \{1, \ldots, n\}). \end{cases}$$

Theorem (Devadoss–Heath–Vipismakul, 2011)

Let G be a finite simple graph. Then the two facets F_I and F_J of the graph cubeahedron $\square_G$ intersect iff one of the following holds:

- $I, J \in \mathcal{I}_G$ and we have either $I \subset J$ or $J \subset I$ or $I \cup J \notin \mathcal{I}_G$.
- One of I and J, say I, is in $\mathcal{I}_G$ and $J = \{\bar{j}\}$ $\exists j \in \{1, \ldots, n\} \setminus I$.
- $I = \{\bar{i}\}$ and $J = \{\bar{j}\}$ for some $i, j \in \{1, \ldots, n\}$.

Furthermore, $\square_G$ is a flag polytope.

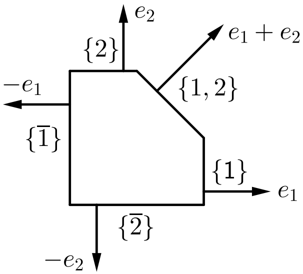
We describe the fan $\Delta_{\square_G}$ of $\square_G$ explicitly. Let

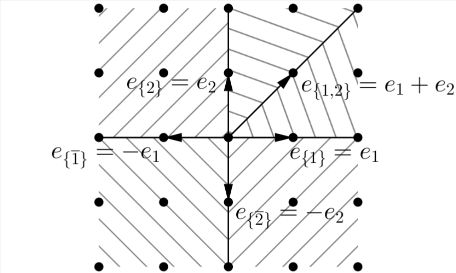
$$\mathcal{N}^{\square}(G) = \{N \subset \mathcal{I}_G \cup \{\{\bar{1}\}, \ldots, \{\bar{n}\}\} \mid F_I \cap F_J \neq \emptyset \text{ for any } I, J \in N\}.$$

For $N \in \mathcal{N}^{\square}(G)$, we denote by σ_N the $|N|$-dimensional cone $\sum_{I \in N} \mathbb{R}_{\geq 0} e_I$ in $\mathbb{R}^n$. Then $\Delta_{\square_G} = \{\sigma_N \mid N \in \mathcal{N}^{\square}(G)\}$.

Example

The fan $\Delta_{\square_{P_2}}$ of $\square_{P_2}$ is illustrated in the right figure and thus the associated toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at one point.





3. Toric Fano varieties

X: a nonsingular projective algebraic variety.

Definition

- X is Fano $\Leftrightarrow$ the anticanonical divisor $-K_X$ is ample.
- X is weak Fano $\Leftrightarrow$ $-K_X$ is nef and big.
    - A divisor D is nef $\Leftrightarrow$ $(D.C) \geq 0$ for any curve $C \subset X$.
    - D is big $\Leftrightarrow$ the Iitaka dimension $\kappa(X, D)$ is equal to $\dim X$.

Theorem

There are a finite number of isomorphism classes of toric (weak) Fano varieties in any given dimension.

dimension	1	2	3	4	5	6
# of toric Fano varieties	1	5	18	124	866	7622
# of toric weak Fano varieties	1	16	?	?	?	?

Let $\Delta(r)$ be the set of r-dimensional cones of Δ for $0 \leq r \leq n$.

Proposition

$$\Delta(n-1) \stackrel{1:1}{\longleftrightarrow} \{\text{torus-invariant curves on } X(\Delta)\},$$
$$\tau \mapsto V(\tau).$$

Proposition

Let $X(\Delta)$ be an n-dimensional nonsingular projective toric variety.

- $X(\Delta)$ is Fano $\Leftrightarrow (-K_{X(\Delta)}.V(\tau)) > 0 \quad \forall \tau \in \Delta(n-1)$.
- $X(\Delta)$ is weak Fano $\Leftrightarrow (-K_{X(\Delta)}.V(\tau)) \geq 0 \quad \forall \tau \in \Delta(n-1)$.

Let Δ be a nonsingular complete fan in $\mathbb{R}^n$.

Proposition

Let $\tau = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_{n-1} \in \Delta(n-1)$, where $v_1, \ldots, v_{n-1}$ are primitive vectors in $\mathbb{Z}^n$, and let v and v' be the distinct primitive vectors in $\mathbb{Z}^n$ s.t. $\tau + \mathbb{R}_{\geq 0} v, \tau + \mathbb{R}_{\geq 0} v' \in \Delta(n)$. Then:

- $\exists a_1, \ldots, a_{n-1} \in \mathbb{Z}$ s.t. $v + v' + a_1 v_1 + \cdots + a_{n-1} v_{n-1} = 0$.
- $(-K_{X(\Delta)}.V(\tau)) = 2 + a_1 + \cdots + a_{n-1}$.

4. Main results

Let G be a finite simple graph.

Theorem 1 (S)

$X(\Delta_{\square_G})$ is Fano $\Leftrightarrow$ each connected component of G has ≤ 2 nodes. In particular, if $X(\Delta_{\square_G})$ is Fano, then it is a product of copies of $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at one point.

Sketch of the Proof Let $n = |V(G)| = \dim X(\Delta_{\square_G})$.

($\Rightarrow$) If G has a connected component with ≥ 3 nodes, then we can find $\sigma_N \in \Delta(n-1)$ such that $(-K_{X(\Delta_{\square_G})}.V(\sigma_N)) = 0$. Therefore $X(\Delta_{\square_G})$ is not Fano.

($\Leftarrow$) Since the union of graphs corresponds to the product of associated toric varieties, it suffices to show that $X(\Delta_{\square_G})$ is Fano if G is connected and $|V(G)| \leq 2$.

(i) If $G = P_1$, then $X(\Delta_{\square_G})$ is $\mathbb{P}^1$.

(ii) If $G = P_2$, then $X(\Delta_{\square_G})$ is $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at one point.

Thus $X(\Delta_{\square_G})$ is Fano in every case. $\square$

Let G be a finite simple graph.

Theorem 2 (S)

The following are equivalent:

- $X(\Delta_{\Box_G})$ is weak Fano.
- $\forall I \subset V(G)$, $G|_I$ is not isomorphic to any of the following:
  (i) A cycle with ≥ 4 nodes.
  (ii) A diamond graph (the graph obtained by removing an edge from a complete graph with four nodes).
  (iii) A claw (a star with three edges).

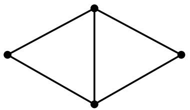


Figure: a diamond graph.

Examples

- If G is a path or a complete graph, then $X(\Delta_{\square_G})$ is weak Fano.
- If G is a graph obtained by connecting more than two graphs with one node, then $X(\Delta_{\square_G})$ is not weak Fano.
- The toric variety associated to the graph cubeahedron of the graph below is weak Fano.

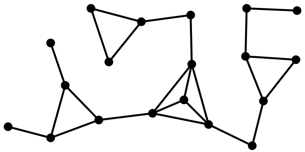


Figure: an example.

dimension	1	2	3	4	5	6
# of connected graphs	1	1	2	6	21	112
weak Fano	1	1	2	3	6	11

Sketch of the Proof Let $n = |V(G)| = \dim X(\Delta_{\Box_G})$.
($\Rightarrow$) Suppose that $\exists I \subset V(G)$ such that $G|_I$ is a cycle of length ≥ 4, a diamond graph, or a claw. Then we can find $\sigma_N \in \Delta(n-1)$ such that $(-K_{X(\Delta_{\Box_G})}.V(\sigma_N)) = -1$. Therefore $X(\Delta_{\Box_G})$ is not weak Fano.
($\Leftarrow$) Suppose that $X(\Delta_{\Box_G})$ is not weak Fano. Then $\exists N \in \mathcal{N}^{\Box}(G)$ such that $|N| = n-1$ and $(-K_{X(\Delta_{\Box_G})}.V(\sigma_N)) \leq -1$, and $\exists \{J, J'\} \subset (\mathcal{I}_G \cup \{\{\overline{1}\}, \ldots, \{\overline{n}\}\}) \setminus N$ such that $N \cup \{J\}$ and $N \cup \{J'\}$ are distinct maximal elements of $\mathcal{N}^{\Box}(G)$. Since $\Box_G$ is flag, $\{J, J'\}$ is not in $\mathcal{N}^{\Box}(G)$. Thus we must have $J \in \mathcal{I}_G$ or $J' \in \mathcal{I}_G$. We may assume $J \in \mathcal{I}_G$.

- If $J' \in \mathcal{I}_G$, then we can find $I \subset V(G)$ such that $G|_I$ is a cycle of length ≥ 4 or a diamond graph.
- If $J' = \{\bar{j}\}$ for $j \in \{1, \ldots, n\}$, then we can find $I \subset V(G)$ such that $G|_I$ is a claw.

$\Box$

Thank you for your attention!

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