On Gelfand-Cetlin Polytopes

Yunhyung Cho

Sungkyunkwan University

JCCA - Mini symposium
May 23, 2018

Outline

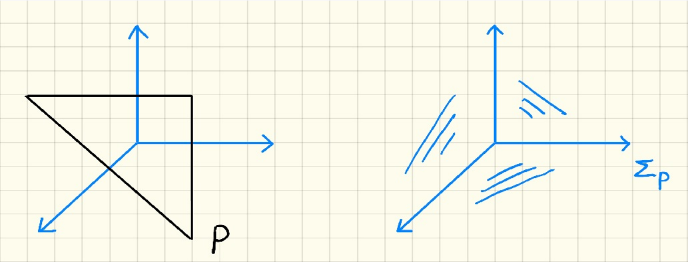
1. **Polytopes and Geometry**
2. **Gelfand-Cetlin Polytopes**
3. **String Polytopes**

1. Polytopes and Geometry

Polytopes and Geometry

From polytopes to algebraic varieties :

- Given convex polytope $P \subset \mathbb{R}^N$, the corresponding normal fan Σ_P :

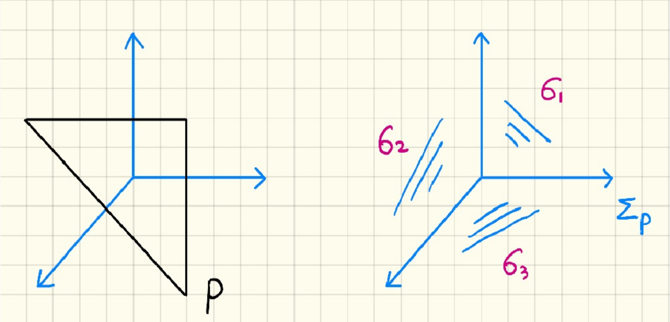


- Each cone corresponds to an affine toric variety
- Intersection of cones $\Rightarrow$ gluing varieties $\Rightarrow$ obtain X_P (proj. toric var.)

Polytopes and Geometry

From polytopes to algebraic varieties :

- Given convex polytope $P \subset \mathbb{R}^N$, the corresponding normal fan Σ_P :

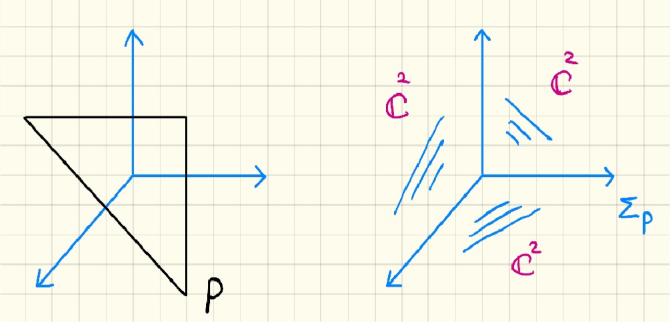


- Each cone corresponds to an affine toric variety
- Intersection of cones $\Rightarrow$ gluing varieties

Polytopes and Geometry

From polytopes to algebraic varieties :

- Given convex polytope $P \subset \mathbb{R}^N$, the corresponding normal fan Σ_P :

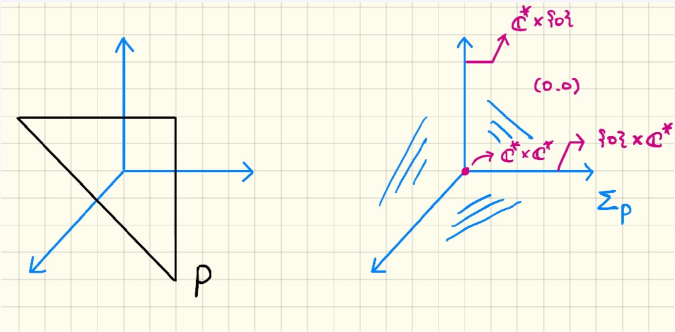


- Each cone corresponds to an affine toric variety
- Intersection of cones $\Rightarrow$ gluing varieties

Polytopes and Geometry

From polytopes to algebraic varieties :

- Given convex polytope $P \subset \mathbb{R}^N$, the corresponding normal fan Σ_P :



- Each cone corresponds to an affine toric variety
- Intersection of cones $\Rightarrow$ gluing varieties

Polytopes and Geometry

From algebraic varieties to polytopes :

- Let $T_{\mathbb{C}} := (\mathbb{C}^*)^n$: n-dimensional complex torus.
- For any $S = \{m_1, \cdots, m_N\} \subset \mathbb{Z}^n$ given, consider the torus embedding

$$\begin{array}{cccccc} i & : & T_{\mathbb{C}} & \hookrightarrow & \mathbb{P}^{N-1} \\ & & (t_1, \cdots, t_n) & \mapsto & [t^{m_1}, \cdots, t^{m_N}] \end{array} \qquad t^m := t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}, m = (a_1, \cdots, a_n).$$

- The Zariski closure of $i(T_{\mathbb{C}})$ is called a **projective toric variety** and denoted by X_S
- For the map

$$\begin{array}{cccc} \mu & : & \mathbb{P}^N & \rightarrow & \Delta^N \\ & & [z_1, \cdots, z_N] & \mapsto & \left(\frac{|z_1|^2}{|z_1|^2 + \cdots + |z_N|^2}, \cdots, \frac{|z_N|^2}{|z_1|^2 + \cdots + |z_N|^2} \right), \end{array}$$

with the projection map $\pi : \mathbb{R}^N \to \mathbb{R}^n$ represented by the matrix $(m_1 \cdots m_N)$, the image $\pi \circ \mu(X_S)$ is a convex polytope, called the moment polytope of X_S, denoted by Δ_{X_S}

Polytopes and Geometry

Theorem

(1) For X : projective toric variety with a moment polytope Δ_X, we have $X \cong X_{\Delta_X}$

(2) X is a smooth if and only if Δ_X is a non-singular simple integral polytope.

Philosophy

Any T-invariant topological and geometric invariants of X are encoded in Δ_X

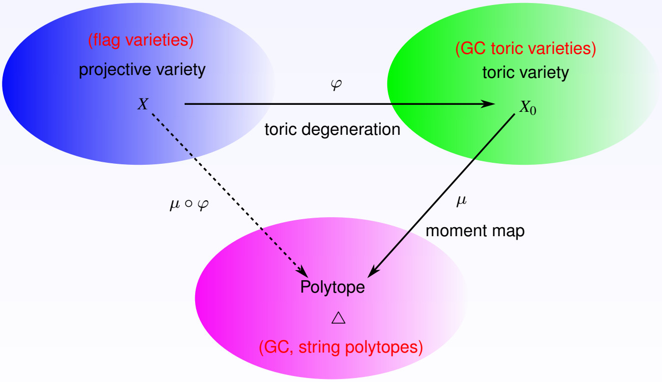
Example :

- Betti numbers of X $\Leftrightarrow$ h-vector of Δ_X
- (Equivariant) (co)homology of X $\Leftrightarrow$ Stanley-Reisner ring of Δ_X
- Quantum cohomology $\Leftrightarrow$ defining equations of Δ_X (Batyrev)
- $\underbrace{\text{open Gromov-Witten invariants}}_{\text{counts of hol. discs bounded by } \mu^{-1}(\mathbf{b})}$ $\Leftrightarrow$ defining equations of Δ_X (FOOO)

Polytopes and Geometry

Recent Progress : Given smooth projective variety X,

- (in many case) one can associate a polytope $\triangle$ (Newton-Okounkov body),
- $\exists$ many similarities between X and X_0.



2. Gelfand-Cetlin Polytopes

Gelfand-Cetlin polytopes

Definition : Given $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_n\}$: sequence of real numbers, assign a polytope

$$\Delta_\lambda = \{(x^{i,j})\} \subset \mathbb{R}^N, \qquad i,j > 0, \quad 2 \leq i+j \leq n+1, \quad N = \frac{n(n+1)}{2}$$

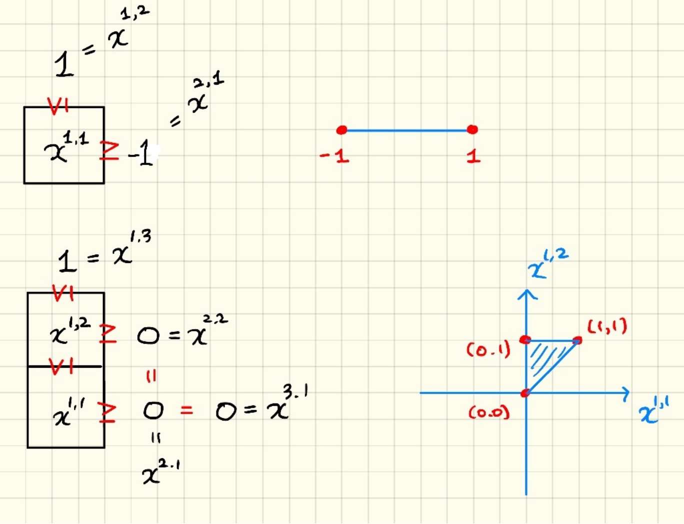
such that

- $\lambda_1 = x^{1,n}, \lambda_2 = x^{2,n-1}, \cdots, \lambda_n = x^{n,1}$
- $x^{i,j} \geq x^{i+1,j}$
- $x^{i,j+1} \geq x^{i,j}$

Such Δ_λ is called a **Gelfand-Cetlin polytope**

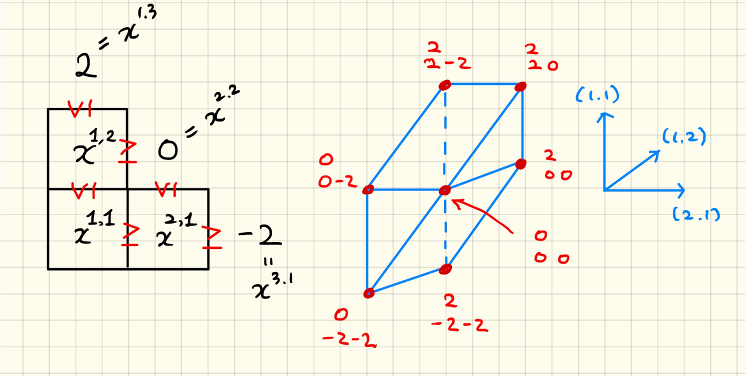
Gelfand-Cetlin polytopes

Example : For $\lambda = (1, -1)$ and $\lambda = (1, 0, 0)$,



Gelfand-Cetlin polytopes

Example : Let $\lambda = (2, 0, -2)$: Fill $\square^{(1,3)}, \square^{(2,2)}, \square^{(3,1)}$ with $\lambda_1, \lambda_2, \lambda_3$

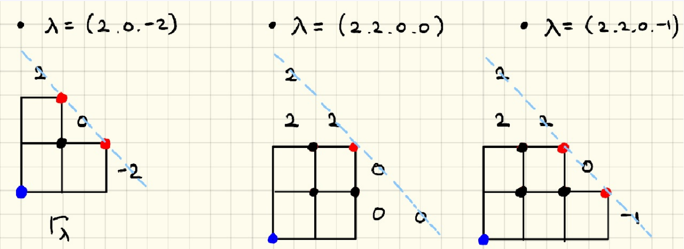


Gelfand-Cetlin polytopes

Face structure of Δ_λ :

The **ladder diagram** Γ_λ is a grid graph defined by

$$\Gamma_\lambda := \bigcup \square^{(i,j)}, \qquad x^{i,j} \neq \text{const. in } \Delta_\lambda.$$



- Blue dot is called the **origin**
- Red dots are called **terminal vertices** (farthest vertices from the origin)

A **positive path** is a shortest path from the origin to some terminal vertex.

Gelfand-Cetlin polytopes

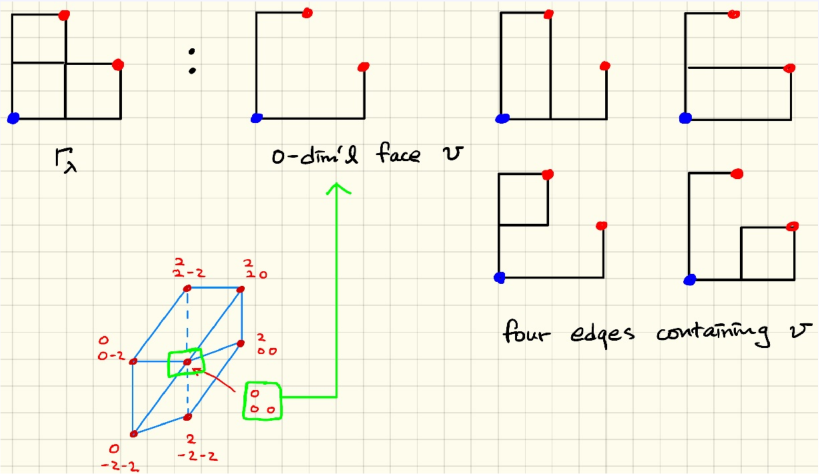
A **face** of Γ_λ is a subgraph γ of Γ_λ such that

- γ is a union of shortest paths
- γ contains all terminal vertices

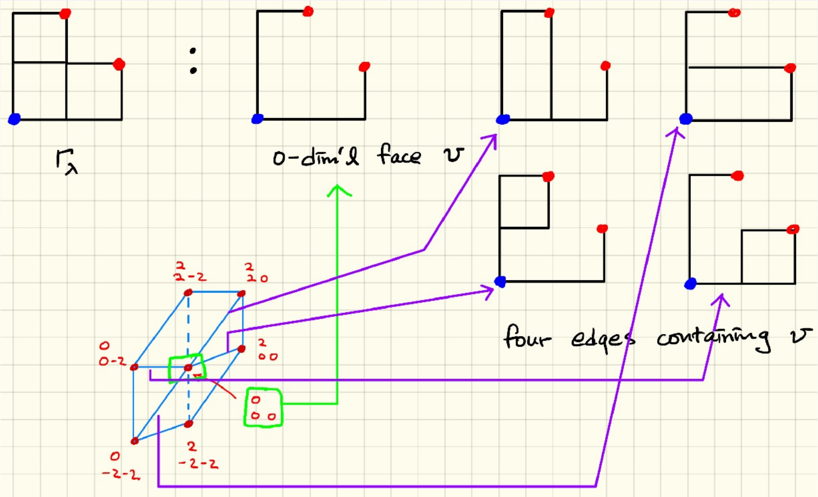
A **dimension** of γ is defined to be the number of minimal cycles in γ.

Gelfand-Cetlin polytopes

Example : $\lambda = (2, 0, -2)$



Gelfand-Cetlin polytopes



Gelfand-Cetlin polytopes

Theorem (An-C.-Kim) Face structure of Δ_λ is equivalent to face structure of Γ_λ.

Theorem (An-C.-Kim) Let $\mathbf{F_k}(t)$ be the f-polynomial for λ where

$$\lambda_1 = \cdots = \lambda_{k_1} > \lambda_{k_1+1} = \cdots = \lambda_{k_1+k_2} > \cdots > \lambda_{k_1+\cdots k_{s-1}+1} = \lambda_{k_1+\cdots k_s}$$

and $\mathbf{k} = (k_1, \cdots, k_s) \in (\mathbb{Z}_{\geq 0})^s$. Then $\mathbf{F_k}(t)$ satisfies the following recurrence relation :

$$\mathbf{F_k}(t) = \sum_{\mathbf{w} \in W_{s-1}} \mathbf{F}_{r_{\mathbf{w}}(\mathbf{k}) * \widetilde{\mathbf{w}}}(t) \cdot t^{|\mathbf{w}|}$$

where

- W_{s-1} : set of sequences of length $s-1$ on the set $\{(0,1), (1,0), (1,1)\}$
- $(x_1, x_2, \cdots, x_m) * (y_1, \cdots, y_{m-1}) := (x_1, y_1, \cdots, x_{m-1}, y_{m-1}, x_m)$

and for $\mathbf{w} = ((\alpha_1, \beta_1), \cdots, (\alpha_{s-1}, \beta_{s-1})) \in W_{s-1}$,

- $r_{\mathbf{w}}(\mathbf{k}) = (k_1', \cdots, k_s')$ with $k_i' := k_i + 1 - \alpha_i - \beta_{i-1}$ ($\alpha_s = \beta_0 = 1$)
- $\widetilde{\mathbf{w}} = (\alpha_1\beta_1, \cdots, \alpha_{s-1}\beta_{s-1})$
- $|\widetilde{\mathbf{w}}| = \sum_{i=1}^{s-1} \alpha_i\beta_i$.

Gelfand-Cetlin polytopes

Theorem (An-C.-Kim) If we denote by

$$\Psi_s(x_1, \cdots, x_s; t) := \sum_{\mathbf{k}\geq 0} \mathbf{F}_{\mathbf{k}}(t) \frac{x_1^{k_1} \cdots x_s^{k_s}}{k_1! \cdots k_s!}$$

the exponential generating function, then $\{\Psi_s\}$ satisfies

$$\mathcal{D}_s \left(\Psi_{2s-1}(x_1, y_1, \cdots, x_{s-1}, y_{s-1}, x_s; t)\right)|_{y_1=\cdots=y_{s-1}=0} = 0$$

where

$$\mathcal{D}_s = \frac{\partial^s}{\partial x_1 \cdots \partial x_s} - \prod_{i=1}^{s-1} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{i+1}} + t \cdot \frac{\partial}{\partial y_i} \right)$$

Gelfand-Cetlin polytopes

Example : For $\mathbf{k} = (1,1)$ (i.e., the case of $\lambda = (1,0)$),

$$\begin{aligned} \mathbf{F}_{\mathbf{k}}(t) &= \sum_{\mathbf{w} \in W_{s-1}} \mathbf{F}_{r_{\mathbf{w}}(\mathbf{k}) * \widetilde{\mathbf{w}}}(t) \cdot t^{|\mathbf{w}|} \\ &= \mathbf{F}_{(1,0)*(0)}(t)t^0 + \mathbf{F}_{(0,1)*(0)}(t)t^0 + \mathbf{F}_{(0,0)*(1)}(t)t^1 \\ &= t+2 \end{aligned}$$

Example : For $\mathbf{k} = (1,1,1)$ (i.e., the case of $\lambda = (2,0,-2)$),

$$\begin{aligned} \mathbf{F}_{\mathbf{k}}(t) &= \sum_{\mathbf{w} \in W_{s-1}} \mathbf{F}_{r_{\mathbf{w}}(\mathbf{k}) * \widetilde{\mathbf{w}}}(t) \cdot t^{|\mathbf{w}|} \\ &= \mathbf{F}_{(0,0,1,0,1)}t^0 + \mathbf{F}_{(0,0,2,0,0)}t^0 + \mathbf{F}_{(0,0,1,1,0)}t^1 + \mathbf{F}_{(1,0,0,0,1)}t^0 + \mathbf{F}_{(1,0,1,0,0)}t^0 \\ &\quad + \mathbf{F}_{(1,0,0,1,0)}t^1 + \mathbf{F}_{(0,1,0,0,1)}t^1 + \mathbf{F}_{(0,1,1,0,0)}t^1 + \mathbf{F}_{(0,1,0,1,0)}t^2 \\ &= (t+2)t^0 + t^0 + (t+2)t^1 + (t+2)t^0 + (t+2)t^0 \\ &\quad + (t+2)t^1 + (t+2)t^1 + (t+2)t^1 + (t+2)t^2 \\ &= t^3 + 6t^2 + 11t + 7 \end{aligned}$$

Gelfand-Cetlin polytopes

Gelfand-Cetlin systems: The GC polytope Δ_λ can be also obtained as follows :

- Let $\mathcal{O}_\lambda$: set of $(n \times n)$ Hermitian matrices having spectra $\lambda = (\lambda_1, \cdots, \lambda_n)$. $\mathcal{O}_\lambda$ is called a **flag manifold of type A**
- Define

$$\Phi_\lambda := \left(\Phi_\lambda^{i,j}\right), \qquad \Phi_\lambda^{i,j}(A) := i\text{-th largest eigenvalue of } A^{(i+j-1)}$$

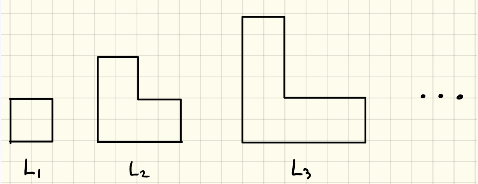
where $A^{(i)}$ is the i-th leading principal minor matrix of A. (**E.g.** $\Phi_\lambda^{1,1}(A) = a_{11}$.) We call Φ_λ a **Gelfand-Cetlin system**.

Theorem : $\Delta_\lambda = \mathrm{Im}(\Phi_\lambda)$ and $\dim \Delta_\lambda = \dim_{\mathbb{C}} \mathcal{O}_\lambda$.

Gelfand-Cetlin polytopes

Theorem(C.-Kim-Oh) "Topology of fibers $\Phi_\lambda^{-1}(\mathbf{u})$ and dimensions"

Let f be a face of Δ_λ and let γ_f be the corresponding face of Γ_λ.
Let's play a Tetris game on γ_f using only "L-blocks" where L-blocks are given as

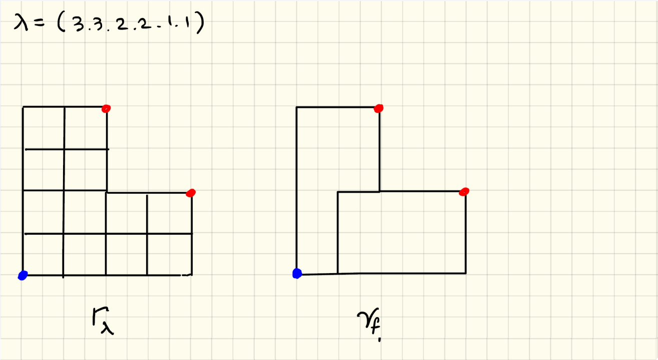


Then, fill γ_f using L-blocks obeying the following rules :

- the top and the rightmost edges of an L-block should overlap an edge of γ_f
- No edge of γ_f is in the interior of an L-block.

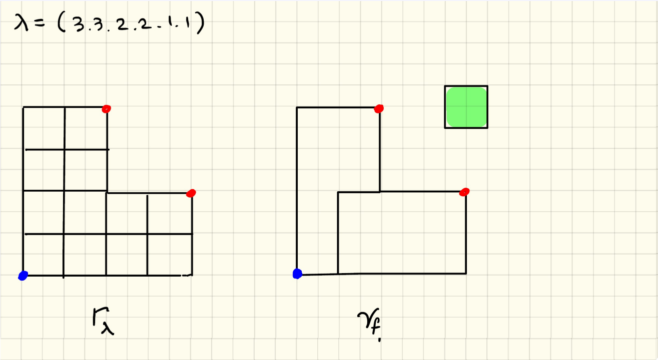
Gelfand-Cetlin polytopes

Theorem(C.-Kim-Oh) “Topology of fibers $\Phi_{\lambda}^{-1}(\mathbf{u})$ and dimensions”



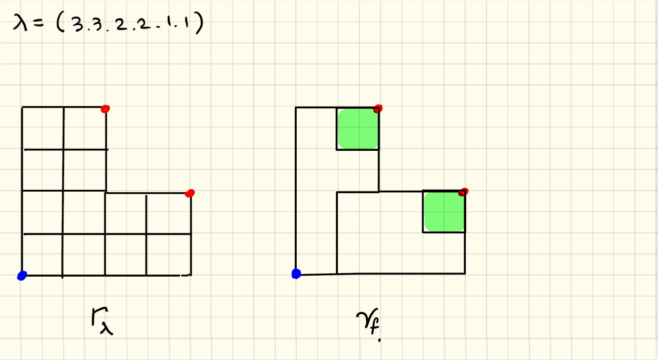
Gelfand-Cetlin polytopes

Theorem(C.-Kim-Oh) "Topology of fibers $\Phi_\lambda^{-1}(\mathbf{u})$ and dimensions"



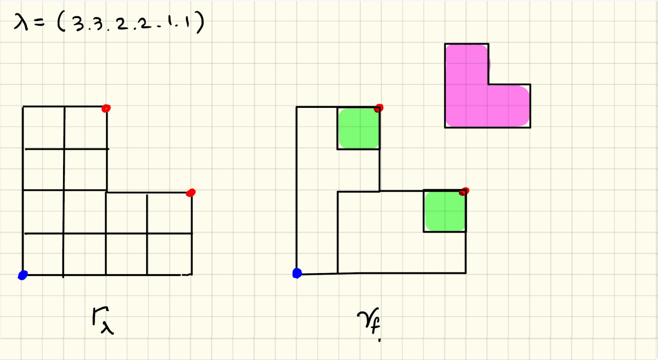
Gelfand-Cetlin polytopes

Theorem(C.-Kim-Oh) “Topology of fibers $\Phi_\lambda^{-1}(\mathbf{u})$ and dimensions”



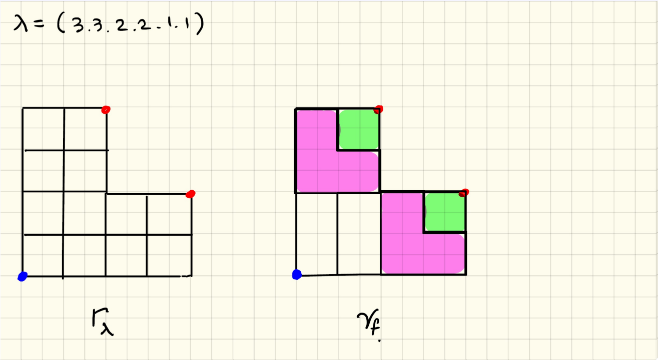
Gelfand-Cetlin polytopes

Theorem(C.-Kim-Oh) "Topology of fibers $\Phi_\lambda^{-1}(\mathbf{u})$ and dimensions"



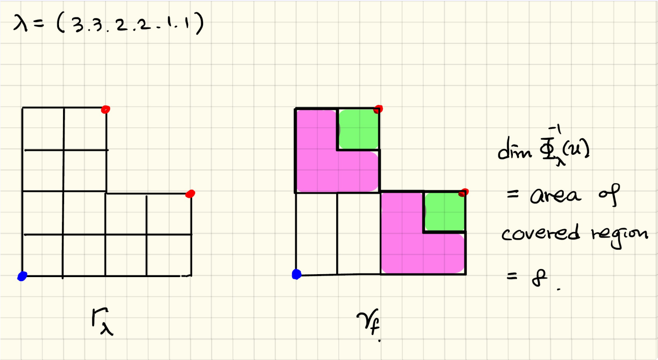
Gelfand-Cetlin polytopes

Theorem(C.-Kim-Oh) “Topology of fibers $\Phi_\lambda^{-1}(\mathbf{u})$ and dimensions”



Gelfand-Cetlin polytopes

Theorem(C.-Kim-Oh) “Topology of fibers $\Phi_\lambda^{-1}(\mathbf{u})$ and dimensions”



Gelfand-Cetlin polytopes

Theorem(C.-Kim-Oh) For any $\mathbf{u}$ in the relative interior of the face f of Δ_λ, the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is a smooth submanifold diffeomorphic to

$$\Phi_\lambda^{-1}(\mathbf{u}) \cong (S^1)^{\dim f} \times Y_f, \qquad Y_f : \text{ some iterated bundle of product of odd spheres}$$

and its dimension equals the area of the region covered by L-blocks.

Remark : When L-blocks covers whole γ_f, then

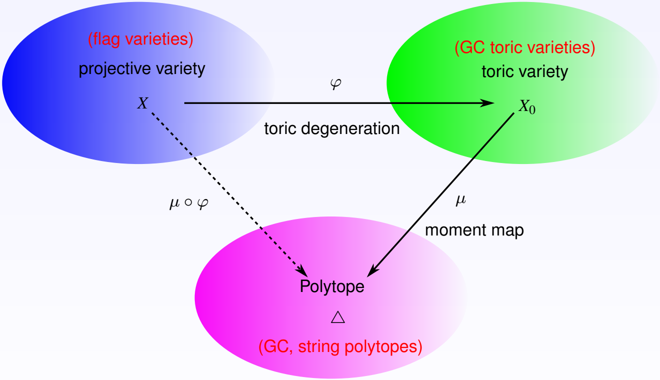
$$\dim \Phi_\lambda^{-1}(\mathbf{u}) = \dim_{\mathbb{C}} \mathcal{O}_\lambda, \quad \forall \mathbf{u} \in \mathring{f}.$$

Such fiber is called a Lagrangian and it is a main object in the study of symplectic manifolds, a candidate for generating the Fukaya category of $\mathcal{O}_\lambda$.

3. String polytopes

String polytopes

Main problems : Let X be a smooth projective variety over $\mathbb{C}$



String polytopes

Main problems : Let X be a smooth projective variety over $\mathbb{C}$

- Find a toric degeneration of X : (problem in commutative algebra)

  Find a flat homomorphism $\mathbb{C}[t] \to \mathbb{C}[X, t]$ such that

  - $\mathbb{C}[X, 1] = \mathbb{C}[X]$

  - $\mathbb{C}[X, 0]$: toric

  (E.g. $\mathbb{C}[X] = \mathbb{C}[x, y, z]/\langle y^2 z = x^3 + z^3\rangle$ and $\mathbb{C}[X, t] := \mathbb{C}[x, y, z]/\langle y^2 z = x^3 + t^6 z^3\rangle$)

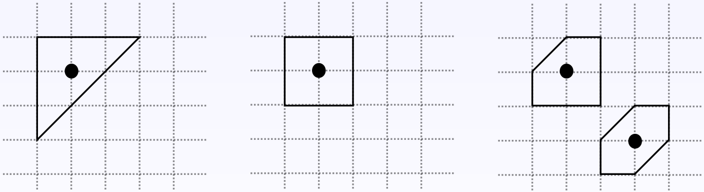
- Determine whether $X_0 := \mathrm{Spec}\ \mathbb{C}[X, 0]$ is nice to study X :

  - Δ_{X_0} is reflexive

  - Δ_{X_0} admits a small resolution

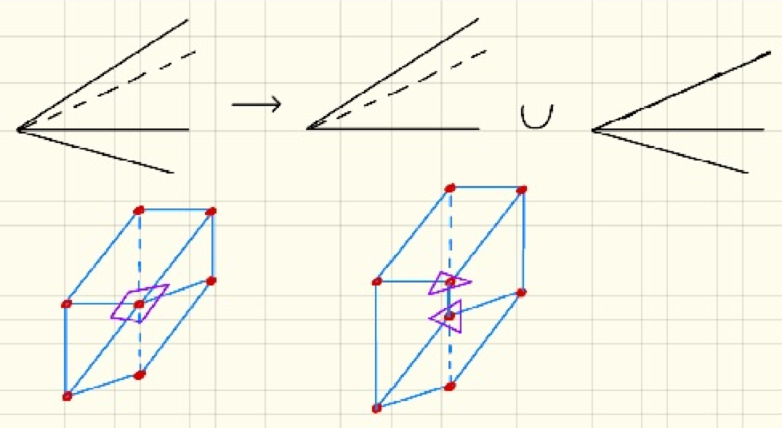
String polytopes

Reflexive polytope : Lattice polytope containing O whose dual is also a lattice polytope.



String polytopes

Small resolution : We say that a polytope P admits a small resolution if the normal fan has a smooth refinement. That is, each maximal cone of the normal fan can be decomposed into smooth cones **without inserting any ray**.



String polytopes

Theorem (Nishinou-Nohara-Ueda) If X admits a toric degeneration to a Fano toric variety admitting small resolution, then many information of X (such as an open GW-invariant and a potential function) can be recovered from Δ_{X_0}.

Theorem (Batyrev - Ciocan-Fontanine - Kim - van Straten) For a proper λ, the Gelfand-Cetlin polytope Δ_λ is a reflexive polytope and admits a small resolution.

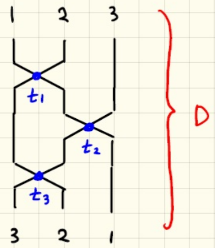
String polytopes

String polytopes : Let $W \cong S_{n-1}$ be the Weyl group of $U(n)$ and let $s_1, \cdots, s_{n-1}$ be the simple transposition (corresponding to a base). Let w_0 be the longest element of W and let $\underline{w_0} = (s_{i_1} s_{i_2} \cdots s_{i_N})$ be a reduced expression of w_0.

For each dominant weight λ, the **string polytope** $\Delta_{\underline{w_0}}(\lambda)$ is a convex rational polytope whose integral points parametrize certain basis (called "crystal basis") of a irreducible $U(n)$ representation with highest weight λ.

Gleizer - Postnikov description

$W_0 = s_1 s_2 s_1$:

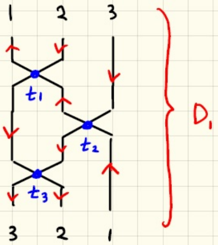


Gleizer - Postnikov description

$W_0 = s_1 s_2 s_1$:

D_k : oriented string diagram s.t.

- $1, 2, \cdots, k$ strings go up
- $k+1, \cdots, n$ strings go down

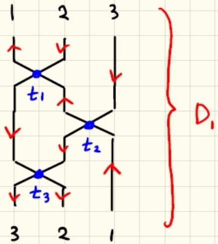


Gleizer - Postnikov description

$W_0 = s_1 s_2 s_1$:

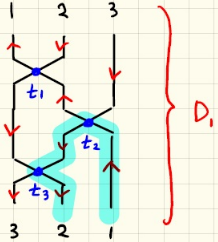
- For each k, find all paths from k to $k+1$ avoiding

 and 



Gleizer - Postnikov description

$W_0 = s_1 s_2 s_1$:



- For each k, find all paths from k to $k+1$ avoiding

 and 

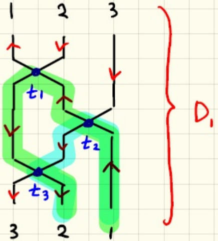
$\rightsquigarrow t_2 - t_3 \geq 0$

- Get $\sum a_i t_i \geq 0$ where

$$a_i = \begin{cases} 1 & \text{when turning from "small one" to "big one"} \\ -1 & \text{otherwise} \end{cases}$$

Gleizer - Postnikov description

$W_0 = s_1 s_2 s_1$:



- For each k, find all paths from k to $k+1$ avoiding

- Get $\sum a_i t_i \geq 0$ where

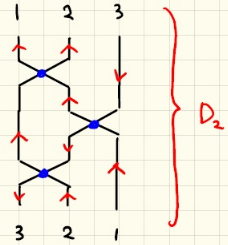
⇝ $t_2 - t_3 \geq 0$

⇝ $t_1 \geq 0$

$$a_i = \begin{cases} 1 & \text{when turning from "small one" to "big one"} \\ -1 & \text{otherwise} \end{cases}$$

Gleizer - Postnikov description

$W_0 = s_1 s_2 s_1$:

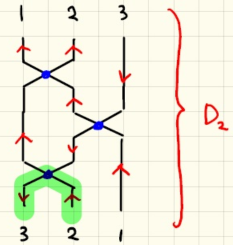


$\rightsquigarrow$ $t_2 - t_3 \geq 0$

$\rightsquigarrow$ $t_1 \geq 0$

Gleizer - Postnikov description

$W_0 = s_1 s_2 s_1$:



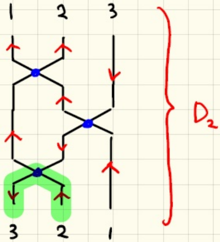
$\rightsquigarrow$ $t_2 - t_3 \geq 0$

$\rightsquigarrow$ $t_1 \geq 0$

$\rightsquigarrow$ $t_3 \geq 0$

Gleizer - Postnikov description

$W_0 = s_1 s_2 s_1$:

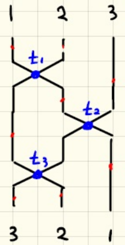


string inequalities

- ⇝ $t_2 - t_3 \geq 0$
- ⇝ $t_1 \geq 0$
- ⇝ $t_3 \geq 0$

Gleizer - Postnikov description

$W_0 = S_1 S_2 S_1$:



λ- inequalities are

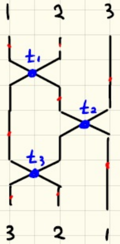
$$\left\{ t_i \leq \lambda_i + \sum_{j>i} c_j t_j \right\}$$

where

$$c_j = \begin{cases} 1 & \text{if } t_j \text{ is one column to the left / right of } t_i \\ -2 & \text{if } t_j \text{ is in the same column as } t_i \\ 0 & \text{otherwise} \end{cases}$$

Gleizer - Postnikov description

$W_0 = S_1 S_2 S_1$:



$t_1 \le \lambda_1 + t_2 - 2t_3$

$t_2 \le \lambda_2 + t_3$

$t_3 \le \lambda_3$

λ- inequalities are

$$\left\{ t_i \le \lambda_i + \sum_{j>i} c_j t_j \right\}$$

where

$$c_j = \begin{cases} 1 & \text{if } t_j \text{ is one column to the left / right of } t_i \\ -2 & \text{if } t_j \text{ is in the same column as } t_i \\ 0 & \text{otherwise} \end{cases}$$

String polytopes

Theorem String polytope $\Delta_{\underline{w_0}}(\lambda)$ is the intersection of the string cone and the λ-cone.

Theorem (C.-Kim-Lee-Park) (Alexeev-Brion Conjecture, 2004) For a proper λ, any string polytope $\Delta_{\underline{w_0}}(\lambda)$ is a reflexive polytope and admits a small resolution.

String polytopes

Open questions :

- We know that any reduced expression of w_0 can be obtained by a sequence of 2-moves and 3-moves starting from $s_1 s_2 s_1 \cdots s_{n-1} \cdots s_1$. Moreover, Berenstein and Zelevinsky described how the defining equations change along 2 or 3 moves. **How does the f-vector (or h-vector) change along 2 or 3 move?**

- **Can we construct a map $\Phi : \mathcal{H}_\lambda \to \Delta_{\underline{w_0}}(\lambda)$ explicitly?**

Thank you!