Rainbow connection parameters and forbidden subgraphs

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Outline

1. Chromatic number
2. Rainbow connection
3. Rainbow vertex-connection
4. 3-rainbow index
5. Some open problems

Forbidden subgraphs

- Many parameters of graphs have natural lower bounds in terms of various graph invariants, such as order, size, minimum degree, diameter, clique number and independence number, etc. However, they can be much larger than the lower bounds in most cases.

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- Many parameters of graphs have natural lower bounds in terms of various graph invariants, such as order, size, minimum degree, diameter, clique number and independence number, etc. However, they can be much larger than the lower bounds in most cases.
- But, if we forbid some special substructures, then they may be bounded in a way such as by a function of those graph invariants.
- Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is $\mathcal{F}$-free if G does not contain any induced subgraph isomorphic to a graph from $\mathcal{F}$.

Outline

Chromatic number

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- The chromatic number $\chi(G)$ of G is the minimum positive integer k such that G admits a k-coloring.
- A clique of a graph is a set of mutually adjacent vertices. The maximum size of a clique in a graph G is denoted $\omega(G)$, which is called the clique number of G.

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Theorem 1.1

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- Mycielski, Sur le coloriage des graphes, Colloq. Math. 3(1955) 161-162 gave a constructive proof of the above theorem.

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Question 1.2

For which family $\mathcal{F}$ of graphs, there exists a function f such that $\chi(G) \leq f(\omega(G))$ if a graph G is $\mathcal{F}$-free ?

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Theorem 1.4 (A. Scott, P. Seymour, JCTB 2016)

Let G be a graph with no odd hole. Then $\chi(G) \leq 2^{2^{\omega(G)+2}}$.

Chromatic number

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Theorem 1.5 (Chudnovsky et al., Ann. Math. 2006)

A graph is perfect if and only if it contains neither induced odd cycles of length at least 5 (odd holes) nor their complements.

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Theorem 1.6 (L. W. Beineke, JCT 9(1970))

A graph is a line graph if and only if it contains none of the 9 Beineke graphs, i.e., it forbids these 9 subgraphs.

- H. Kierstead told me that 3 of these forbidden subgraphs are enough to guarantee the upper bound $\omega(L(G)) + 1$.

- Except for proper (vertex) coloring, there are other colorings, say, edge-coloring, list coloring, rainbow connection coloring, etc.

- Except for proper (vertex) coloring, there are other colorings, say, edge-coloring, list coloring, rainbow connection coloring, etc.
- Some of these chromatic numbers also have obvious lower bounds in terms of some other parameters, and upper bounds as a function of the parameters under the condition of some forbidden subgraphs.

Outline

Rainbow connection

- G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohem. 133(1)(2008) 85-98 introduced the concept of rainbow connection number, denoted by $rc(G)$.

Definitions

- A path in an edge-colored graph is a rainbow path if no two edges of the path are colored with a same color.

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- An edge-colored graph is rainbow connected, if for any two vertices of the graph, there is a rainbow path connecting them. And, the edge-coloring is called a rainbow connection coloring.
- For a connected graph G, the rainbow connection number, denoted by $rc(G)$, is defined to be the minimum number of colors needed in an edge-coloring of G to make G rainbow connected.

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Proposition 2.1

Let G be a connected graph with n vertices. Then
(a) $1 \leq rc(G) \leq n-1$,
(b) $rc(G) = 1$ *if and only if* G *is complete,*
(c) $rc(G) = n-1$ *if and only if* G *is a tree,*
(d) $rc(G) \geq diam(G)$,
(e) if G *is a cycle of length* $n \geq 4$, *then* $rc(G) = \lceil \frac{n}{2} \rceil$.

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(e) if G *is a cycle of length* $n \geq 4$, *then* $rc(G) = \lceil \frac{n}{2} \rceil$.

- Note that the difference $rc(G) - diam(G)$ can be arbitrarily large. In fact, if $G = K_{1,n}$, then we have $rc(G) - diam(G) = n - 2$, since every edge requires a single color. Bridges have to receive distinct colors in any rainbow connection coloring.

Motivation

- Therefore, connected bridgeless graphs have been studied in M. Basavaraju, L.S. Chandran, D. Rajendraprasad, D. Ramaswamy, Rainbow connection number and radius, Graphs & Combin. 30(2014) 275-285 and they obtained the following result.

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For every connected bridgeless graph G with radius r,

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For every connected bridgeless graph G with radius r,

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- Note that, since $rad(G) \leq diam(G)$, the above theorem gives, for bridgeless graphs, an upper bound on $rc(G)$ which is quadratic in terms of the diameter $diam(G)$ of G. The upper bound is best possible even for graphs with high connectivity.

Question

P. Holub, Z. Ryjáček, I. Schiermeyer, P. Vrána, Rainbow connection and forbidden subgraphs, Discrete Math. 338(10)(2015) 1706-1713 considered forbidden families $\mathcal{F}$ implying a linear upper bound on $rc(G)$ in terms of $diam(G)$.

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Question 2.2

For which families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}, k_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $rc(G) \leq q_{\mathcal{F}} \cdot diam(G) + k_{\mathcal{F}}$?

Results when $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 1$

- When $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 1$ or 2, they gave a complete answer.

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Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $rc(G) \leq diam(G) + k_X$, if and only if $X = P_3$.

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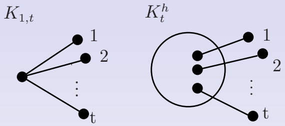
- Since the proof is short, we give its details, just to show the main idea for other complicated proofs.
- **Proof.** $\Longleftarrow$) If $X = P_3$, then G is complete, implying $rc(G) = diam(G) = 1$.

Proof of Theorem 2.3

$\Longrightarrow$) Let $t \geq k_X + 3$. Consider the graphs $K_{1,t}$ and K_t^h shown in Figure 1.



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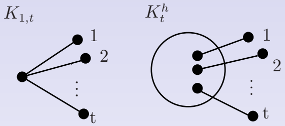

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Since $diam(K_{1,t}) = 2$ and $rc(K_{1,t}) = t > diam(K_{1,t}) + k_X$, then X must be an induced subgraph of $K_{1,t}$, and hence X is isomorphic to $K_{1,r}$ for some $r \leq t$.

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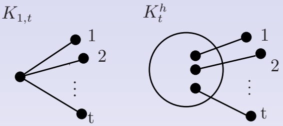

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On the other hand, since $diam(K_t^h) = 3$ and $rc(K_t^h) = t + 1 > diam(K_t^h) + k_X$, then K_t^h must contain an induced copy of X. Since K_t^h is $K_{1,3}$-free and $X = K_{1,r}$, then X must be a $K_{1,1}$ or $K_{1,2}$. Since if any graph free of $K_{1,1} = K_2$ is an empty graph, X must be $K_{1,2} = P_3$.

Results when $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 2$

Theorem 2.4

Let X, Y be connected graphs such that $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rc(G) \leq diam(G) + k_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$ $(r \geq 4)$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of the net N.

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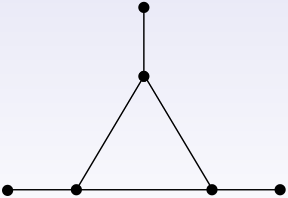

Figure 2. The graph N

Results for general $q_{\mathcal{F}}$

- They found that there are no more families with $|\mathcal{F}| \leq 2$ for general $q_{\mathcal{F}}$.

Result when $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 3$

- J. Brousek, P. Holub, Z. Ryjáček, P. Vrána, *Finite families of forbidden subgraphs for rainbow connection in graphs*, Discrete Math. 339(9)(2016) 2304-2312 considered further when $q_{\mathcal{F}} = 1$, and finalized their previous results.

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- Let $H \overset{\text{IND}}{\subseteq} G$ denote that H is an induced subgraph of G. Set:

$$\mathfrak{F}_1 = \{\{P_3\}\},$$
$$\mathfrak{F}_2 = \{\{X, Y\} | \{X, Y\} \overset{\text{IND}}{\subseteq} \{K_{1,3}, N\}\},$$
$$\mathfrak{F}_3 = \{\{K_{1,r}, P_4\} | r \geq 4\}.$$

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Theorem 2.5

Let $\mathcal{F}$ be a family of connected graphs with $|\mathcal{F}| = 3$ such that $\mathcal{F} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$. Then there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G satisfies $rc(G) \leq diam(G) + k_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathfrak{F}_4 \cup \mathfrak{F}_5 \cup \mathfrak{F}_6$.

Result when $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 3$

Set:

$$\overline{\mathfrak{F}_4} = \{\{K_{1,3}, K_s^h, N_{1,j,k}\} | s > 3, 1 \leq j \leq k, j + k > 2\},$$
$$\overline{\mathfrak{F}_5} = \{\{K_{1,r}, K_s^h, P_\ell\} | r > 3, s > 3, \ell > 4\},$$
$$\overline{\mathfrak{F}_6} = \{\{K_{1,r}, S_{1,j,k}, N\} | r > 3, 1 \leq j \leq k, j + k > 2\},$$

and

$$\mathfrak{F}_i = \{\{X, Y, Z\} | \{X, Y, Z\} \overset{\text{IND}}{\subseteq} \mathcal{F} \text{ for some } \mathcal{F} \in \overline{\mathfrak{F}_i}\}, i = 4, 5, 6,$$

where $K_{1,t}, K_t^h, S_{i,j,k}$ and $N_{i,j,k}$ with $t, i, j, k \in \mathbb{N}$ are shown in Figure 3.

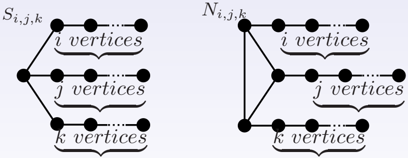

Figure 3: The graphs $S_{i,j,k}$, $N_{i,j,k}$

Result when $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 4$

Set:

$$\overline{\mathfrak{F}_7} = \{\{K_{1,r}, K_s^h, N_{1,j,k}, S_{1,\bar{j},\bar{k}}\} | r > 3, s > 3, 1 \leq j \leq k, j+k > 2, 1 \leq \bar{j} \leq \bar{k}, \bar{j} + \bar{k} > 2\},$$

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Theorem 2.6

Let $\mathcal{F}$ be a family of connected graphs with $|\mathcal{F}| = 4$ such that $\mathcal{F} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \cdots \cup \mathfrak{F}_6$. Then there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G satisfies $rc(G) \leq diam(G) + k_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathfrak{F}_7$.

Final result when $q_{\mathcal{F}} = 1$

- Finally, they gave a complete answer to Question 2.2 for any finite family $\mathcal{F}$ when $q_{\mathcal{F}} = 1$.

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Theorem 2.7

Let $\mathcal{F}$ be a finite family of connected graphs. Then there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G satisfies $rc(G) \leq diam(G) + k_{\mathcal{F}}$, if and only if $\mathcal{F}$ contains a subfamily $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \cdots \cup \mathfrak{F}_7$.

Question under the assumption $\delta(G) \geq 2$

- P. Holub, Z. Ryjáček, I. Schiermeyer, P. Vrána, *On forbidden subgraphs and rainbow connection in graphs with minimum degree 2*, Discrete Math. 338(3)(2015) 1-8 added an assumption $\delta(G) \geq 2$ to Question 2.2 (i.e., pendant edges are not allowed).

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Question 2.8

For which families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}, k_{\mathcal{F}}$ such that a connected graph G with $\delta(G) \geq 2$ being $\mathcal{F}$-free implies $rc(G) \leq q_{\mathcal{F}} \cdot diam(G) + k_{\mathcal{F}}$?

Results when $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 1$

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Theorem 2.9

Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G with minimum degree $\delta(G) \geq 2$ satisfies $rc(G) \leq diam(G) + k_X$, if and only if X is an induced subgraph of P_5.

Results when $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 2$

- P. Holub, Z. Ryjáček, I. Schiermeyer, P. Vrána, *Characterizing forbidden pairs for rainbow connection in graphs with minimum degree 2*, Discrete Math. 339(2016) 1058-1068 gave a complete characterization when $|\mathcal{F}| = 2$.

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Theorem 2.10

Let $X, Y \overset{IND}{\nsubseteq} P_5$ be a pair of connected graphs. Then there is a constant k_{XY} such that every connected (X,Y)-free graph G with $\delta(G) \geq 2$ satisfies $rc(G) \leq diam(G) + k_{XY}$, if and only if either $\{X,Y\} \overset{IND}{\subseteq} \{P_6, Z_1^r\}$ for some $r \in \mathbb{N}$ or $\{X,Y\} \overset{IND}{\subseteq} \{Z_3, P_7\}$, or $\{X,Y\} \overset{IND}{\subseteq} \{Z_3, S_{1,1,4}\}$, or $\{X,Y\} \overset{IND}{\subseteq} \{Z_3, S_{3,3,3}\}$, or $\{X,Y\} \overset{IND}{\subseteq} \{S_{2,2,2}, N_{2,2,2}\}$, where Z_1^r and Z_3 are shown in Figure 4.

Results when $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 2$

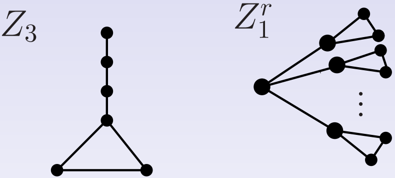

Figure 4: The graphs Z_3 and Z_1^r

Results for general $q_{\mathcal{F}}$

- When $|\mathcal{F}| = 1$ and 2, they found the answers for general $q_{\mathcal{F}}$, which are the same as in the case $q_{\mathcal{F}} = 1$.

Outline

Rainbow vertex-connection

- As a natural idea, M. Krivelevich, R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree, J. Graph Theory 63(2010) 185-191 introduced the vertex-version of rainbow connection number, called rainbow vertex-connection number, denoted by $rvc(G)$.

Definitions

- A path in a vertex-colored graph is a vertex-rainbow path if any two internal vertices of the path have distinct colors.

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- A path in a vertex-colored graph is a vertex-rainbow path if any two internal vertices of the path have distinct colors.
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- For a connected graph G, the rainbow vertex-connection number, denoted by $rvc(G)$, is defined to be the minimum number of colors needed in a vertex-coloring of G to make G rainbow vertex-connected.

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Let G be a connected graph with n vertices. Then
(a) $diam(G) - 1 \leq rvc(G) \leq n - 2$;
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- Note that the difference $rvc(G) - diam(G)$ can be arbitrarily large. In fact, if G is a subdivision of a star $K_{1,n}$, then we have $rvc(G) - diam(G) = (n + 1) - 4 = n - 3$, since every internal vertex (cut vertex) requires a single color.

Question

- We consider a similar question to Question 2.2 concerning rainbow vertex-connection number, that is, characterize forbidden families $\mathcal{F}$ which imply that $rvc(G)$ is upper bounded in terms of the diameter.

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Results when $|\mathcal{F}| \leq 2$

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Theorem 3.2 (W. Li, X. Li, J. Zhang, 38(1)(2018) 143–154)

Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $rvc(G) \leq diam(G) + k_X$, if and only if $X = P_3$ or P_4.

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Theorem 3.3 (W. Li, X. Li, J. Zhang, DMGT 38(1)(2018) 143–154)

Let $X, Y \neq P_3$ or P_4 be a pair of connected graphs. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rvc(G) \leq diam(G) + k_{XY}$, if and only if $\{X, Y\} \overset{IND}{\subseteq} \{P_5, K_r^h\}$ $(r \geq 4)$, or $\{X, Y\} \overset{IND}{\subseteq} \{S_{1,2,2}, N\}$.

Outline

1. Chromatic number
2. Rainbow connection
3. Rainbow vertex-connection
4. 3-rainbow index
5. Some open problems

3-rainbow index

- G. Chartrand, F. Okamoto, P. Zhang, Rainbow trees in graphs and generalized connectivity, Networks 55(2010) 360-367 generalized the concept of rainbow path to rainbow tree and proposed the parameter rainbow k-index, denoted by $rx_k(G)$.

Definitions

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- A tree in an edge-colored connected graph is a rainbow tree if no two edges of the tree are assigned the same color.
- For a vertex subset $S \subseteq V(G)$, a rainbow tree is called a rainbow S-tree if it connects (or contains) the vertices of S in G.
- Given an integer $k \geq 2$, a rainbow k-index coloring of G is an edge-coloring of G having the property that for every k-subset S of $V(G)$, there exists one rainbow S-tree in G. In this case, the graph G is called rainbow k-index connected.

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- For a connected graph G, the rainbow k-index, denoted by $rx_k(G)$, is defined to be the minimum number of colors needed in a rainbow k-index coloring of G.
- $rc(G) = rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G) \leq n-1.$

Special graph classes

G. Chartrand, F. Okamoto, P. Zhang, Rainbow trees in graphs and generalized connectivity, Networks 55(2010) 360-367 determined the rainbow k-index of trees, cycles, unicyclic graphs and complete graphs.

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Theorem 4.2

For integers k and n with $3 \leq k \leq n$,

$$rx_k(C_n) = \begin{cases} n-2 & \text{if } k = 3 \text{ and } n \geq 4 \\ n-1 & \text{if } k = n = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

Special graph classes

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Theorem 4.3

If G is a unicyclic graph of order $n \geq 3$ and girth g, then

$$rx_k(G) = \begin{cases} n-2 & \text{if } k = 3 \text{ and } g \geq 4 \\ n-1 & \text{if } g = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

Special graph classes

G. Chartrand, F. Okamoto, P. Zhang, Rainbow trees in graphs and generalized connectivity, Networks 55(2010) 360-367 investigated the rainbow 3-index of complete graphs:

Theorem 4.4

$$rx_3(K_n) = \begin{cases} 2 & \text{if } 3 \leq n \leq 5 \\ 3 & \text{if } n \geq 6. \end{cases}$$

Observations

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For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$.

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For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $k-1 \leq sdiam_k(G) \leq rx_k(G) \leq n-1$.

- Notice that for a fixed integer k with $k \geq 3$, the difference $rx_k(G) - sdiam_k(G)$ can be arbitrarily large. In fact, if G is a star $K_{1,n}$, then we have $rx_k(G) - sdiam_k(G) = n - k$, since every edge (cut edge) requires a single color.

Question

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For which families $\mathcal{F}$ of connected graphs, there is a constant $C_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $rx_k(G) \leq sdiam_k(G) + C_{\mathcal{F}}$?

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- For general k, the question is very difficult. For $k = 4$, there are very few results on rainbow 4-index, even if the exact value of $rx_4(K_n)$ has not been determined. So we pay our attention on the case $k = 3$. Fortunately, we have completely solved the question for any finite family $\mathcal{F}$.

Results when $|\mathcal{F}| \leq 2$

Theorem 4.7 (W. Li, X. Li, J. Zhang, Graphs & Combin. 33(4)(2017) 999–1008)

Let X be a connected graph. Then there is a constant C_X such that every connected X-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_X$, if and only if $X = P_3$.

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Theorem 4.8 (W. Li, X. Li, J. Zhang, Graphs & Combin. 33(4)(2017) 999–1008)

Let $X, Y \neq P_3$ be a pair of connected graphs. Then there is a constant C_{XY} such that every connected (X, Y)-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{XY}$, if and only if (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.

Results when $|\mathcal{F}| = 3$

- We set:

$$\mathfrak{F}_1 = \{\{P_3\}\},$$
$$\mathfrak{F}_2 = \{\{K_{1,r}, P_4\} | r \geq 3\},$$
$$\mathfrak{F}_3 = \{\{K_{1,r}, Y, P_\ell\} | r \geq 3, Y \overset{\text{IND}}{\subseteq} K_s^h, s \geq 3, \ell > 4\}.$$

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Theorem 4.9 (W. Li, X. Li, J. Zhang, Graphs & Combin. 33(4)(2017) 999–1008)

Let $\mathcal{F}$ be a family of connected graphs with $|\mathcal{F}| = 3$ such that $\mathcal{F} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2$. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathfrak{F}_3$.

Final result

- Finally, we give a complete answer to Question 4.6 for any finite family $\mathcal{F}$ with $k = 3$.

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Theorem 4.10 (W. Li, X. Li, J. Zhang, Graphs & Combin. 33(4)(2017) 999–1008)

Let $\mathcal{F}$ be a finite family of connected graphs. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$, if and only if $\mathcal{F}$ contains a subfamily $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.

Outline

Open problems

- At the end, we list some unsolved problems.

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Open problem 5.1

For which (finite) families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}$ and $C_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $rvc(G) \leq q_{\mathcal{F}} \cdot diam(G) + C_{\mathcal{F}}$?

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- We gave an answer for $q_{\mathcal{F}} = 1$ and $|\mathcal{F}| = 1$ or 2. How about the families $\mathcal{F}$ for general (finite) $|\mathcal{F}|$ and $q_{\mathcal{F}}$?

Open problems

- Another question is:

Open problem 5.2

For which (finite) families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}$ and $C_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $rx_k(G) \leq q_{\mathcal{F}} \cdot sdiam_k(G) + C_{\mathcal{F}}$?

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For which (finite) families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}$ and $C_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $rx_k(G) \leq q_{\mathcal{F}} \cdot sdiam_k(G) + C_{\mathcal{F}}$?

- We gave a complete answer for $q_{\mathcal{F}} = 1$. Are there more (finite) families $\mathcal{F}$ for general $q_{\mathcal{F}}$?

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- Just like the characterization of perfect graphs.

Thank you!