A diagram associated with the subconstituent algebra of a distance-regular graph

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Introduction

In this work we consider a distance-regular graph Γ.

Fix a vertex x of Γ and consider the corresponding subconstituent algebra (or Terwilliger algebra) T.

The algebra T is the $\mathbb{C}$-algebra generated by the Bose-Mesner algebra M of Γ and the dual Bose-Mesner algebra M^* of Γ with respect to x.

The algebra T is finite-dimensional and semisimple.

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The algebra T is finite-dimensional and semisimple.
So it is natural to compute the irreducible T-modules.
These modules are important in the study of

- hypercubes,
- dual polar graphs,
- spin models,
- codes,
- the bipartite property,
- the almost-bipartite property,
- the Q-polynomial property,
- the thin property, etc.

Distance-regular graph

Let $\Gamma = (X, \mathcal{E})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{E}$.
Define the diameter $D := \max\{\partial(x, y) | x, y \in X\}$.

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Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y.
We abbreviate $k_i := p^0_{ii}$ $(0 \leq i \leq D)$.

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For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = i, \\ 0 & \text{if} \quad \partial(x,y) \neq i, \end{cases} \qquad x, y \in X.$$

We call A_i the *i-th distance matrix* of Γ.
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Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A.
The matrices $A_0, A_1, ..., A_D$ form a basis for M. (See [Brouwer, Cohen, Neumaier 1989].)
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M has a basis $E_0, E_1, ..., E_D$ such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $E_i^t = E_i$ $(0 \leq i \leq D)$; (iv) $\overline{E_i} = E_i$ $(0 \leq i \leq D)$; (v) $E_iE_j = \delta_{ij}E_i$ $(0 \leq i, j \leq D)$. (See [Bannai, Ito 1984].)
The matrices $E_0, E_1, ..., E_D$ are called the primitive idempotents of Γ.

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Fix a vertex $x \in X$.
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$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if} \quad \partial(x, y) = i, \\ 0 & \text{if} \quad \partial(x, y) \neq i, \end{cases} \qquad y \in X.$$

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We call E_i^* the *i-th dual idempotent* of Γ with respect to x.
Let $M^* = M^*(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ with basis $E_0^*, E_1^*, ..., E_D^*$.
We call M^* the dual Bose-Mesner algebra of Γ with respect to x.

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$$(A_i^*)_{yy} = |X|(E_i)_{xy} \qquad y \in X.$$

We call A_i^* the *dual distance matrix* of Γ with respect to x and E_i.
The matrices $A_0^*, A_1^*, ..., A_D^*$ form a basis for M^*. (See [Terwilliger 1992].)

The subconstituent algebra T

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We will use the following notation.
For any two subspaces $\mathcal{R}, \mathcal{S}$ of $\mathrm{Mat}_X(\mathbb{C})$ we define

$$\mathcal{R}\mathcal{S} = Span\{RS | R \in \mathcal{R}, S \in \mathcal{S}\}.$$

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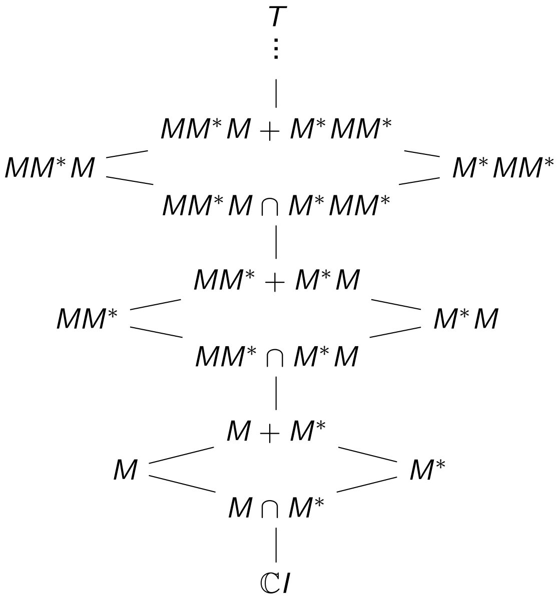
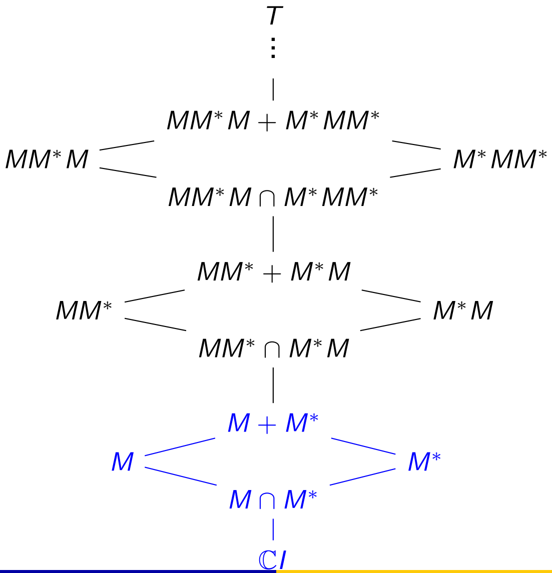
Consider the subspaces $M, M^*, MM^*, M^*M, MM^*M, M^*MM^*, \ldots$ along with their intersections and sums; see diagram.
We describe the diagram up to $MM^* + M^*M$.


T
$\vdots$
$MM^*M + M^*MM^*$
MM^*M
M^*MM^*
$MM^*M \cap M^*MM^*$
$MM^* + M^*M$
MM^*
M^*M
$MM^* \cap M^*M$
$M + M^*$
M
M^*
$M \cap M^*$
$\mathbb{C}I$

The subspace $M + M^*$



For $0 \leq i, j \leq D$, we have

$$\langle A_i, A_j \rangle = \delta_{ij} k_i |X|, \qquad \langle E_i, E_j \rangle = \delta_{ij} m_i,$$
$$\langle A_i^*, A_j^* \rangle = \delta_{ij} m_i |X|, \qquad \langle E_i^*, E_j^* \rangle = \delta_{ij} k_i,$$

where m_i is the rank of E_i.

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$$\langle A^*_i, A^*_j \rangle = \delta_{ij} m_i |X|, \qquad \langle E^*_i, E^*_j \rangle = \delta_{ij} k_i,$$

where m_i is the rank of E_i.

Lemma 1.

Each of the following is an orthogonal basis for M:

$$\{A_i\}_{i=0}^{D}, \qquad \{E_i\}_{i=0}^{D}.$$

Moreover, each of the following is an orthogonal basis for M^*:

$$\{A^*_i\}_{i=0}^{D}, \qquad \{E^*_i\}_{i=0}^{D}.$$

Lemma 2.

For $0 \leq i, j \leq D$, $\langle A_i, A_j^* \rangle = \delta_{i0}\delta_{0j}|X|k_i$.

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Lemma 3.

The following is an orthogonal basis for $M + M^*$*:*

$$A_D, \ldots, A_1, I, A_1^* \ldots, A_D^*.$$

So $dim(M + M^*) = 2D + 1$.

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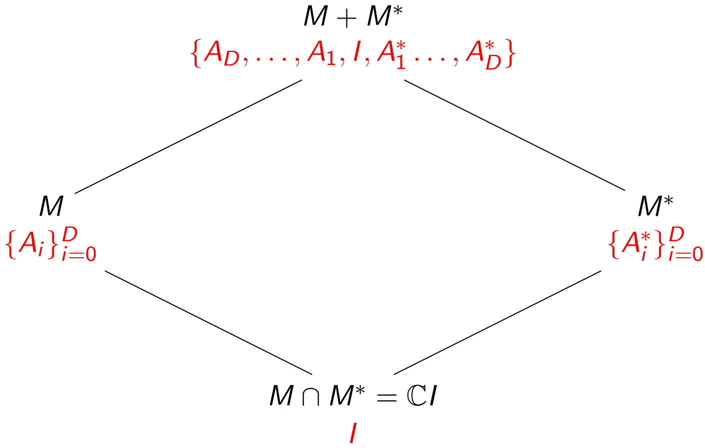
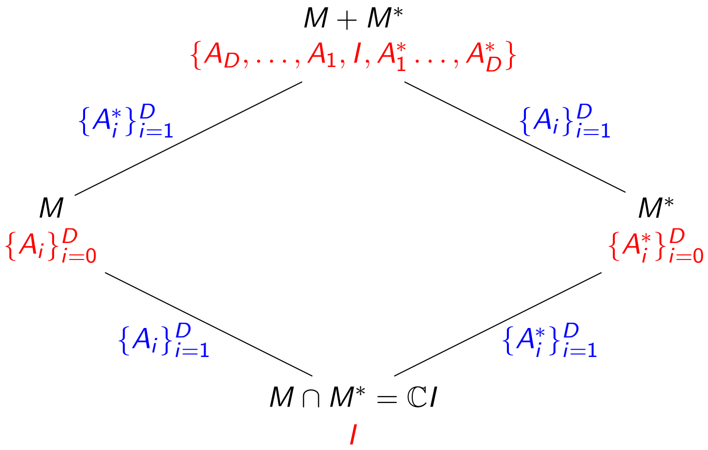
Lemma 4.

We have $M \cap M^* = \mathbb{C}I$ *and* $dim(M \cap M^*) = 1$.



For each edge $U \subseteq W$ shown in the diagram, we desire to find an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement.
$(U^\perp = \{w \in W | \langle w, u \rangle = 0 \text{ for all } u \in U\})$


$M + M^*$
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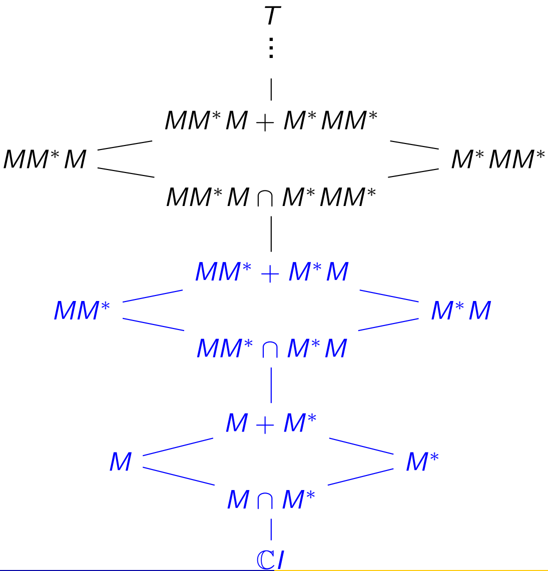
I

The subspace $MM^* + M^*M$



For $0 \le i \le D$ let θ_i denote an eigenvalue of $A = A_1$ associated with E_i. Let λ be an indeterminate. Define polynomials $\{u_i\}_{i=0}^{D}$ in $\mathbb{C}[\lambda]$ by $u_0 = 1$, $u_1 = \lambda/k$, and $\lambda u_i = p^i_{1,i-1} u_{i-1} + p^i_{1i} u_i + p^i_{1,i+1} u_{i+1}$ $(1 \le i \le D-1)$.

$$A_j = k_j \sum_{i=0}^{D} u_j(\theta_i) E_i, \qquad E_j = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) A_i \qquad (0 \le j \le D).$$

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Lemma 5.

For $0 \leq i, j, r, s \leq D$,

1. $\langle A_iA_j^*, A_r^*A_s\rangle = \delta_{is}\delta_{jr}|X|k_im_ju_i(\theta_j)$,
2. $\langle A_iA_j^*, A_rA_s^*\rangle = \delta_{ir}\delta_{js}|X|k_im_j$.

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2. $\langle A_iA^*_j, A_rA^*_s\rangle = \delta_{ir}\delta_{js}|X|k_im_j$.

Lemma 6.

1. *The matrices* $\{A_iA^*_j | 0 \le i,j \le D\}$ *form an orthogonal basis for* MM^*.
2. *The matrices* $\{A^*_jA_i | 0 \le i,j \le D\}$ *form an orthogonal basis for* M^*M.
3. $dim(MM^*) = (D+1)^2$, $dim(M^*M) = (D+1)^2$.

Lemma 7.

$$MM^* + M^*M = \sum_{i=0}^{D}\sum_{j=0}^{D} Span(A_iA_j^*, A_j^*A_i) \qquad (orthogonal\ direct\ sum).$$

$$Thus\ dim(MM^* + M^*M) = \sum_{i=0}^{D}\sum_{j=0}^{D} dim(Span(A_iA_j^*, A_j^*A_i)).$$

Definition 8.

For $0 \leq i, j \leq D$ let $H_{i,j}$ denote the 2×2 matrix of inner products for $A_i A_j^*, A_j^* A_i$.

Lemma 9.

For $0 \leq i, j \leq D$,

$$H_{i,j} = |X| k_i m_j \begin{pmatrix} 1 & u_i(\theta_j) \\ u_i(\theta_j) & 1 \end{pmatrix}.$$

So $det(H_{i,j}) = |X|^2 k_i^2 m_j^2 (1 - (u_i(\theta_j))^2)$ *and* $det(H_{i,j}) = 0$ *if and only if* $u_i(\theta_j) = \pm 1$.

Lemma 10.

The following elements are orthogonal:

$$A_iA_j^* + A_j^*A_i, \qquad A_iA_j^* - A_j^*A_i.$$

Moreover

$$||A_iA_j^* + A_j^*A_i||^2 = 2|X|k_i m_j(1 + u_i(\theta_j)),$$
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Lemma 11.

The following (i)–(iii) hold for $0 \leq i, j \leq D$:

1. *Assume $u_i(\theta_j) = 1$. Then $A_iA_j^* = A_j^*A_i$ and this common value is nonzero.*
2. *Assume $u_i(\theta_j) = -1$. Then $A_iA_j^* = -A_j^*A_i$ and this common value is nonzero.*
3. *Assume $u_i(\theta_j) \neq \pm 1$. Then $A_iA_j^*, A_j^*A_i$ are linearly independent.*

Lemma 12.

For $0 \leq i, j \leq D$ *we give an orthogonal basis for* $Span(A_iA_j^*, A_j^*A_i)$.

case	*orthogonal basis*	*dimension*
$u_i(\theta_j) = \pm 1$	$A_iA_j^*$	1
$u_i(\theta_j) \neq \pm 1$	$A_iA_j^* + A_j^*A_i, \quad A_iA_j^* - A_j^*A_i$	2

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Corollary 13.

The following is an orthogonal basis for $MM^* + M^*M$:

$$\{A_iA_j^* + A_j^*A_i, A_iA_j^* - A_j^*A_i | 0 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$$

$$\cup\{A_iA_j^* | 0 \leq i, j \leq D, u_i(\theta_j) = \pm 1\}.$$

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2. $dim(MM^* \cap M^*M) = 2D + 1 + P.$

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Lemma 16.

The following is an orthogonal basis for $MM^* \cap M^*M$:

$$\{A_i A_j^* | 0 \leq i,j \leq D, u_i(\theta_j) = \pm 1\}.$$

$$MM^* + M^*M$$

$$\{A_iA_j^* + A_j^*A_i, A_iA_j^* - A_j^*A_i | 0 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$$
$$\cup\{A_iA_j^* | 0 \leq i, j \leq D, u_i(\theta_j) = \pm 1\}$$

$$MM^*$$

$$\{A_iA_j^* | 0 \leq i, j \leq D\}$$

$$M^*M$$

$$\{A_j^*A_i | 0 \leq i, j \leq D\}$$

$$MM^* \cap M^*M$$

$$\{A_iA_j^* | 0 \leq i, j \leq D, u_i(\theta_j) = \pm 1\}$$

$$M + M^*$$

$$\{A_D, \ldots, A_1, I, A_1^* \ldots, A_D^*\}$$

$$MM^* + M^*M$$

$$\{A_iA_j^* + A_j^*A_i, A_iA_j^* - A_j^*A_i \mid 0 \leq i,j \leq D, u_i(\theta_j) \neq \pm 1\}$$
$$\cup \{A_iA_j^* \mid 0 \leq i,j \leq D, u_i(\theta_j) = \pm 1\}$$

$$\{u_i(\theta_j)A_iA_j^* - A_j^*A_i \mid 1 \leq i,j \leq D, u_i(\theta_j) \neq \pm 1\}$$

$$\{A_iA_j^* - u_i(\theta_j)A_j^*A_i \mid 1 \leq i,j \leq D, u_i(\theta_j) \neq \pm 1\}$$

$$MM^*$$

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$$M + M^*$$

$$\{A_D, \ldots, A_1, I, A_1^* \ldots, A_D^*\}$$

Summary of main results

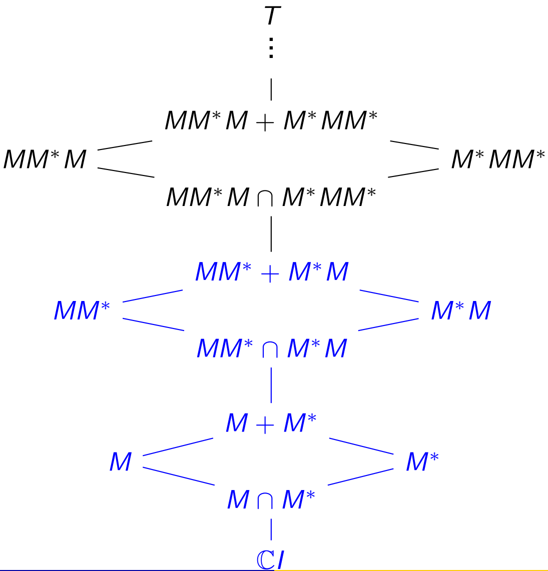
In each row of the table below we describe a subspace U in the diagram. We give an orthogonal basis for U along with the dimension of U.

subspace U	orthogonal basis for U	dimension of U
$M \cap M^*$	I	1
M	$\{A_i\}_{i=0}^D$	$D+1$
M^*	$\{A_i^*\}_{i=0}^D$	$D+1$
$M+M^*$	$\{A_D, \ldots, A_1, I, A_1^*, \ldots, A_D^*\}$	$2D+1$
$MM^* \cap M^*M$	$\{A_iA_j^* \mid 0 \leq i,j \leq D, u_i(\theta_j) = \pm 1\}$	$2D+1+P$
MM^*	$\{A_iA_j^* \mid 0 \leq i,j \leq D\}$	$(D+1)^2$
M^*M	$\{A_j^*A_i \mid 0 \leq i,j \leq D\}$	$(D+1)^2$
MM^*+M^*M	$\{A_iA_j^* + A_j^*A_i, A_iA_j^* - A_j^*A_i$	$2D^2+2D+1-P$
	$\mid 0 \leq i,j \leq D, u_i(\theta_j) \neq \pm 1\}$	
	$\cup \{A_iA_j^* \mid 0 \leq i,j \leq D, u_i(\theta_j) = \pm 1\}$	

We give an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement.

U	W	orthogonal basis for $U^{\perp} \cap W$	dimension
$M \cap M^*$	M	$\{A_i\}_{i=1}^{D}$	D
$M \cap M^*$	M^*	$\{A_i^*\}_{i=1}^{D}$	D
M	$M + M^*$	$\{A_i^*\}_{i=1}^{D}$	D
M^*	$M + M^*$	$\{A_i\}_{i=1}^{D}$	D
$M + M^*$	$MM^* \cap M^*M$	$\{A_iA_j^* \mid 1 \leq i, j \leq D, u_i(\theta_j) = \pm 1\}$	P
$MM^* \cap M^*M$	MM^*	$\{A_iA_j^* \mid 1 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
$MM^* \cap M^*M$	M^*M	$\{A_j^*A_i \mid 1 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
MM^*	$MM^* + M^*M$	$\{u_i(\theta_j)A_iA_j^* - A_j^*A_i \mid 1 \leq i, j \leq D,$ $u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
M^*M	$MM^* + M^*M$	$\{A_iA_j^* - u_i(\theta_j)A_j^*A_i \mid 1 \leq i, j \leq D,$ $u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$

Open problems



Lemma. (Terwilliger, 1992)

For $0 \leq h, i, j, r, s, t \leq D$,

1. $\langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \rangle = \delta_{ir}\delta_{js}\delta_{ht} k_h p_{ij}^h$,
2. $\langle E_i A_j^* E_h, E_r A_s^* E_t \rangle = \delta_{ir}\delta_{js}\delta_{ht} m_h q_{ij}^h$.

Corollary.

For $0 \leq h, i, j \leq D$,

1. $E_i^* A_h E_j^* = 0$ if and only if $p_{ij}^h = 0$,
2. $E_i A_h^* E_j = 0$ if and only if $q_{ij}^h = 0$.

The subspace $M^* M M^*$ has an orthogonal basis $\{E_i^* A_j E_h^* | 0 \leq h, i, j \leq D, p_{ij}^h \neq 0\}$.
Similarly, the subspace $M M^* M$ has an orthogonal basis $\{E_i A_j^* E_h | 0 \leq h, i, j \leq D, q_{ij}^h \neq 0\}$.

Problem 17.

Find an orthogonal basis for the following subspaces:

(i) $MM^*M \cap M^*MM^*$,

(ii) $MM^*M + M^*MM^*$.

Problem 18.

In each row of the table below we give an edge $U \subseteq W$ from the diagram. Find an orthogonal basis for the orthogonal complement of U in W.

U	W
$MM^* + M^*M$	$MM^*M \cap M^*MM^*$
$MM^*M \cap M^*MM^*$	MM^*M
$MM^*M \cap M^*MM^*$	M^*MM^*
MM^*M	$MM^*M + M^*MM^*$
M^*MM^*	$MM^*M + M^*MM^*$

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MM^*M	$MM^*M + M^*MM^*$
M^*MM^*	$MM^*M + M^*MM^*$

Thank you for your attention.