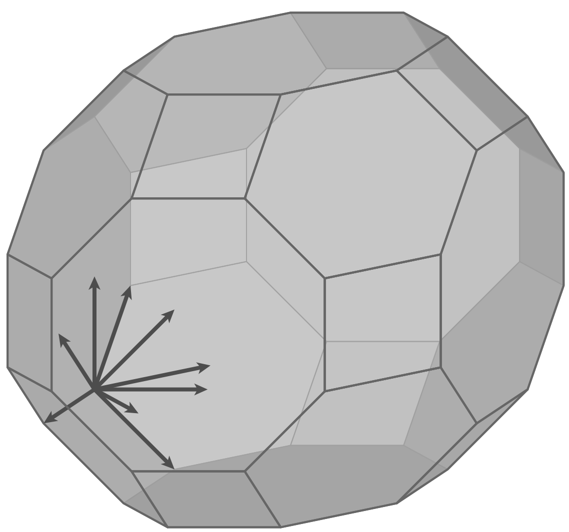
Classification of Combinatorial Polynomials (in particular, Ehrhart Polynomials of Zonotopes)



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Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, $\mathrm{ehr}_{\mathcal{P}}(t) := \left| t\mathcal{P} \cap \mathbb{Z}^d \right|$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\mathrm{vol}\,\mathcal{P}$ and constant term 1.

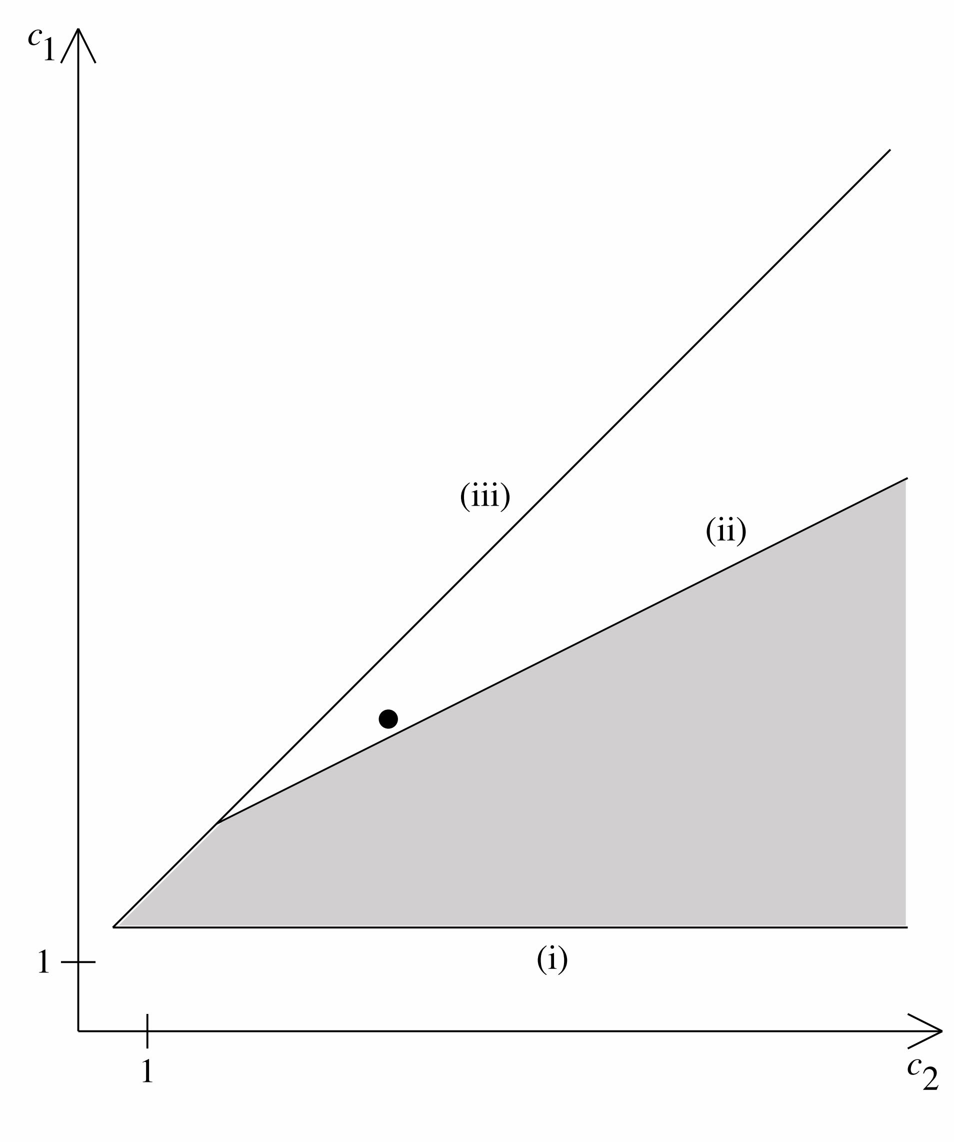
$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \ge 1} \mathrm{ehr}_{\mathcal{P}}(t)\, z^t = \frac{h^*(z)}{(1-z)^{d+1}}$$

Equivalent descriptions of an Ehrhart polynomial:

- $\mathrm{ehr}_{\mathcal{P}}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_0$
- via roots of $\mathrm{ehr}_{\mathcal{P}}(t)$
- $\mathrm{Ehr}_{\mathcal{P}}(z) \quad \longrightarrow \quad \mathrm{ehr}_{\mathcal{P}}(t) = h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d}$

(Wide) Open Problem Classify Ehrhart polynomials.

Two-dimensional Ehrhart Polynomials



Essentially due to Pick (1899) and Scott (1976)

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$$\longrightarrow \qquad \mathrm{ehr}_{\mathcal{P}}(t) \ = \ h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d}$$

Theorem (Macdonald 1971) $(-1)^d\, \mathrm{ehr}_{\mathcal{P}}(-t)$ enumerates the interior lattice points in $t\mathcal{P}$. Equivalently,

$$\mathrm{ehr}_{\mathcal{P}^\circ}(t) \ = \ h_d^* \binom{t+d-1}{d} + h_{d-1}^* \binom{t+d-2}{d} + \cdots + h_0^* \binom{t-1}{d}$$

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

Corollary If $h_{d+1-k}^* > 0$ then $k\mathcal{P}^\circ$ contains an integer point.

Positivity Among Ehrhart Polynomials

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Theorem (Betke–McMullen 1985, Stapledon 2009) If $h_d^* > 0$ then

$$h^*(z) \;=\; a(z) + z\, b(z)$$

where $a(z) = z^d\, a(\frac{1}{z})$ and $b(z) = z^{d-1}\, b(\frac{1}{z})$ with nonnegative coefficients.

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Open Problem Try to prove the analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.

Unimodality & Real-rooted Polynomials

The polynomial $h(z) = \sum_{j=0}^{d} h_j z^j$ is unimodal if for some $k \in \{0, 1, \dots, d\}$

$$h_0 \le h_1 \le \cdots \le h_k \ge \cdots \ge h_d$$

Crucial Example $h(z)$ has only real roots

Conjectures $h^*(z)$ is unimodal/real-rooted for

- hypersimplices
- order polytopes
- alcoved polytopes
- lattice polytopes with unimodular triangulations
- IDP polytopes (integer decomposition property)

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Crucial Example $h(z)$ has only real roots

Conjecture (Stanley 1989) $h^*(z)$ is unimodal for IDP polytopes.

Classic Example $\mathcal{P} = [0,1]^d$ comes with the Eulerian polynomial $h^*(z)$

Theorem (Schepers–Van Langenhoven 2013) $h^*(z)$ is unimodal for lattice parallelepipeds.

Zonotopes

The zonotope generated by $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ is $\left\{ \sum_{j=1}^{n} \lambda_j \mathbf{v}_j : \, 0 \le \lambda_j \le 1 \right\}$

Theorem (MB–Jochemko–McCullough) $h^*(z)$ is real rooted for lattice zonotopes.

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Theorem (MB–Jochemko–McCullough) $h^*(z)$ is real rooted for lattice zonotopes.

Theorem (MB–Jochemko–McCullough) The convex hull of the h^*-polynomials of all d-dimensional lattice zonotopes is the d-dimensional simplicial cone

$$A_1(d+1,z) + \mathbb{R}_{\ge 0}\, A_2(d+1,z) + \cdots + \mathbb{R}_{\ge 0}\, A_{d+1}(d+1,z)$$

where we define an (A,j)-Eulerian polynomial as

$$A_j(d,z) \ := \ \sum_{k=0}^{d-1} \left| \{ \sigma \in S_d : \ \sigma(d) = d+1-j \text{ and } \operatorname{des}(\sigma) = k \} \right| z^k$$

Eulerian Polynomials

The (type A) Eulerian polynomials are

$$A(d,z) \ := \ \sum_{k=0}^{d-1} |\{\sigma \in S_d : \, \mathrm{des}(\sigma) = k\}| \, z^k$$

where $\mathrm{des}(\sigma)$ is the number of descents $\sigma(j+1) < \sigma(j)$

$A(d,z)$ is symmetric, real rooted, and $\displaystyle\sum_{t \ge 0} (t+1)^d \, z^t \ = \ \frac{A(d,z)}{(1-z)^{d+1}}$

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My favorite proof Compute the Ehrhart series of

$$[0,1]^d \ = \bigsqcup_{\sigma \in S_d} \left\{ \mathbf{x} \in \mathbb{R}^d : \begin{array}{l} 0 \le x_{\sigma(d)} \le x_{\sigma(d-1)} \le \cdots \le x_{\sigma(1)} \le 1 \\ x_{\sigma(j+1)} < x_{\sigma(j)} \text{ if } j \in \mathrm{Des}(\sigma) \end{array} \right\}$$

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$$A_j(d,z) \ := \ \sum_{k=0}^{d-1} \left|\{\sigma \in S_d : \ \sigma(d) = d+1-j \text{ and } \mathrm{des}(\sigma) = k\}\right| z^k$$

seem to have first been used by Brenti–Welker (2008). They are not all symmetric but unimodal (Kubitzke–Nevo 2009) and real rooted (Savage–Visontai 2015).

The Geometry of Refined Eulerian Polynomials

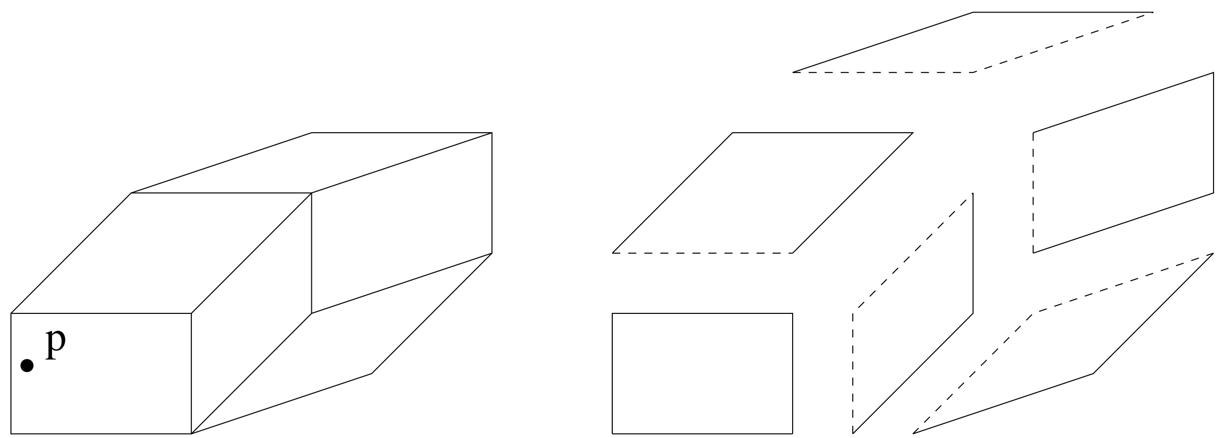
Lemma 1 $A_j(d,z) = \sum_{k=0}^{d-1} \left| \{ \sigma \in S_d : \sigma(d) = d+1-j \text{ and } \operatorname{des}(\sigma) = k \} \right| z^k$

is the h^*-polynomial of the half-open cube

$$C_j^d := [0,1]^d \setminus \left\{ \mathbf{x} \in \mathbb{R}^d : x_d = x_{d-1} = \cdots = x_{d+1-j} = 1 \right\}$$

Lemma 2 The h^*-polynomial of a half-open lattice parallelepiped is a linear combination of $A_j(d,z)$.

Lemma 3



Zonotopal h^*-polynomials

Theorem (MB–Jochemko–McCullough) $h^*(z)$ is real rooted for lattice zonotopes.

Theorem (MB–Jochemko–McCullough) The convex hull of the h^*-polynomials of all d-dimensional lattice zonotopes is the d-dimensional simplicial cone

$$\mathcal{K} \ := \ A_1(d+1,z) + \mathbb{R}_{\geq 0}\, A_2(d+1,z) + \cdots + \mathbb{R}_{\geq 0}\, A_{d+1}(d+1,z)$$

Open Problem Classify h^*-polynomials of d-dimensional lattice zonotopes.

This is nontrivial: we can prove that each h^*-polynomial is actually in

$$A_1(d+1,z) + \mathbb{Z}_{\geq 0}\, A_2(d+1,z) + \cdots + \mathbb{Z}_{\geq 0}\, A_{d+1}(d+1,z)$$

however, $\mathcal{K}$ is not IDP. (And the above is not complete either.)

Valuations

A $\mathbb{Z}^d$-valuation φ satisfies $\varphi(\varnothing) = 0$,

$$\varphi(\mathcal{P} \cup \mathcal{Q}) \ = \ \varphi(\mathcal{P}) + \varphi(\mathcal{Q}) - \varphi(\mathcal{P} \cap \mathcal{Q})$$

whenever $\mathcal{P}, \mathcal{Q}, \mathcal{P} \cup \mathcal{Q}, \mathcal{P} \cap \mathcal{Q}$ are lattice polytopes, and $\varphi(\mathcal{P} + \mathbf{x}) = \varphi(\mathcal{P})$ for all $\mathbf{x} \in \mathbb{Z}^d$.

Theorem (McMullen 1977) For any lattice polytope $\mathcal{P}$

$$\sum_{t \geq 0} \varphi(t\mathcal{P})\, z^t \ = \ \frac{h_0^\varphi + h_1^\varphi\, z + \cdots + h_d^\varphi(P)\, z^d}{(1-z)^{d+1}}$$

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Theorem (Jochemko–Sanyal 2016) A $\mathbb{Z}^d$-valuation φ satisfies $h^\varphi \geq 0$ for every lattice polytope if and only if $\varphi(\Delta^\circ) \geq 0$ for all lattice simplices Δ.

Theorem (MB–Jochemko–McCullough) $h^\varphi(z)$ is real rooted for any lattice zonotope and any combinatorially positive valuation φ.

Type B

Conjecture (Schepers–Van Langenhoven 2013) An IDP polytope with interior lattice points has an alternatingly increasing h^*-polynomial.

Theorem (MB–Jochemko–McCullough) The Schepers–Van Langenhoven Conjecture holds for type-B zonotopes $\left\{\sum_{j=1}^{n} \lambda_j \mathbf{v}_j : -1 \le \lambda_j \le 1\right\}$

Main tool Type-B Eulerian polynomials stemming from signed permutations

$$\sum_{t \ge 0} (2t+1)^d \, z^t \;=\; \frac{B(d,z)}{(1-z)^{d+1}}$$

Theorem (Brenti 1994) $B(d,z)$ is real rooted.

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Main tool We define the (B, l)-Eulerian polynomials

$$B_l(d, z) := \sum_{k=0}^{d} \left| \{ (\sigma, \epsilon) \in B_d : \epsilon_d \sigma(d) = d + 1 - l \text{ and } \operatorname{des}(\sigma, \epsilon) = k \} \right| z^k,$$

prove that they are real rooted and alternatingly increasing, and realize them as h^*-polynomials of half-open ± 1-cubes.