Gallai-Ramsey Number for K4

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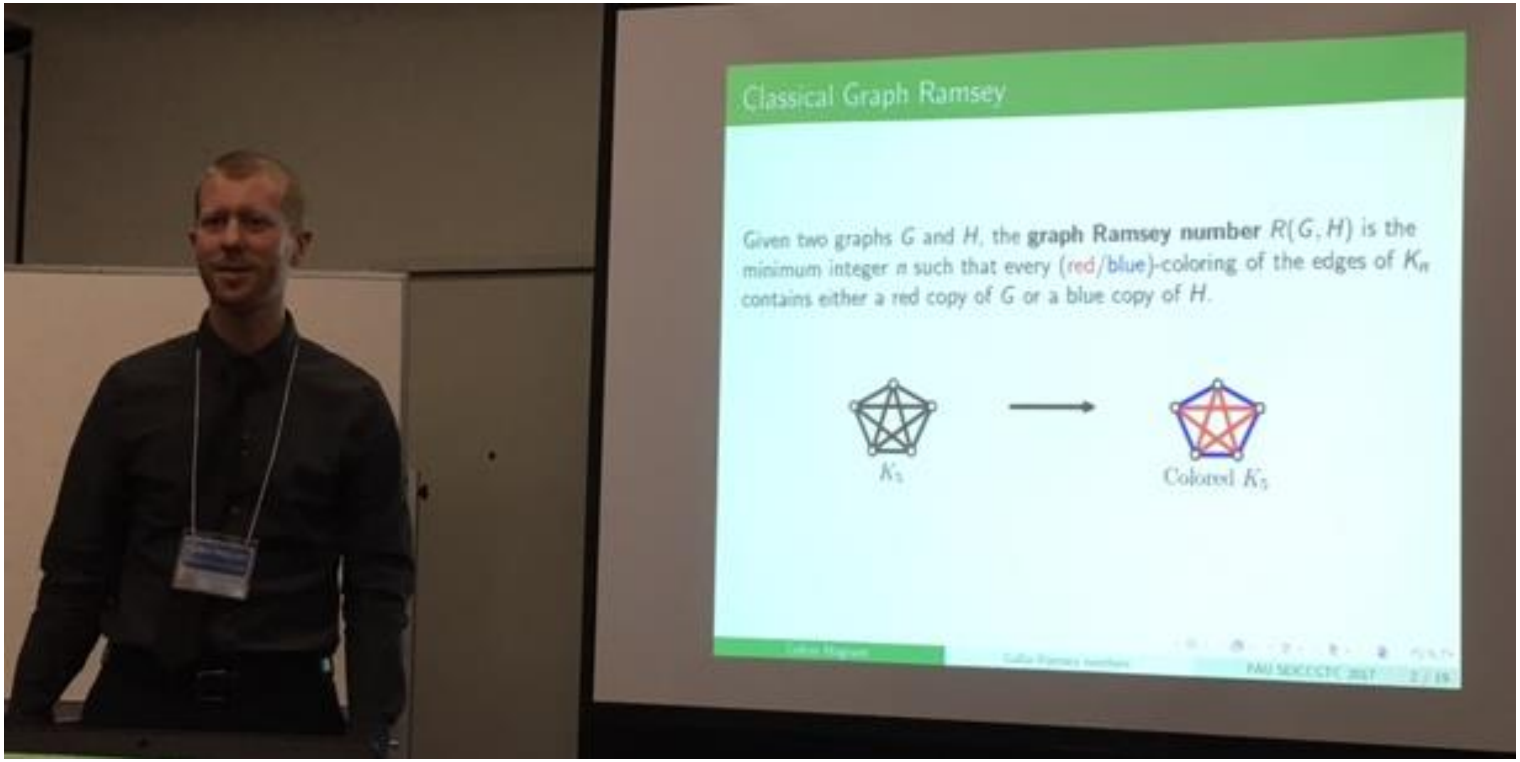
Gallai-Ramsey Number

Given two graphs G and H, the Ramsey number R(G,H) is the minimum integer n such that every red/blue-colouring of the edges of the complete graph on n vertices contains either a red copy of G or a blue copy of H.

Gallai-Ramsey Number



Gallai-Ramsey Number

$$R(K_3, K_3) = 6$$

$$R(K_4, K_4) = 18$$

$$43 \leq R(K_5, K_5) \leq 48$$

$$102 \leq R(K_6, K_6) \leq 165$$

Gallai-Ramsey Number

Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Paul Erdos

Gallai-Ramsey Number

$$R(K_3, K_4) = 9$$

$$R(K_3, K_5) = 13$$

$$R(K_4, K_5) = 25$$

Gallai-Ramsey Number

Given a graph H, the k-coloured Gallai-Ramsey number is defined to be the minimum integer n such that every k-colouring (using all k colours) of the complete graph on n vertices contains either a rainbow triangle or a monchromatic copy of H.

notation: $gr_k(K_3 : H)$

Gallai-Ramsey Number

Theorem (Chung and Graham, 1983)

$$gr_k(K_3 : K_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

$$R_2(K_3) = R(K_3, K_3) = 6$$

$$R_3(K_3) = R(K_3, K_3, K_3) = 17 \text{ (Greenwood and Gleason, 1955)}$$

Gallai-Ramsey Number

Theorem (Faudree, Gould, Jacobson, Magnant, 2010)

$$gr_k(K_3 : C_4) = k + 4 \text{ for } k \geq 2.$$

$$gr_k(K_3 : P_n) \geq \left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \text{ for } n, k \geq 1.$$

$$gr_k(K_3 : P_4) = k + 3 \text{ for } k \geq 1.$$

$$gr_k(K_3 : P_5) = k + 4 \text{ for } k \geq 1.$$

$$gr_k(K_3 : P_6) = 2k + 4 \text{ for } k \geq 1.$$

Gallai-Ramsey Number

Theorem (Hall, Magnant, Ozeki, Tsugaki, 2014)

Given integers $n \geq 3$ and $k \geq 1$,

$$\left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \leq gr_k(K_3 : P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor k + 3\left\lceil \frac{n}{2} \right\rceil.$$

Given integers $n \geq 2$ and $k \geq 1$,

$$(n-1)k + n + 1 \leq gr_k(K_3 : C_{2n}) \leq (n-1)k + 3n,$$

$$n2^k + 1 \leq gr_k(K_3 : C_{2n+1}) \leq (2^{k+3} - 3)n \log n.$$

Gallai-Ramsey Number

Theorem (Fujita and Magnant, 2011)

$$gr_k(K_3 : C_6) = 2k + 4 \text{ for } k \geq 1.$$

$$gr_k(K_3 : C_5) = 2 \cdot 2^k + 1 \text{ for } k \geq 1.$$

Theorem (Gregory, Magnant and Magnant, 2016)

$$gr_k(K_3 : C_8) = 3k + 5 \text{ for } k \geq 1.$$

Theorem (Bruce and Song, Bosse and Song, 2017)

$$gr_k(K_3 : C_7) = 3 \cdot 2^k + 1 \text{ for } k \geq 1.$$

$$gr_k(K_3 : C_9) = 4 \cdot 2^k + 1 \text{ for } k \geq 1.$$

Gallai-Ramsey Number

Theorem (Bruce and Song, 2017)

$$gr_k(K_3 : C_7) = 3 \cdot 2^k + 1 \text{ for } k \geq 1.$$

Theorem (Bosse and Song, 2017)

$$gr_k(K_3 : C_9) = 4 \cdot 2^k + 1 \text{ for } k \geq 1.$$

$$gr_k(K_3 : C_{11}) = 5 \cdot 2^k + 1 \text{ for } k \geq 1.$$

Gallai-Ramsey Number

Theorem (Zhang, Lei, Shi, and Song, 2017)

$$gr_k(K_3 : C_{10}) = 4k + 5 \text{ for } k \geq 1.$$

$$gr_k(K_3 : P_{10}) = 4k + 5 \text{ for } k \geq 1.$$

$$gr_k(K_3 : P_{2n+1}) = (n-1) \cdot k + n + 2 \text{ for } n \in \{3,4\} \text{ and } k \geq 1.$$

Gallai-Ramsey Number

Conjecture (Fox, Grinshpun, and Pach, 2015)

$$gr_k(K_3 : K_p) = \begin{cases} (r(p)-1)^{k/2} + 1 & \text{if } k \text{ is even,} \\ (p-1)(r(p)-1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd,} \end{cases}$$

where $r(p) = R(K_p, K_p)$

Gallai-Ramsey Number

Theorem (Magnant, Saito, and I.S., 2017)

$$gr_k(K_3 : K_4) = \begin{cases} 17^{k/2} + 1, & \text{if } k \text{ is even} \\ 3 \cdot 17^{(k-1)/2} + 1, & \text{if } k \text{ is odd} \end{cases}$$

Gallai-Ramsey Number

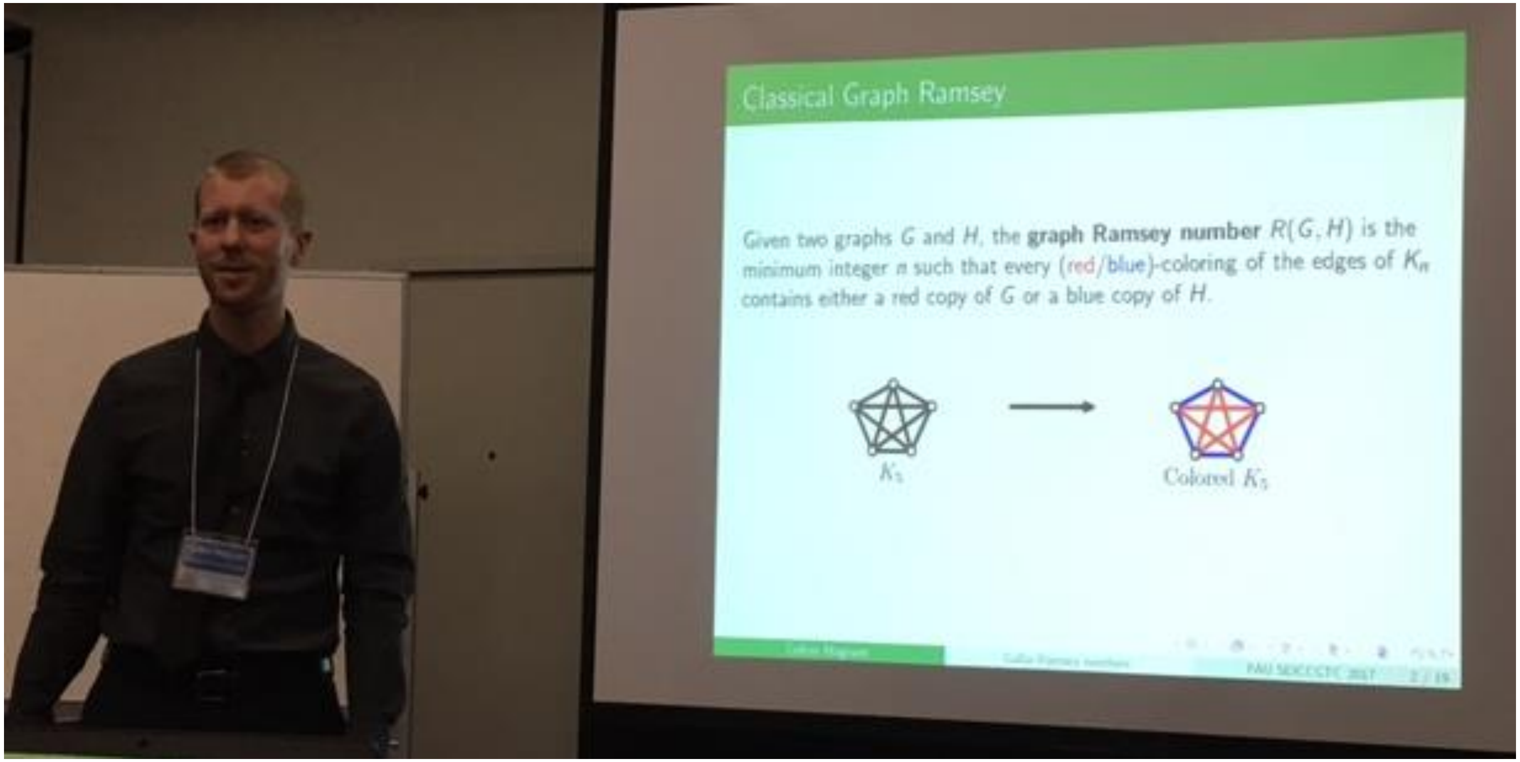
Theorem (Gallai, 1967)
In any edge-colouring of the complete graph with no rainbow triangle, there exists a partition of the vertices into at least two parts (called a Gallai partition or G-partition for short) such that, there are at most two colours on the edges between the parts, and only one colour on the edges between each pair of parts.

Gallai-Ramsey Number

Proof approach

- For $k \geq 3$ we apply induction on k
- Consider a Gallai partition with two colours red and blue. Let t be the number of the parts, and let t be minimal. Since $R(K_4, K_4) = 18$, we have $2 \leq t \leq 17$.

Gallai-Ramsey Number

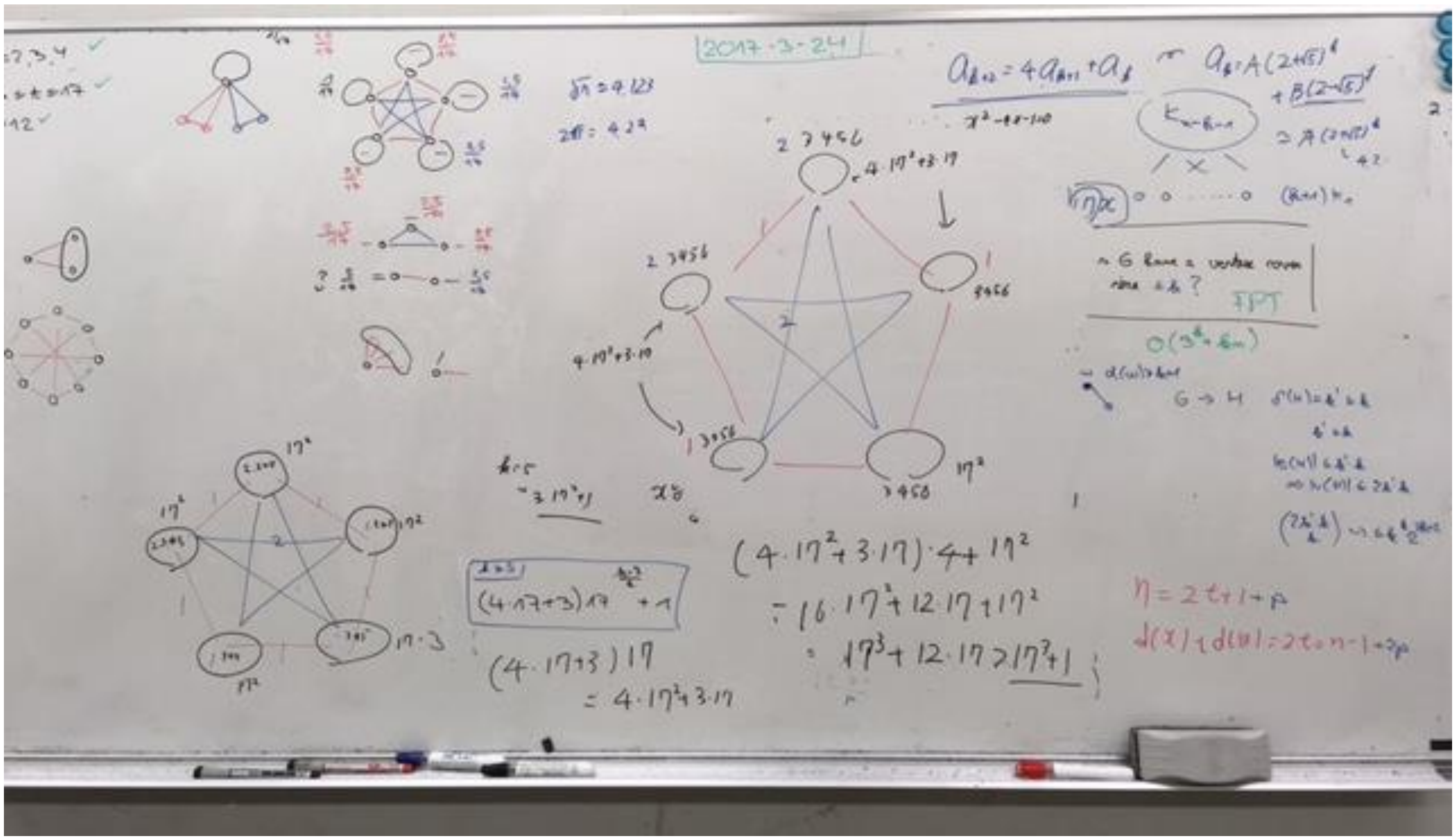


Gallai-Ramsey Number

Proof in progress – Nihon university, march 2017

Gallai-Ramsey Number

Proof in progress – Nihon university, march 24, 2017



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Theorem (Magnant, Saito, and I.S., 2017)

Let $k \geq 1$, and s be an integer with $0 \leq s \leq k$. Then

$gr_k(K_3 : sK_4, (k-s)K_3) = g(k, s)$, where

$$g(k,s) = \begin{cases} 17^{s/2} \cdot 5^{(k-s)/2} + 1 & \text{if } s \text{ and } (k-s) \text{ are both even,} \\ 2 \cdot 17^{s/2} \cdot 5^{(k-s-1)/2} + 1 & \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ 3 \cdot 17^{(k-1)/2} + 1 & \text{if } s = k \text{ and } s \text{ is odd,} \\ 8 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-1)/2} + 1 & \text{if } s \text{ and } (k-s) \text{ are both odd,} \\ 16 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-2)/2} + 1 & \text{if } s < k, \text{ and } s \text{ is odd, and } (k-s) \text{ is even.} \end{cases}$$

Gallai-Ramsey Number

The next case: $K5$

We believe that the conjecture can be proved, although the exact value of $R(K5,K5)$ is not (yet) known.

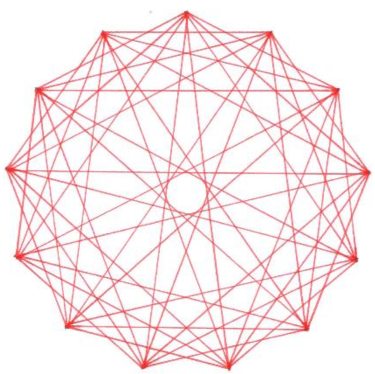
January 16, 2018 – a surprise

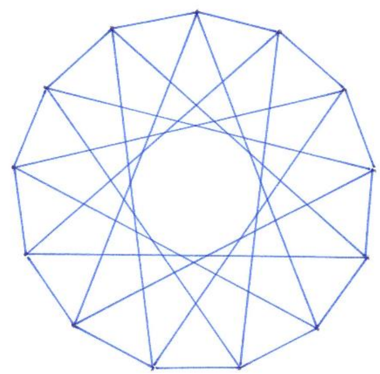
The complete graph K169 can be coloured with 3 colours such that it contains no monochromatic K5 and no rainbow triangle.

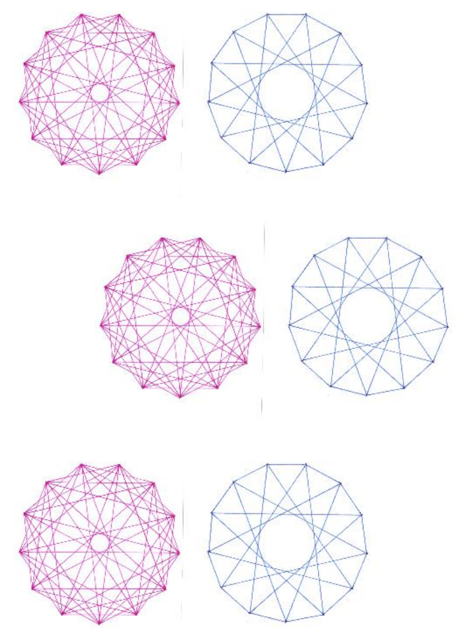
Gallai-Ramsey Number

The next case: K5

The complete graph K169 can be coloured with 3 colours such that it contains no monochromatic K5 and no rainbow triangle.







Gallai-Ramsey Number

The next case: K5

Consequences:

If R(5,5)=43, then the conjectured value for p=5 and k is odd is false.
If 44 <= R(5,5) <= 48, then the conjecture can be true for p=5.

Gallai-Ramsey Number

The next case: K5

Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Paul Erdos

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Cảm ơn bạn đã quan tâm

Thank you very much!

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The end