

Gallai-Ramsey Number for K4

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(joint work with Colton Magnant and Akira Saito)

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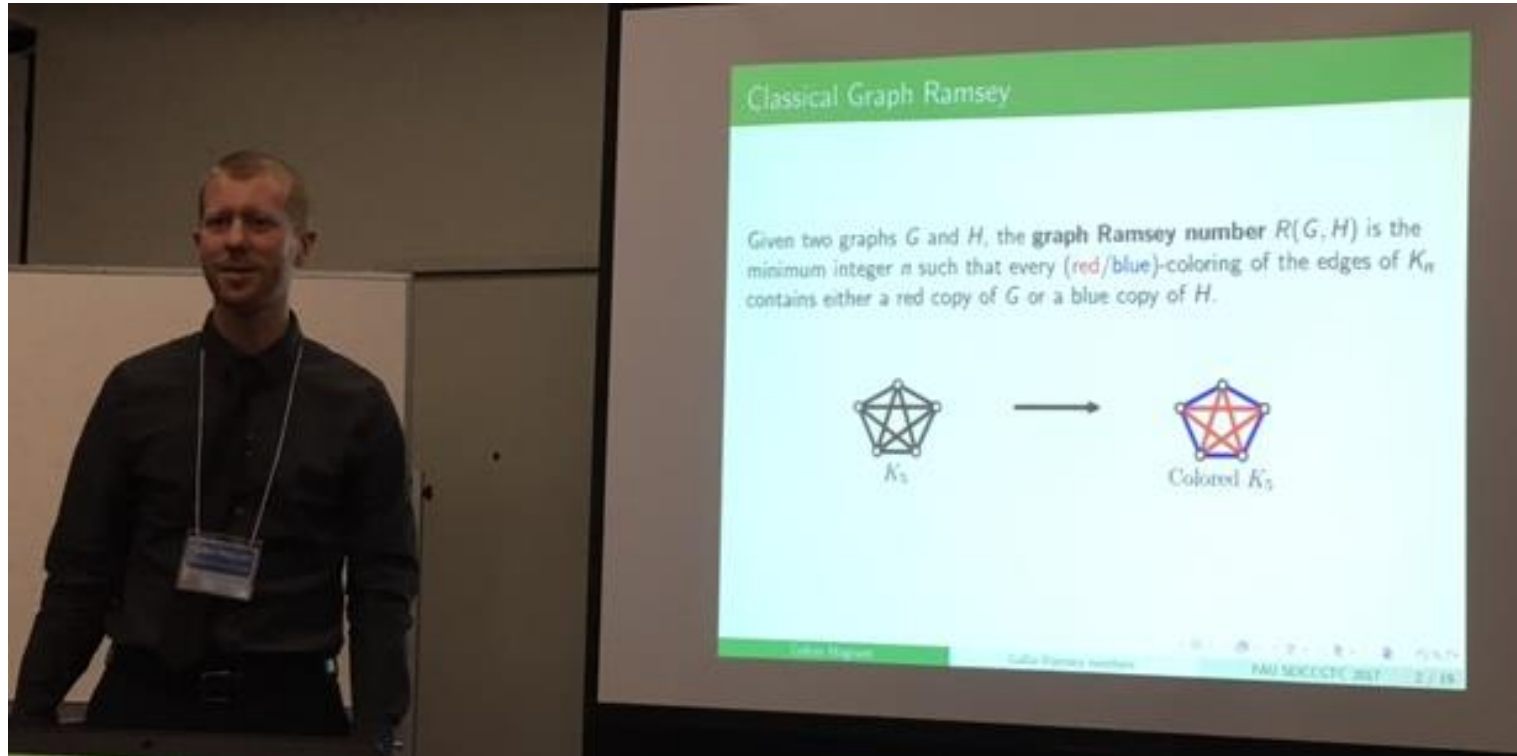
Chromatic Number

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Gallai-Ramsey Number

Given two graphs G and H , the **Ramsey number** $R(G,H)$ is the minimum integer n such that every **red/blue**-colouring of the edges of the complete graph on n vertices contains either a red copy of G or a blue copy of H .

Gallai-Ramsey Number



The slide on the screen is titled "Classical Graph Ramsey" and contains the following text: "Given two graphs G and H , the graph Ramsey number $R(G, H)$ is the minimum integer n such that every (red/blue)-coloring of the edges of K_n contains either a red copy of G or a blue copy of H ."

Below the text, there are two diagrams. The first diagram shows a complete graph K_5 with five vertices and all possible edges between them. An arrow points to the second diagram, which shows a "Colored K_5 " where the edges are colored in red and blue. In this coloring, a red triangle is visible within the graph.

Gallai-Ramsey Number

$$R(K_3, K_3) = 6$$

$$R(K_4, K_4) = 18$$

$$43 \leq R(K_5, K_5) \leq 48$$

$$102 \leq R(K_6, K_6) \leq 165$$

Gallai-Ramsey Number

Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Paul Erdos



Gallai-Ramsey Number

$$R(K_3, K_4) = 9$$

$$R(K_3, K_5) = 13$$

$$R(K_4, K_5) = 25$$

Gallai-Ramsey Number

Given a graph H , the k -coloured Gallai-Ramsey number is defined to be the minimum integer n such that every k -colouring (using all k colours) of the complete graph on n vertices contains either a rainbow triangle or a monochromatic copy of H .

notation: $gr_k(K_3 : H)$

Gallai-Ramsey Number

Theorem (Chung and Graham, 1983)

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{K}_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

$$R_2(\mathbf{K}_3) = R(\mathbf{K}_3, \mathbf{K}_3) = 6$$

$$R_3(\mathbf{K}_3) = R(\mathbf{K}_3, \mathbf{K}_3, \mathbf{K}_3) = 17 \text{ (Greenwood and Gleason, 1955)}$$

Gallai-Ramsey Number

Theorem (Faudree, Gould, Jacobson, Magnant, 2010)

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{C}_4) = k + 4 \text{ for } k \geq 2.$$

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{P}_n) \geq \left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \text{ for } n, k \geq 1.$$

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{P}_4) = k + 3 \text{ for } k \geq 1.$$

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{P}_5) = k + 4 \text{ for } k \geq 1.$$

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{P}_6) = 2k + 4 \text{ for } k \geq 1.$$

Gallai-Ramsey Number

Theorem (Hall, Magnant, Ozeki, Tsugaki, 2014)

Given integers $n \geq 3$ and $k \geq 1$,

$$\left\lfloor \frac{n-2}{2} \right\rfloor^k + \left\lceil \frac{n}{2} \right\rceil + 1 \leq \text{gr}_k(\mathbf{K}_3 : \mathbf{P}_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor^k + 3 \left\lceil \frac{n}{2} \right\rceil.$$

Given integers $n \geq 2$ and $k \geq 1$,

$$(n-1)k + n + 1 \leq \text{gr}_k(\mathbf{K}_3 : \mathbf{C}_{2n}) \leq (n-1)k + 3n,$$

$$n2^k + 1 \leq \text{gr}_k(\mathbf{K}_3 : \mathbf{C}_{2n+1}) \leq (2^{k+3} - 3)n \log n.$$

Gallai-Ramsey Number

Theorem (Fujita and Magnant, 2011)

$$\text{gr}_k(\mathbf{K}_3 : C_6) = 2k + 4 \text{ for } k \geq 1.$$

$$\text{gr}_k(\mathbf{K}_3 : C_5) = 2 \cdot 2^k + 1 \text{ for } k \geq 1.$$

Theorem (Gregory, Magnant and Magnant, 2016)

$$\text{gr}_k(\mathbf{K}_3 : C_8) = 3k + 5 \text{ for } k \geq 1.$$

Theorem (Bruce and Song, Bosse and Song, 2017)

$$\text{gr}_k(\mathbf{K}_3 : C_7) = 3 \cdot 2^k + 1 \text{ for } k \geq 1.$$

$$\text{gr}_k(\mathbf{K}_3 : C_9) = 4 \cdot 2^k + 1 \text{ for } k \geq 1.$$

Gallai-Ramsey Number

Theorem (Bruce and Song, 2017)

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{C}_7) = 3 \cdot 2^k + 1 \text{ for } k \geq 1.$$

Theorem (Bosse and Song, 2017)

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{C}_9) = 4 \cdot 2^k + 1 \text{ for } k \geq 1.$$

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{C}_{11}) = 5 \cdot 2^k + 1 \text{ for } k \geq 1.$$

Gallai-Ramsey Number

Theorem (Zhang, Lei, Shi, and Song, 2017)

$$\text{gr}_k(K_3 : C_{10}) = 4k + 5 \text{ for } k \geq 1.$$

$$\text{gr}_k(K_3 : P_{10}) = 4k + 5 \text{ for } k \geq 1.$$

$$\text{gr}_k(K_3 : P_{2n+1}) = (n - 1) \cdot k + n + 2 \text{ for } n \in \{3, 4\} \text{ and } k \geq 1.$$

Gallai-Ramsey Number

Conjecture (Fox, Grinshpun, and Pach, 2015)

$$\text{gr}_k(\mathbf{K}_3 : \mathbf{K}_p) = \begin{cases} (r(p)-1)^{k/2} + 1 & \text{if } k \text{ is even,} \\ (p-1)(r(p)-1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd,} \end{cases}$$

where $r(p) = R(\mathbf{K}_p, \mathbf{K}_p)$

Gallai-Ramsey Number

Theorem (Magnant, Saito, and I.S., 2017)

$$\text{gr}_k(K_3 : K_4) = \begin{cases} 17^{k/2} + 1, & \text{if } k \text{ is even} \\ 3 \cdot 17^{(k-1)/2} + 1, & \text{if } k \text{ is odd} \end{cases}$$

Gallai-Ramsey Number

Theorem (Gallai, 1967)

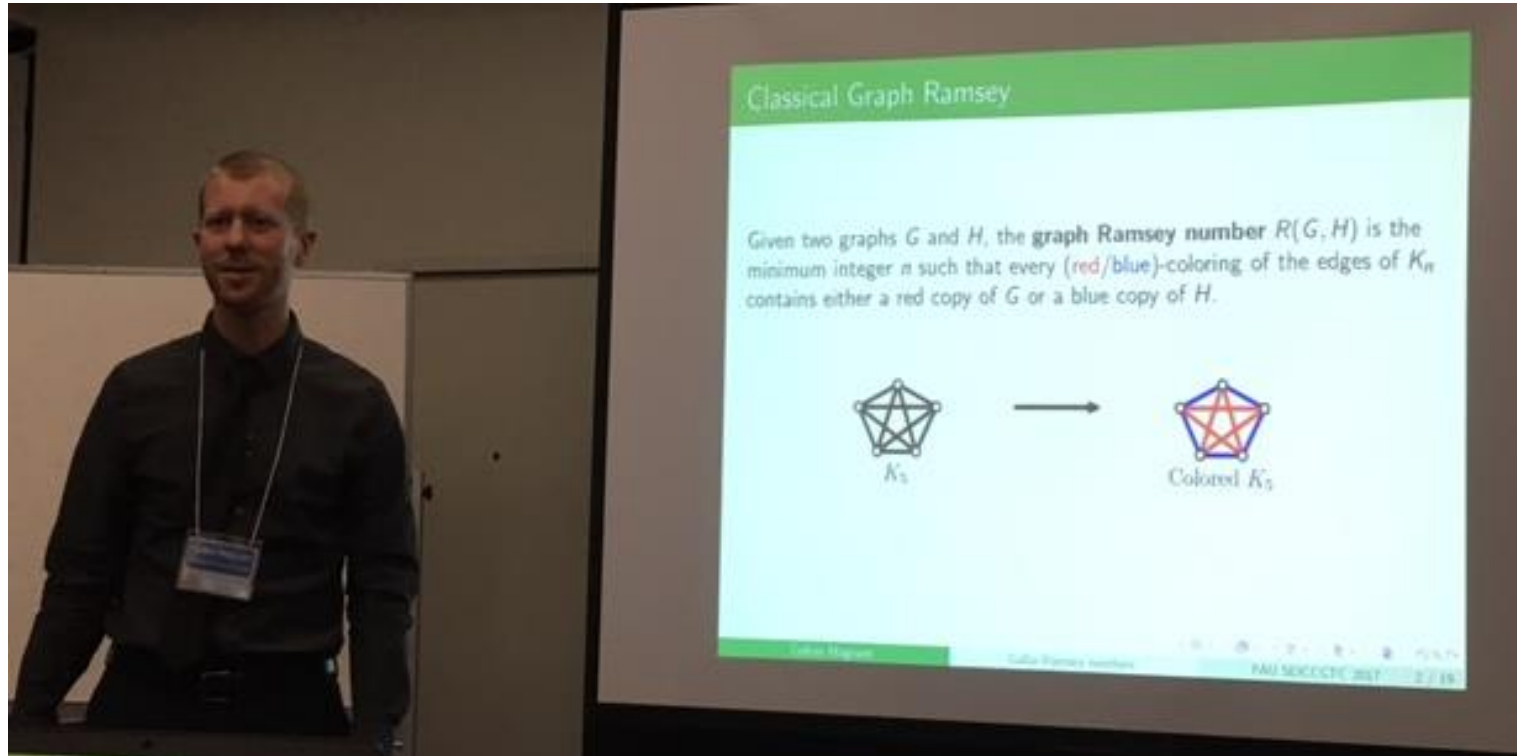
In any edge-colouring of the complete graph with no rainbow triangle, there exists a partition of the vertices into at least two parts (called a **Gallai partition** or G-partition for short) such that, there are at most two colours on the edges between the parts, and only one colour on the edges between each pair of parts.

Gallai-Ramsey Number

Proof approach



- For $k \geq 3$ we apply induction on k
- Consider a Gallai partition with two colours red and blue.
Let t be the number of the parts, and let t be minimal. Since $R(K_4, K_4) = 18$, we have $2 \leq t \leq 17$.

Gallai-Ramsey Number



Classical Graph Ramsey

Given two graphs G and H , the graph Ramsey number $R(G, H)$ is the minimum integer n such that every (red/blue)-coloring of the edges of K_n contains either a red copy of G or a blue copy of H .

 K_5 \longrightarrow  Colored K_5

The slide shows a complete graph K_5 on the left, which is a pentagon with all its diagonals. An arrow points to the right, where the same K_5 graph is shown with its edges colored in red and blue. The bottom of the slide has a navigation bar with icons and the text 'Gallai-Ramsey numbers' and 'FALLSEMESTER 2017 2/16'.

Gallai-Ramsey Number

Proof in progress – Nihon university, march 2017



Gallai-Ramsey Number

Proof in progress – Nihon university, march 24, 2017

The whiteboard contains several diagrams and equations related to the Gallai-Ramsey number. At the top center, the date "2017-3-24" is written. On the left, there are several small graphs, including a triangle and a star graph. In the center, a large graph with 6 vertices is shown, with edges colored in red and blue. The vertices are labeled with numbers like $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$, $4 \cdot 17^2 + 3 \cdot 17$, 17^2 , and $3 \cdot 17^2$. To the right, there are algebraic derivations and a boxed section.

Equations and derivations on the whiteboard include:

- $A_{k+1} = 4A_k + A_1$ and $A_k = A(2k+5)^k + B(2-k)^k$
- $\sqrt{5} = 2.23$ and $2\sqrt{5} = 4.47$
- $\eta = 2t + 1 + p$
- $d(x) + d(y) = 2t + n - 1 + 2p$
- $G \rightarrow H$ with $d(x) = k' \cdot k$ and $k' = k$
- $(2k' \cdot k) \rightarrow k \cdot k' \cdot k$
- $(4 \cdot 17^2 + 3 \cdot 17) \cdot 4 + 17^2 = 16 \cdot 17^2 + 12 \cdot 17 + 17^2 = 17^3 + 12 \cdot 17 + 17^2 + 1$
- $(4 \cdot 17 + 3) \cdot 17 = 4 \cdot 17^2 + 3 \cdot 17$
- $(4 \cdot 17 + 3) \cdot 17^2 + 1$
- $(4 \cdot 17 + 3) \cdot 17^2 + 1$
- $(4 \cdot 17 + 3) \cdot 17^2 + 1$

Gallai-Ramsey Number

Theorem (Magnant, Saito, and I.S., 2017)

Let $k \geq 1$, and s be an integer with $0 \leq s \leq k$. Then

$\text{gr}_k(\mathbf{K}_3 : s\mathbf{K}_4, (k-s)\mathbf{K}_3) = g(k, s)$, where

$$g(k, s) = \begin{cases} 17^{s/2} \cdot 5^{(k-s)/2} + 1 & \text{if } s \text{ and } (k-s) \text{ are both even,} \\ 2 \cdot 17^{s/2} \cdot 5^{(k-s-1)/2} + 1 & \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ 3 \cdot 17^{(k-1)/2} + 1 & \text{if } s = k \text{ and } s \text{ is odd,} \\ 8 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-1)/2} + 1 & \text{if } s \text{ and } (k-s) \text{ are both odd,} \\ 16 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-2)/2} + 1 & \text{if } s < k, \text{ and } s \text{ is odd, and } (k-s) \text{ is even.} \end{cases}$$

Gallai-Ramsey Number

The next case: K_5

We believe that the conjecture can be proved, although the exact value of $R(K_5, K_5)$ is not (yet) known.

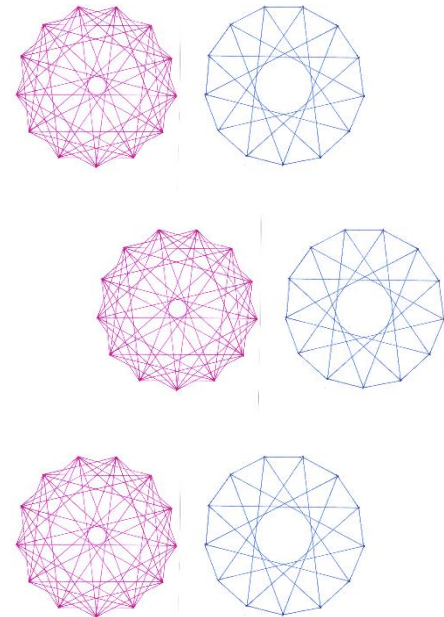
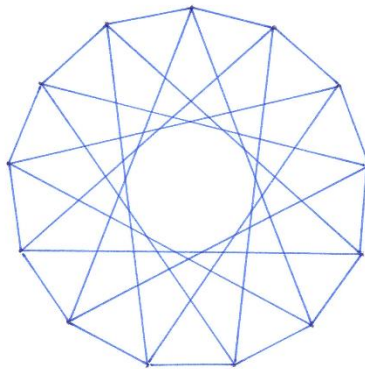
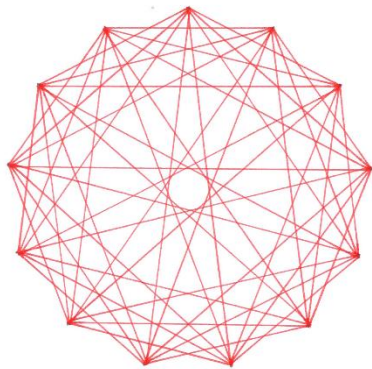
January 16, 2018 – a surprise

The complete graph K_{169} can be coloured with 3 colours such that it contains no monochromatic K_5 and no rainbow triangle.

Gallai-Ramsey Number

The next case: K_5

The complete graph K_{169} can be coloured with 3 colours such that it contains no monochromatic K_5 and no rainbow triangle.



Gallai-Ramsey Number

The next case: K_5

Consequences:

If $R(5,5)=43$, then the conjectured value for $p=5$ and k is odd is false.

If $44 \leq R(5,5) \leq 48$, then the conjecture can be true for $p=5$.

Gallai-Ramsey Number

The next case: K_5

Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Paul Erdos



Gallai-Ramsey Number



Cảm ơn bạn đã quan tâm

Thank you very much!

Gallai-Ramsey Number

The end