Maximizing the order of regular bipartite graphs for given valency and second eigenvalue

Hiroshi Nozaki

Aichi University of Education
Joint work with S.M. Cioabă and J.H. Koolen

JCCA 2018
Sendai International Center
May 23, 2018

Known results

$v(k,\lambda)$: the maximum possible order of a connected k-regular graph G with $\lambda_2 \le \lambda$ (λ_2: second eigenvalue)

Theorem 1 (Cioabă, Koolen, N. and Vermette (2016))

Let λ be the second-largest eigenvalue of matrix $T(t,k,c)$. Then we have

$$v(k,\lambda) \le 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}.$$

Equality holds $\Leftrightarrow$ Distance-regular graph with $g \ge 2d$.
(g: girth, $d+1$: # of eigen.)

For $d \ge 7$, there does not exist a distance-regular graph with $g \ge 2d$ (Damerell–Georgiadcodis (1981), Bannai–Ito (1981))

Second-largest eigenvalue of regular graph

- $G = (V, E)$: a simple k-regular graph.
- $\boldsymbol{A}$: the adjacency matrix of G.

$$\boldsymbol{A}(u, v) = \begin{cases} 1 \text{ if } \{u, v\} \in E, \\ 0 \text{ otherwise.} \end{cases}$$

- $\lambda_1 = k > \lambda_2 > \cdots > \lambda_r$: the distinct eigenvalues of $\boldsymbol{A}$.

Theorem 2 (Alon–Boppana, Serre)

For given k and λ with $\lambda < 2\sqrt{k-1}$, there exist finitely many k-regular graphs with $\lambda_2 \leq \lambda$.

Spectral gap

- *Spectral gap* $\tau(G) = k - \lambda_2$.
- For $\emptyset \neq S \subset V$,

$$\partial S = \{\{u, v\} \in E \mid u \in S, v \in V \setminus S\}.$$

- *Edge expansion ratio*:

$$h(G) = \min_{S \subset V, 1 \leq |S| \leq |V|/2} \frac{|\partial S|}{|S|}.$$

Theorem 3 (Cheeger inequalities, Alon and Milman (1985))

$$\tau(G)/2 \leq h(G) \leq \sqrt{2k\tau(G)}.$$

Small λ_2 (k: fixed) $\longrightarrow$ Large $\tau(G)$, $h(G)$ $\longrightarrow$ High connectivity

Problem

- $v(k, \lambda)$: the maximum possible order of a connected bipartite k-regular graph G with $\lambda_2 \leq \lambda$.

Problem 4

Determine $v(k, \lambda)$, and classify the graphs meeting $v(k, \lambda)$.

Polynomials for regular bipartite graphs

$$\mathscr{F}_0(x) = 1, \qquad \mathscr{F}_1(x) = x - k, \qquad \mathscr{F}_2(x) = x^2 - (3k-2)x + k(k-1)$$

$$\mathscr{F}_i(x) = (x - 2k + 2)\mathscr{F}_{i-1}(x) - (k-1)^2 \mathscr{F}_{i-2}(x) (i \geq 3)$$

Let $\boldsymbol{B}$ be the biadjacency matrix of a k-regular bipartite graph.

$$\boldsymbol{A} = \begin{pmatrix} O & \boldsymbol{B} \\ \boldsymbol{B}^\top & O \end{pmatrix}, \boldsymbol{A}^2 = \begin{pmatrix} \boldsymbol{B}\boldsymbol{B}^\top & O \\ O & \boldsymbol{B}^\top \boldsymbol{B} \end{pmatrix}, \boldsymbol{A}^{2i} = \begin{pmatrix} (\boldsymbol{B}\boldsymbol{B}^\top)^i & O \\ O & (\boldsymbol{B}^\top \boldsymbol{B})^i \end{pmatrix}$$

Each entry of $\mathscr{F}_i(\boldsymbol{B}\boldsymbol{B}^\top)$ is non-negative.

Linear programming bound for regular bipartite graphs

Theorem 5 (Cioabă, Koolen, and N.)

Let $G = (V, E)$ be a connected k-regular bipartite graph. Suppose there exists a polynomial $f(x) = \sum_{i=0}^{s} c_i \mathscr{F}_i(x)$ s.t.

- $f(k^2) > 0$, $f(\lambda^2) \leq 0$ *for each eigenvalue* $\lambda \neq k, -k$ *of* G,
- $c_0 > 0$, *and* $c_i \geq 0$ *for each* $i = 1, \ldots, s$.

Then

$$|V| \leq \frac{2f(k^2)}{c_0}.$$

New bounds for regular bipartite graphs

$$T = T(k,t,c) = \begin{pmatrix} 0 & k & & & & \\ 1 & 0 & k-1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 0 & k-1 & \\ & & & c & 0 & k-c \\ & & & & k & 0 \end{pmatrix}$$

: $t \times t$ tridiagonal matrix for $1 \leq c \leq k$.

Theorem 6 (Cioabă, Koolen, and N.)

Let λ be the second-largest eigenvalue of T. Then we have

$$v(k,\lambda) \leq 2\left(\sum_{i=0}^{t-4}(k-1)^i + \frac{(k-1)^{t-3}}{c} + \frac{(k-1)^{t-2}}{c}\right).$$

This equality holds if and only if the graph is a bipartite distance-regular graph with the intersection matrix $T(k,t,c)$.

Examples attaining the bound

Equality holds $\Leftrightarrow$ $g \geq 2d-2$
where g: girth, $d+1$: $\#$ of distinct eigenvalues.

k	λ	$v(k,\lambda)$	Name
2	$2\cos(2\pi/n)$	n (even)	n-cycle C_n
k	0	$2k$	Complete bipartite graph $K_{k,k}$
k	$\sqrt{k-\lambda}$	$2(1+\frac{k(k-1)}{\lambda})$	Symmetric (v,k,λ)-design
r^2-r+1	r	$2(r^2+1)\times$ (r^2-r+1)	$pg(r^2-r+1, r^2-r+1, (r-1)^2)$
q	$\sqrt{q}$	$2q^2$	$AG(2,q)$ minus a parallel class
$q+1$	$\sqrt{2q}$	$2\sum_{i=0}^{3} q^i$	$GQ(q,q)$
$q+1$	$\sqrt{3q}$	$2\sum_{i=0}^{5} q^i$	$GH(q,q)$
6	2	162	$pg(6,6,2)$

(q: prime power, r: power of 2)

Non-existence of bip. DRG with $g \geq 2d-2$ for large d

Theorem 7 (Cioabă, Koolen, and N.)

Suppose $k \geq 3$. There does not exist a bipartite distance-regular graph Γ with the intersection matrix $T(k, d+1, c)$ for $d \geq 15$ and $d = 11$.

We use a similar manner given by Fuglister (1987) and Bannai–Ito (1981).

- m_θ is the multiplicity of an eigenvalue θ

$$m_\theta = \frac{|V|k(k-1)\big(\phi-4\big)\big((c-1)(k-1)\phi+(k-c)^2\big)}{2\big((k-1)\phi-k^2\big)[(d-1)(c-1)(k-1)\phi+d(k-c)^2+2(c-1)(k-c)]},$$

where $(k-1)\phi = \theta^2$.

- Factorization of the characteristic polynomial mod prime p.

Thank you.

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