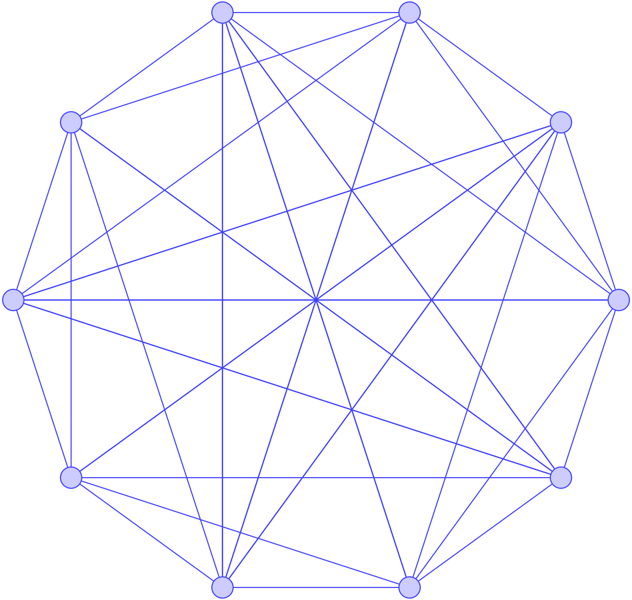
Edge-regular graphs and regular cliques

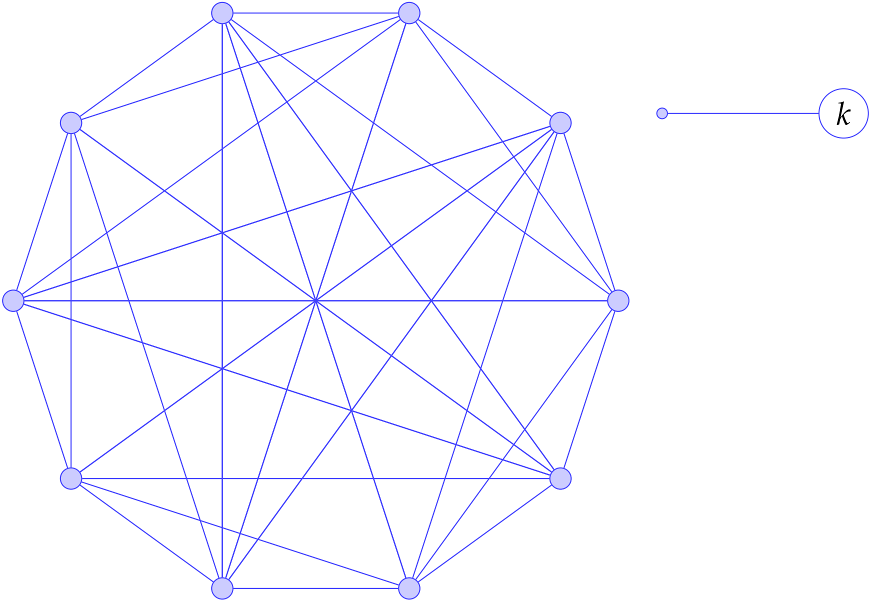
Gary Greaves

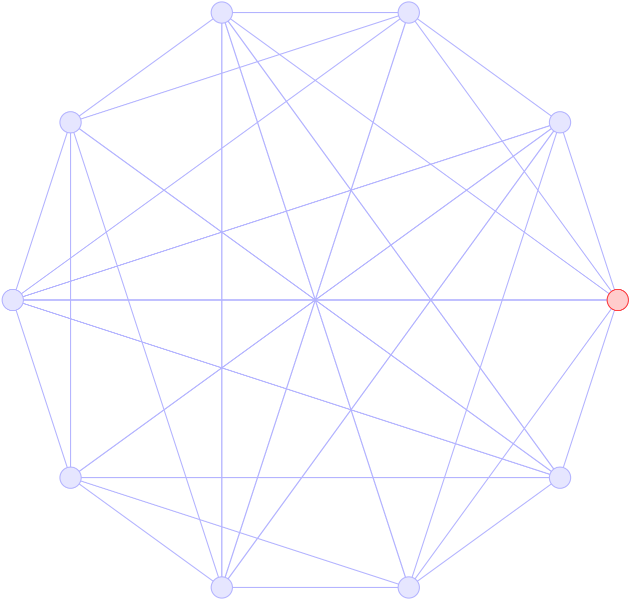
Nanyang Technological University, Singapore

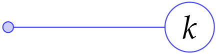
23rd May 2018

joint work with J. H. Koolen

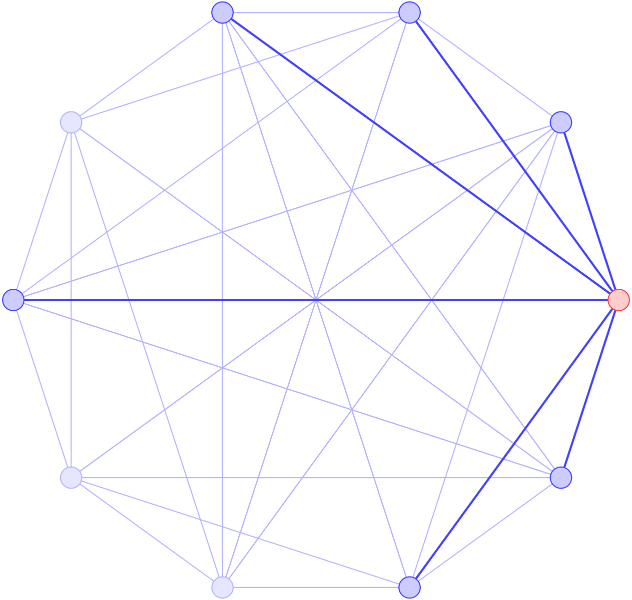


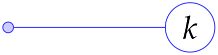

k

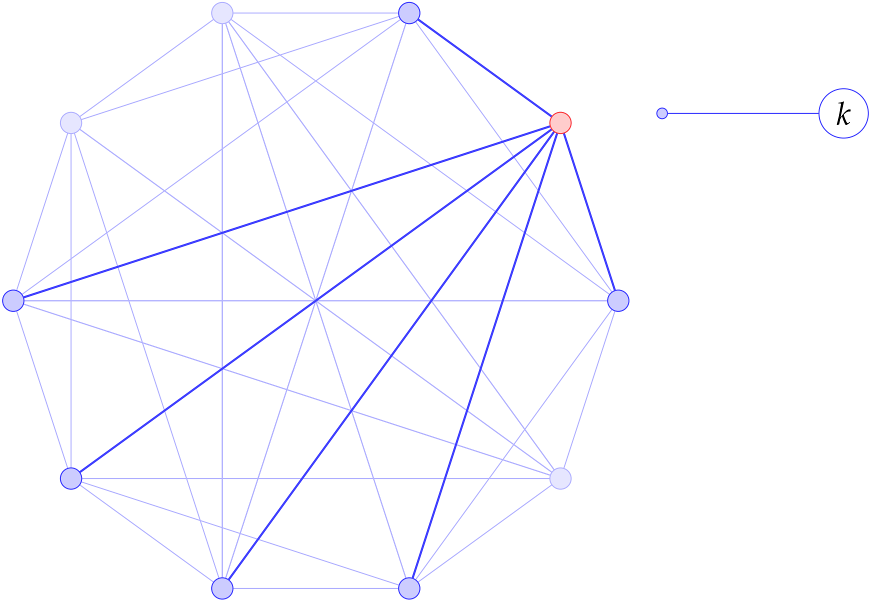


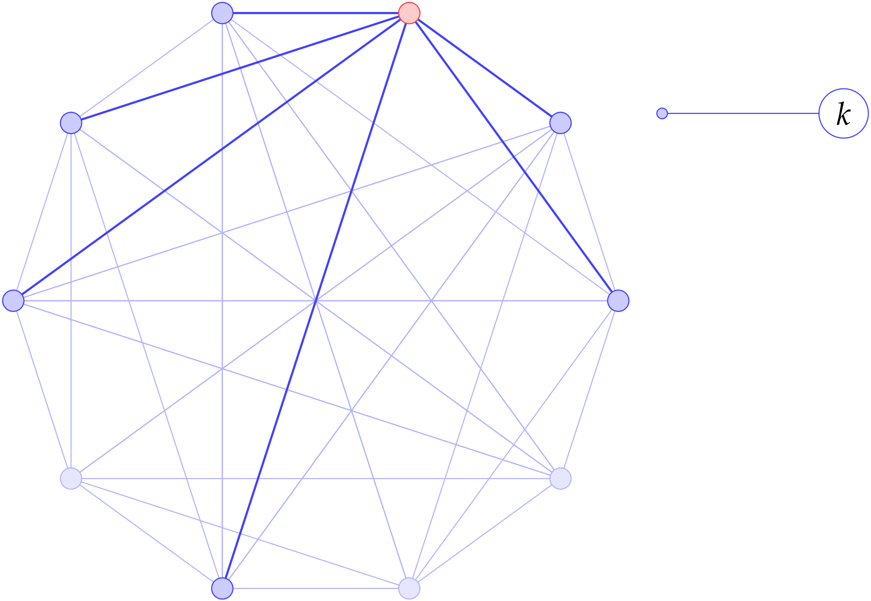


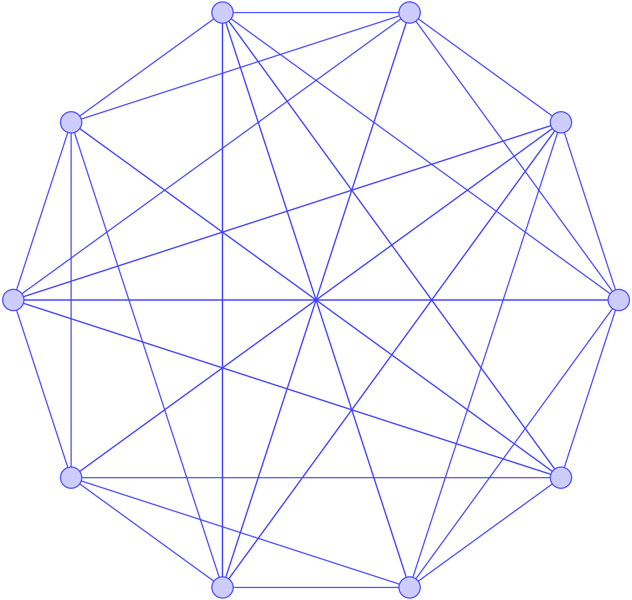
k

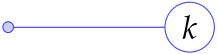



k

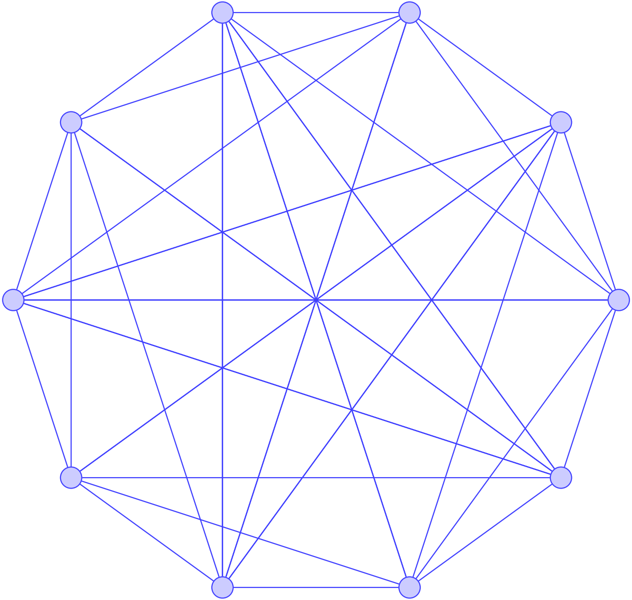

k

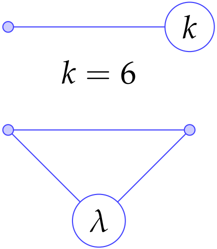

k

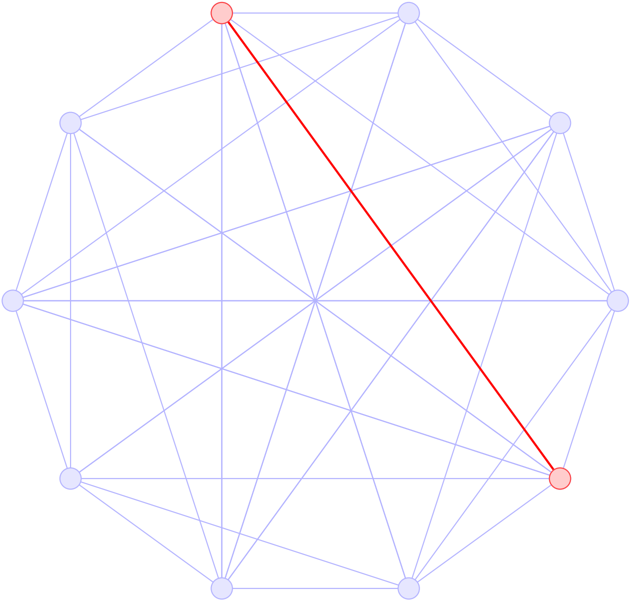


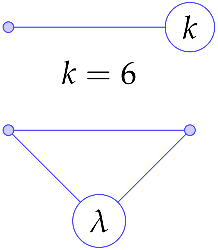


$k = 6$

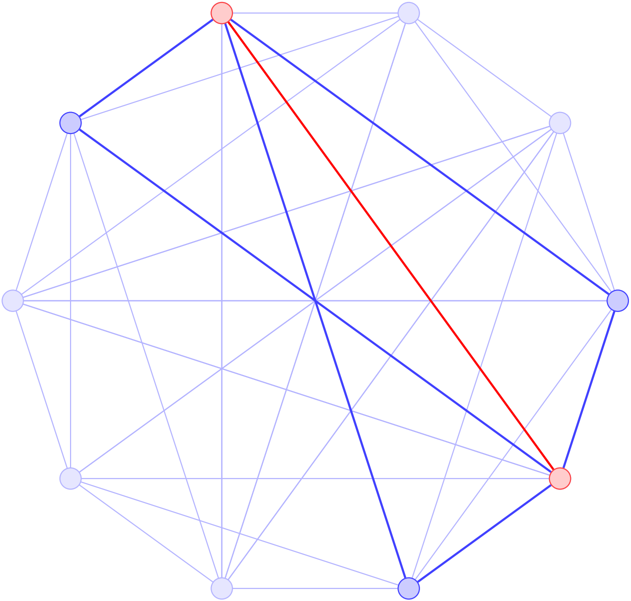


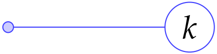

k
$k = 6$
λ



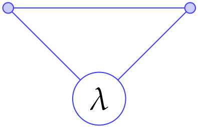


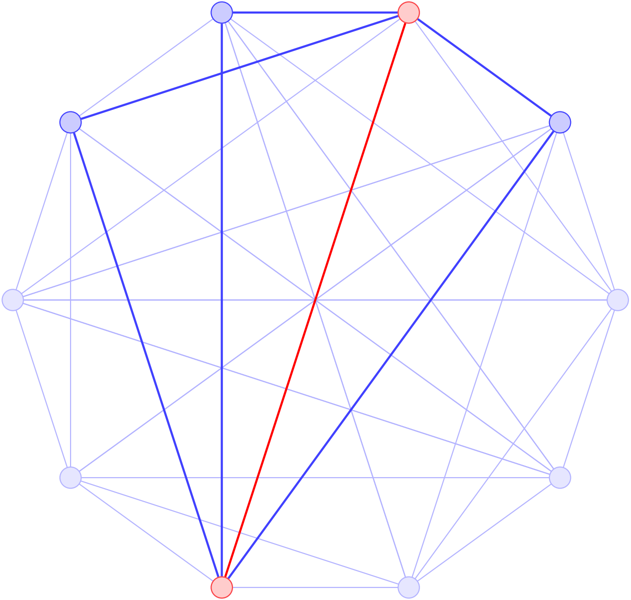
k
$k = 6$
λ

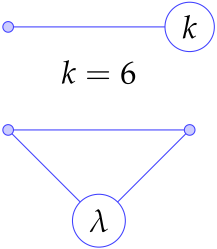


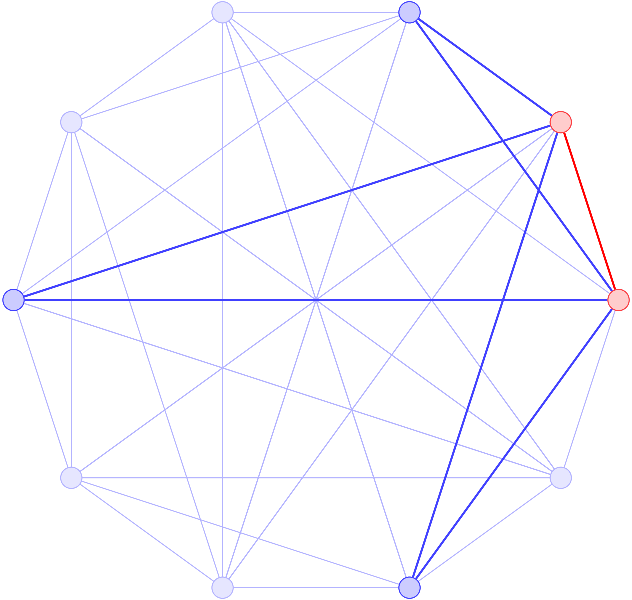


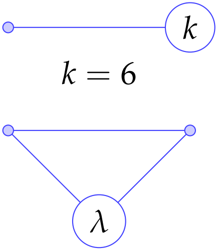
$k = 6$

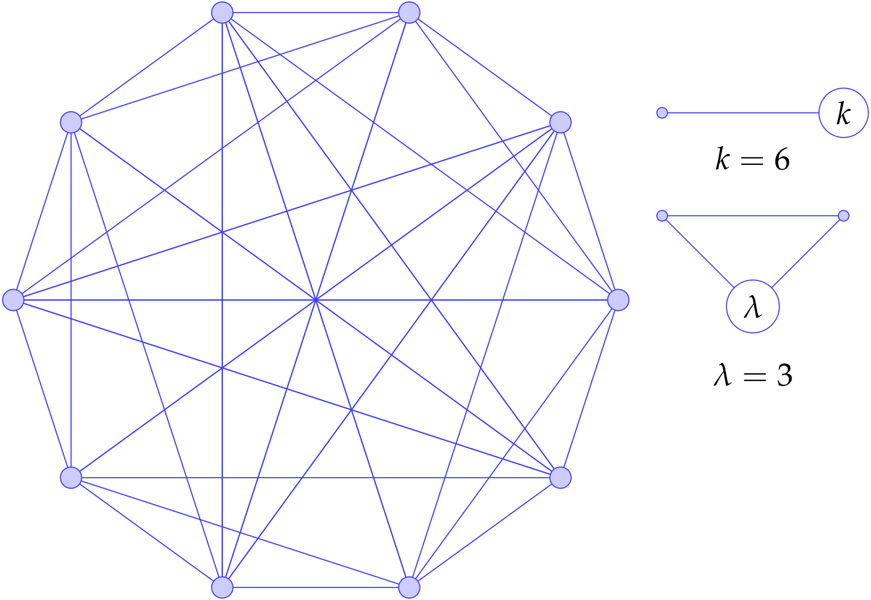


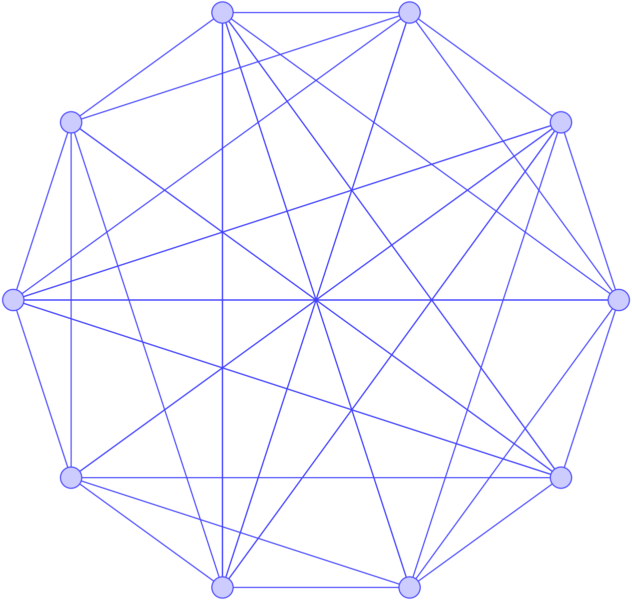


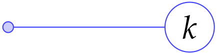

k
$k = 6$
λ



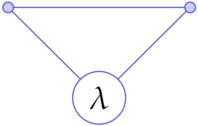

k
$k = 6$
λ


k
$k = 6$
λ
$\lambda = 3$



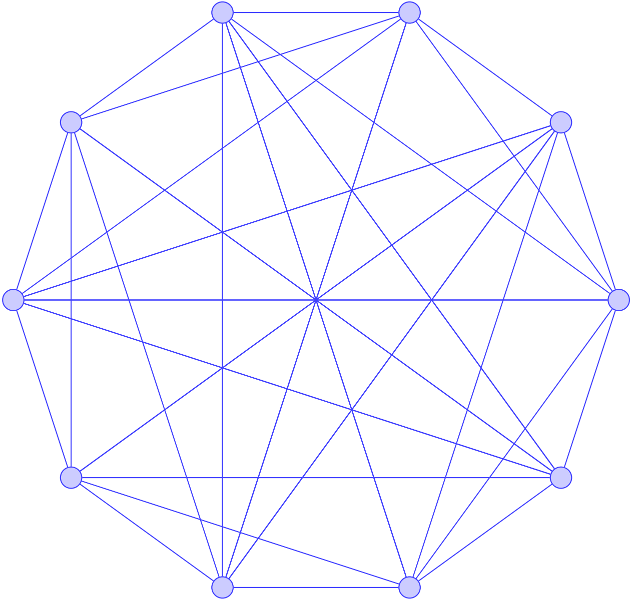


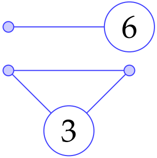
$k = 6$



$\lambda = 3$

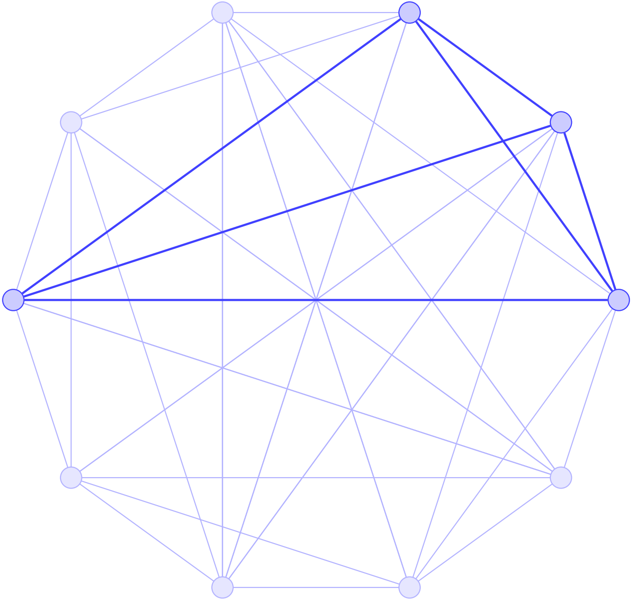
edge-regular
$\mathrm{erg}(10, 6, 3)$

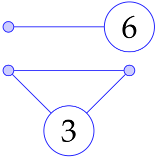




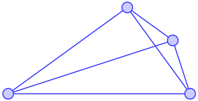
edge-regular

$\mathrm{erg}(10, 6, 3)$

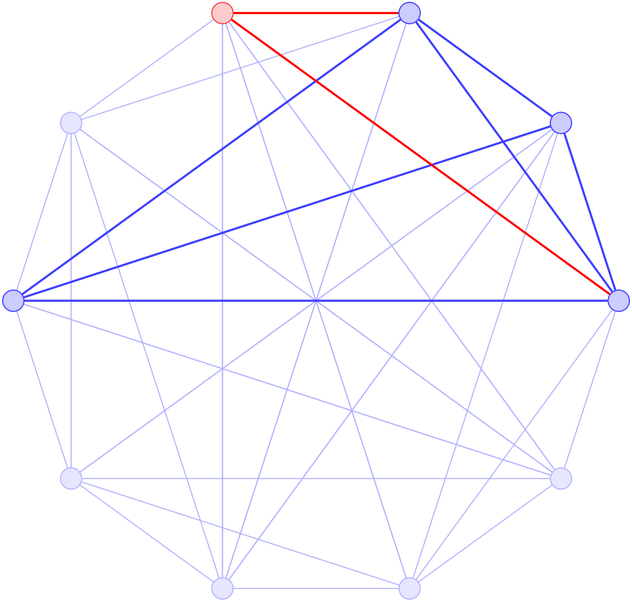


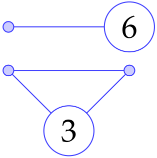


edge-regular
$\mathrm{erg}(10, 6, 3)$



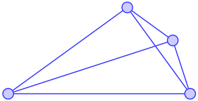
clique
of order 4





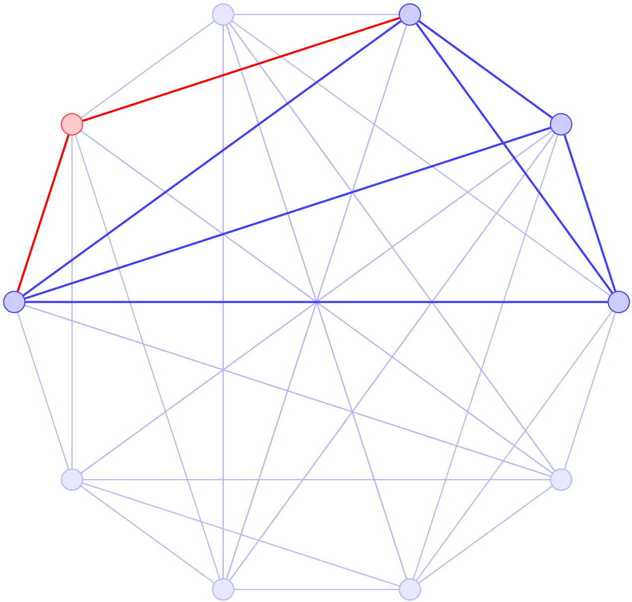
edge-regular

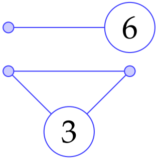
$\mathrm{erg}(10, 6, 3)$



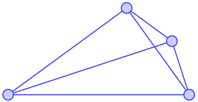
clique

of order 4

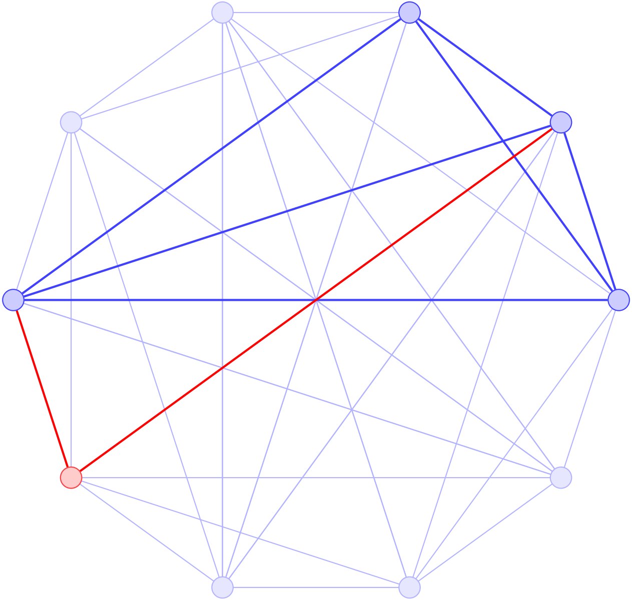


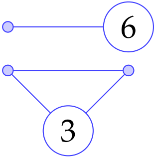


edge-regular
$\mathrm{erg}(10, 6, 3)$



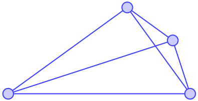
clique
of order 4





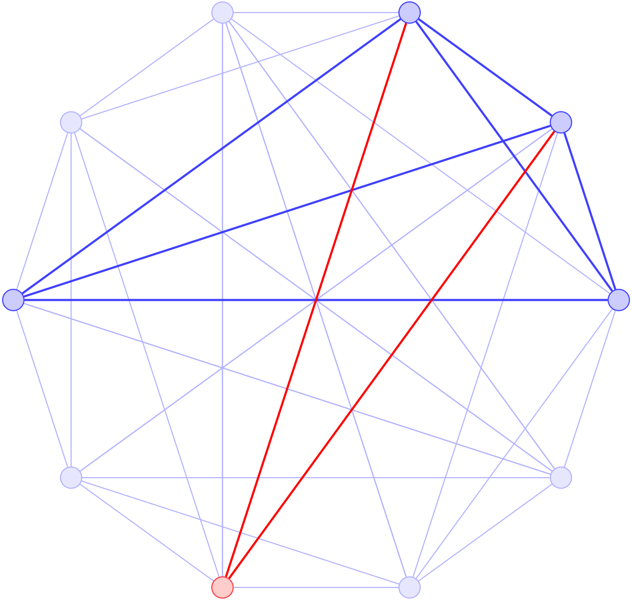
edge-regular

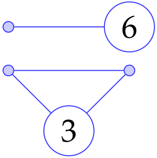
$\mathrm{erg}(10,6,3)$



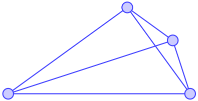
clique

of order 4

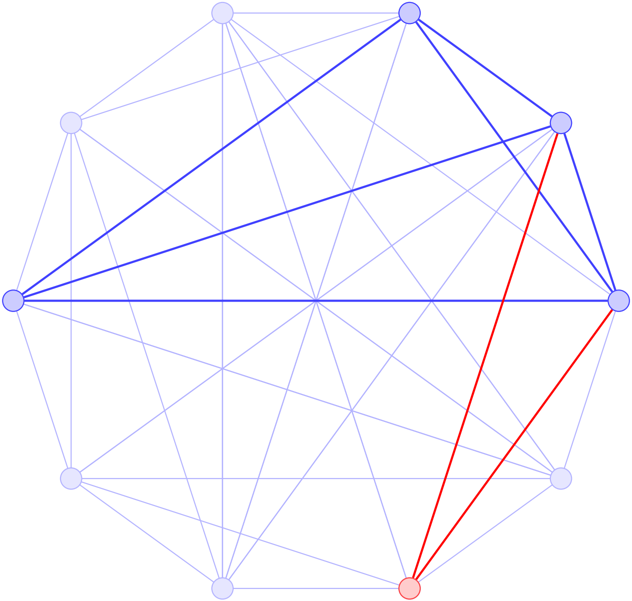


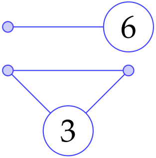


edge-regular
$\mathrm{erg}(10, 6, 3)$



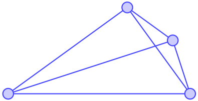
clique
of order 4





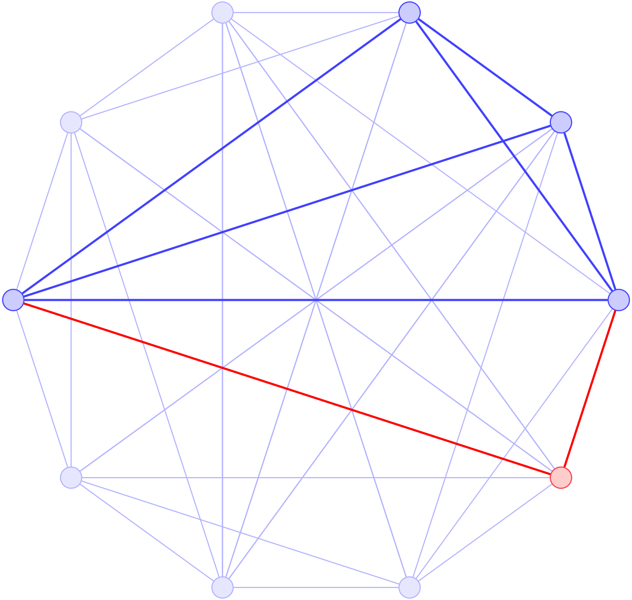
edge-regular

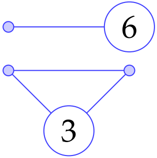
$\mathrm{erg}(10, 6, 3)$



clique

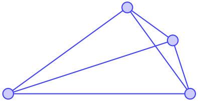
of order 4





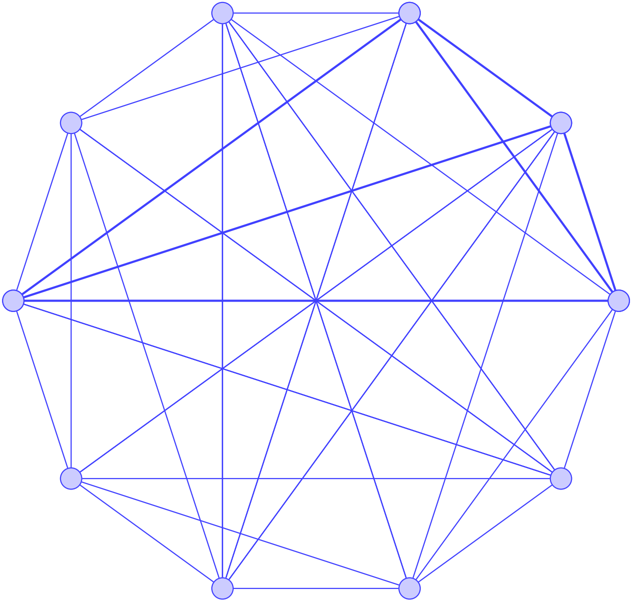
edge-regular

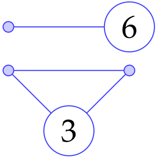
$\mathrm{erg}(10, 6, 3)$



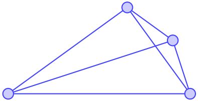
clique

of order 4





edge-regular
$\mathrm{erg}(10,6,3)$



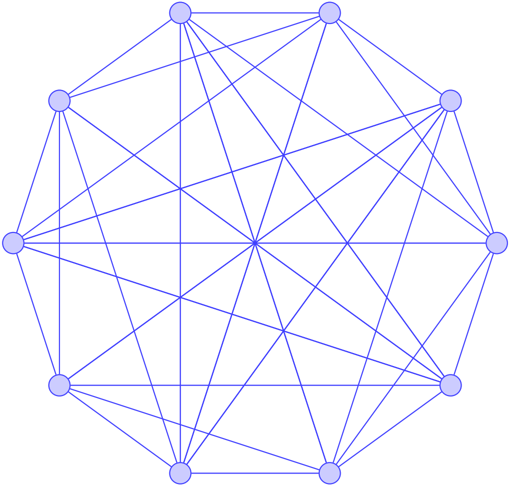
2-**regular** clique
of order 4

Theorem (Neumaier 1981)

Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

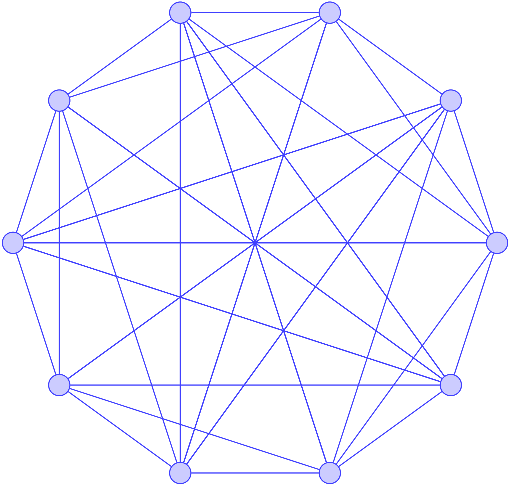
Theorem (Neumaier 1981)

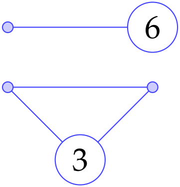
Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.



Theorem (Neumaier 1981)

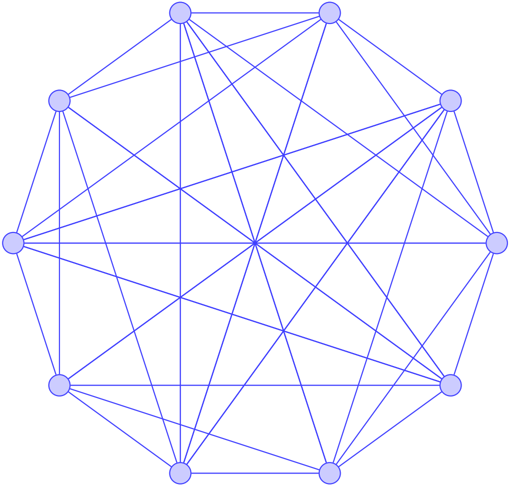
Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

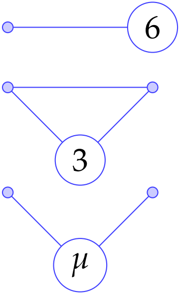




Theorem (Neumaier 1981)

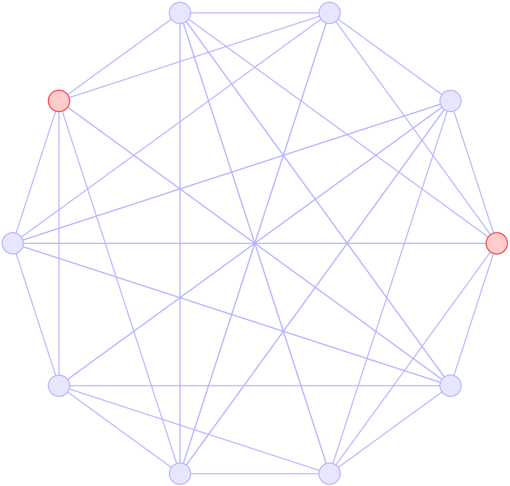
Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

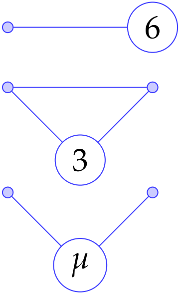




Theorem (Neumaier 1981)

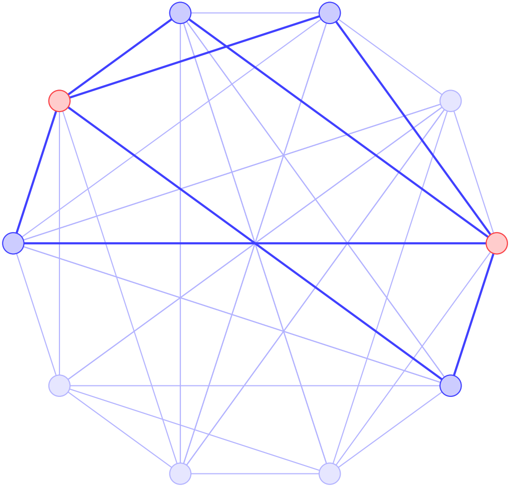
Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

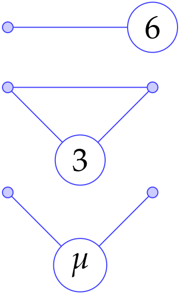




Theorem (Neumaier 1981)

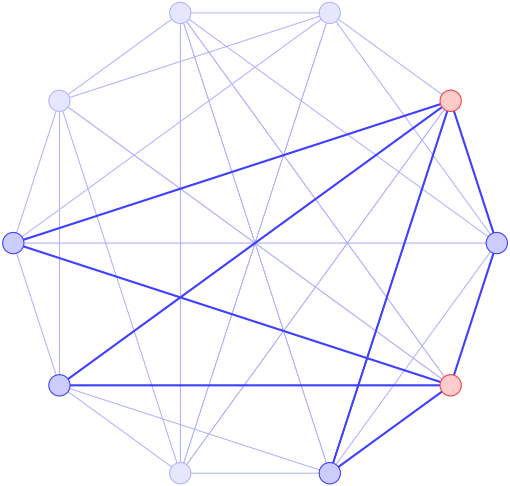
Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

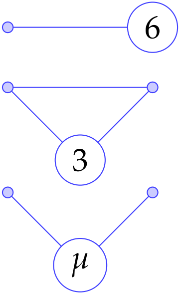




Theorem (Neumaier 1981)

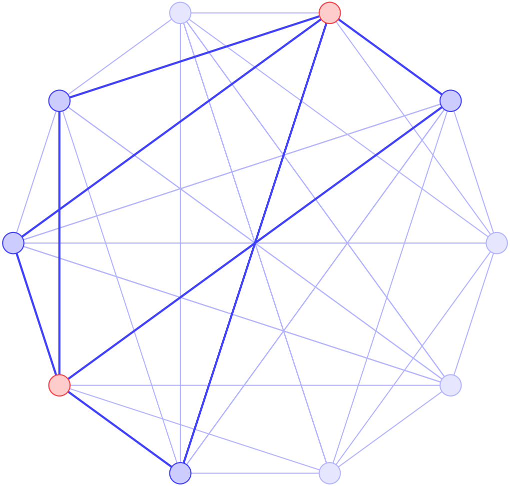
Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

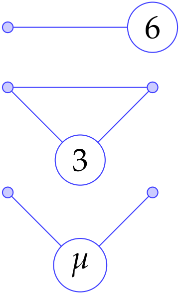




Theorem (Neumaier 1981)

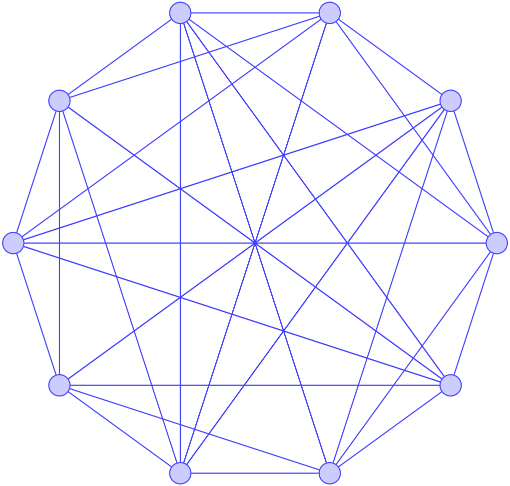
Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

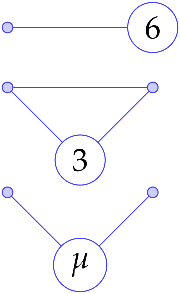




Theorem (Neumaier 1981)

Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

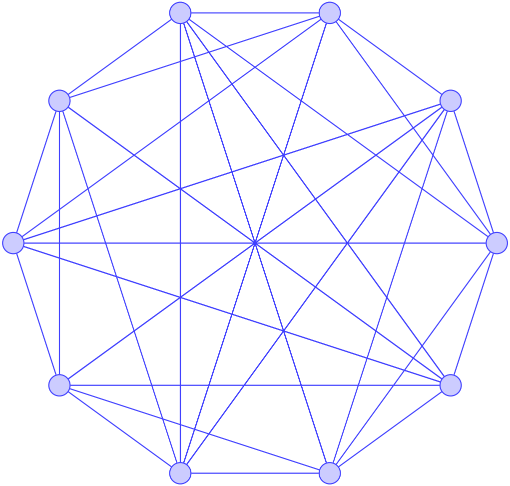


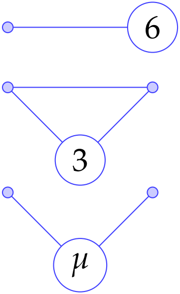


$\mu = 4$

Theorem (Neumaier 1981)

Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.





$\mu = 4$

strongly regular
$\mathrm{srg}(10, 6, 3, 4)$

Question (Neumaier 1981)

Is every edge-regular graph with a regular clique strongly regular?

Question (Neumaier 1981)

Is every edge-regular graph with a regular clique strongly regular?

Answer (GG and Koolen 2018)

No.

Question (Neumaier 1981)

Is every edge-regular graph with a regular clique strongly regular?

Answer (GG and Koolen 2018)

No. There exist infinitely many non-strongly-regular, edge-regular vertex-transitive graphs with regular cliques.

An example

Gary — magma.exe ◂ magma — 94×34

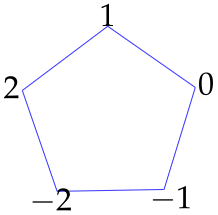
```
Graph
Vertex  Neighbours

1       8 9 14 15 18 19 22 24 27 ;
2       8 9 10 16 19 20 23 25 28 ;
3       9 10 11 17 20 21 22 24 26 ;
4       10 11 12 15 18 21 23 25 27 ;
5       11 12 13 15 16 19 24 26 28 ;
6       12 13 14 16 17 20 22 25 27 ;
7       8 13 14 17 18 21 23 26 28 ;
8       1 2 7 15 17 20 22 25 26 ;
9       1 2 3 16 18 21 23 26 27 ;
10      2 3 4 15 17 19 24 27 28 ;
11      3 4 5 16 18 20 22 25 28 ;
12      4 5 6 17 19 21 22 23 26 ;
13      5 6 7 15 18 20 23 24 27 ;
14      1 6 7 16 19 21 24 25 28 ;
15      1 4 5 8 10 13 22 23 28 ;
16      2 5 6 9 11 14 22 23 24 ;
17      3 6 7 8 10 12 23 24 25 ;
18      1 4 7 9 11 13 24 25 26 ;
19      1 2 5 10 12 14 25 26 27 ;
20      2 3 6 8 11 13 26 27 28 ;
21      3 4 7 9 12 14 22 27 28 ;
22      1 3 6 8 11 12 15 16 21 ;
23      2 4 7 9 12 13 15 16 17 ;
24      1 3 5 10 13 14 16 17 18 ;
25      2 4 6 8 11 14 17 18 19 ;
26      3 5 7 8 9 12 18 19 20 ;
27      1 4 6 9 10 13 19 20 21 ;
28      2 5 7 10 11 14 15 20 21 ;

>
```

Cayley graphs

- Let G be an (additive) group and $S \subseteq G$ a (symmetric) generating subset, i.e., $s \in S \implies -s \in S$ and $G = \langle S \rangle$.
- The **Cayley graph** $\mathrm{Cay}(G, S)$ has vertex set G and edge set
$$\{\{g, g+s\} : g \in G \text{ and } s \in S\}.$$

Example

$\Gamma = \mathrm{Cay}(\mathbb{Z}_5, S)$ Generating set $S = \{-1, 1\}$



An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$

Generating set S

$(01,0)$	$(01,\pm 1)$
$(10,0)$	$(10,\pm 2)$
$(11,0)$	$(11,\pm 3)$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$$\begin{array}{ll} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$$\begin{matrix} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{matrix}$$

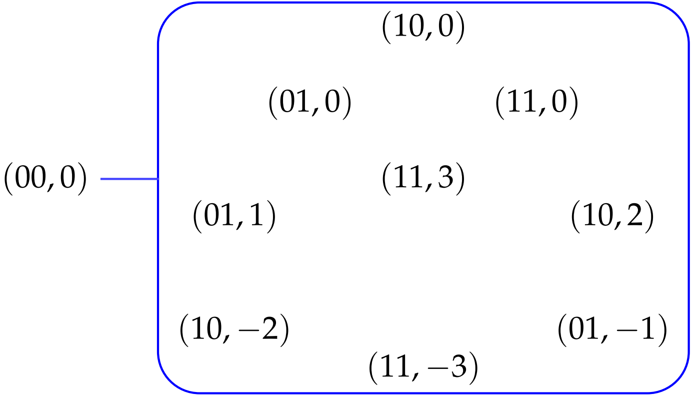
$(00, 0)$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

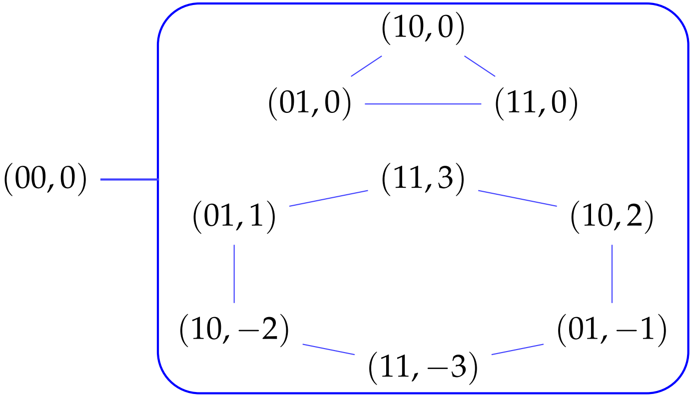


An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$



An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;

- Γ has a 1-**regular** 4-**clique**:

Generating set S

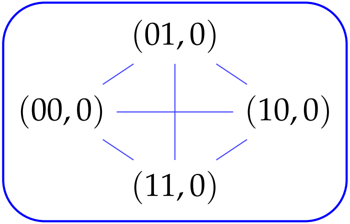
$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-**regular** 4-**clique**:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

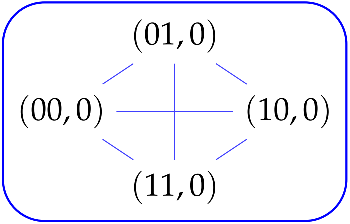


An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-**regular** 4-**clique**:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$



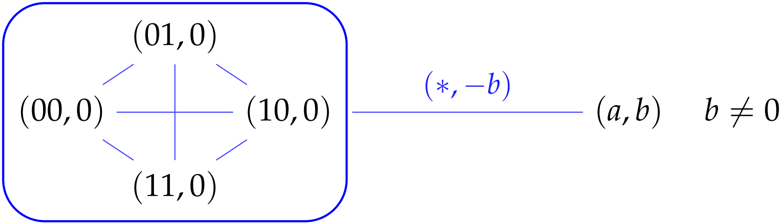
$(a, b) \quad b \neq 0$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-**regular** 4-**clique**:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$



An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular 4-clique;

- Γ is **not strongly regular**:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular 4-clique;

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

- Γ is **not strongly regular**:

$(00, 0)$ —

$(01, 0)$	$(11, 0)$
$(11, 3)$	$(01, 1)$
$(10, -2)$	$(10, 0)$
$(11, -3)$	$(01, -1)$
	$(10, 2)$

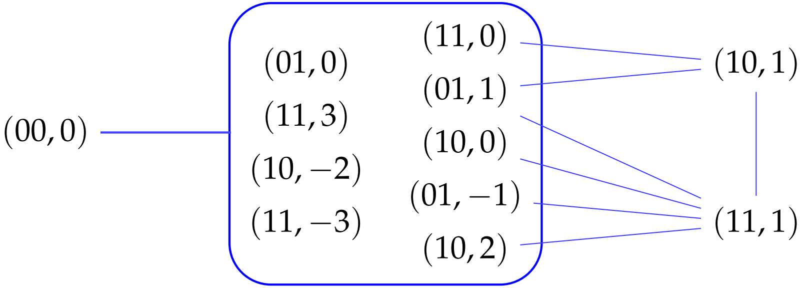
An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular 4-clique;

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

- Γ is **not strongly regular**:



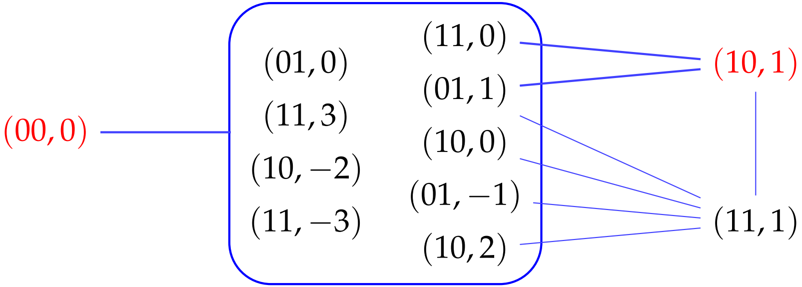
An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular 4-clique;

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

- Γ is **not strongly regular**:



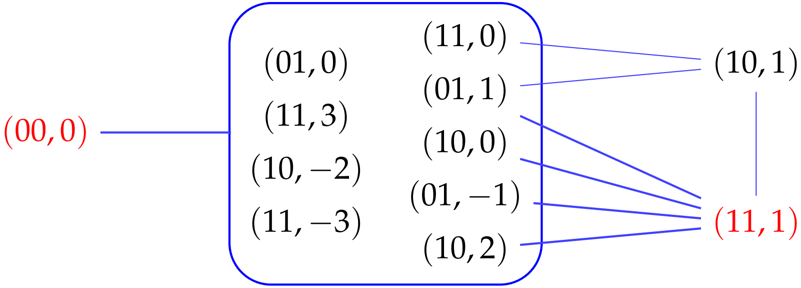
An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular 4-clique;

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

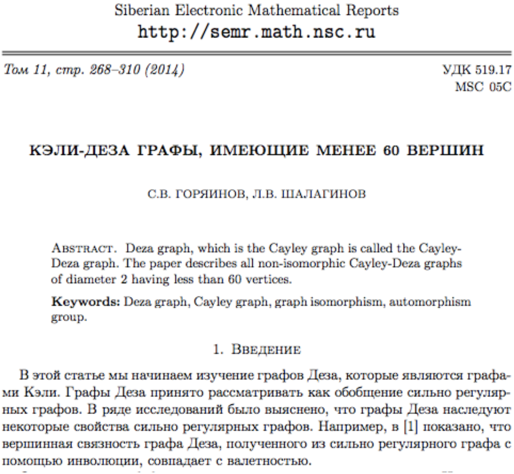
- Γ is **not strongly regular**:



General construction

- Generalise: $\mathbb{Z}_2^2 \oplus \mathbb{Z}_7$ to $\mathbb{Z}_{(c+1)/2} \oplus \mathbb{Z}_2^2 \oplus \mathbb{F}_q$.
- Works for $q \equiv 1 \pmod 6$ such that the 3rd cyclotomic number $c = c_q^3(1,2)$ is odd.
- Then there exists an $\mathrm{erg}(2(c+1)q, 2c+q, 2c)$ having a 1-regular clique of order $2c+2$.
- Take $p \equiv 1 \pmod 3$ a prime s.t. $2 \not\equiv x^3 \pmod p$. Then there exist a such that $c_{p^a}^3(1,2)$ is odd.

Examples in the wild



Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 11, стр. 268–310 (2014) УДК 519.17
MSC 05C

КЭЛИ-ДЕЗА ГРАФЫ, ИМЕЮЩИЕ МЕНЕЕ 60 ВЕРШИН

С.В. ГОРЯИНОВ, Л.В. ШАЛАГИНОВ

Abstract. Deza graph, which is the Cayley graph is called the Cayley-Deza graph. The paper describes all non-isomorphic Cayley-Deza graphs of diameter 2 having less than 60 vertices.

Keywords: Deza graph, Cayley graph, graph isomorphism, automorphism group.

1. Введение

В этой статье мы начинаем изучение графов Деза, которые являются графами Кэли. Графы Деза принято рассматривать как обобщение сильно регулярных графов. В ряде исследований было выяснено, что графы Деза наследуют некоторые свойства сильно регулярных графов. Например, в [1] показано, что вершинная связность графа Деза, полученного из сильно регулярного графа с помощью инволюции, совпадает с валентностью.

Four $\mathrm{erg}(24, 8, 2)$ graphs with a 1-regular clique

Open problems

- Find general construction that includes $\mathrm{erg}(24, 8, 2)$

- Smallest non-strongly-regular, edge-regular graph with regular clique (Neumaier graph)

- All known examples have 1-regular cliques

~~Open~~ Closed problems

- Find general construction that includes $\mathrm{erg}(24, 8, 2)$
  - GG and Koolen (2018+): New infinite construction a-antipodal $\mathrm{erg}(v, k, \lambda)$ to $\mathrm{erg}(v(\lambda+2)/a, k+\lambda+1, \lambda)$.

- Smallest non-strongly-regular, edge-regular graph with regular clique (Neumaier graph)
  - Evans and Goryainov (2018+): Smallest is $\mathrm{erg}(16, 9, 4)$

- All known examples have 1-regular cliques
  - Evans and Goryainov (2018+): 2-regular cliques

Problems

- Find general construction that includes $\mathrm{erg}(24,8,2)$
  - GG and Koolen (2018+): New infinite construction a-antipodal $\mathrm{erg}(v,k,\lambda)$ to $\mathrm{erg}(v(\lambda+2)/a, k+\lambda+1, \lambda)$.
- Smallest non-strongly-regular, edge-regular graph with regular clique (Neumaier graph)
  - Evans and Goryainov (2018+): Smallest is $\mathrm{erg}(16,9,4)$
- All known examples have 1-regular cliques
  - Evans and Goryainov (2018+): 2-regular cliques
- $\exists$ Neumaier graphs with 3-regular cliques?

Problems

- Find general construction that includes $\mathrm{erg}(24, 8, 2)$
  - GG and Koolen (2018+): New infinite construction a-antipodal $\mathrm{erg}(v, k, \lambda)$ to $\mathrm{erg}(v(\lambda+2)/a, k+\lambda+1, \lambda)$.
- Smallest non-strongly-regular, edge-regular graph with regular clique (Neumaier graph)
  - Evans and Goryainov (2018+): Smallest is $\mathrm{erg}(16, 9, 4)$
- All known examples have 1-regular cliques
  - Evans and Goryainov (2018+): 2-regular cliques
- $\exists$ Neumaier graphs with 3-regular cliques?
- $\exists$ Neumaier graphs with diameter $\geqslant 3$?

Thank you for your attention

Further reading:

G. R. W. Greaves and J. H. Koolen, *Edge-regular graphs with regular cliques*,
European J. Combin. **71** (2018), pp. 194–201.