On the Optical and Forwarding Indices of Graphs

Yuan-Hsun Lo

School of Mathematical Sciences
Xiamen University

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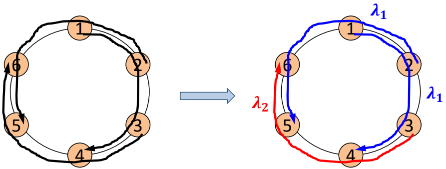
Joint with Hung-Lin Fu (NCTU), Wing Shing Wong (CUHK)
and Yijin Zhang (NJUST)

Outline

Wavelength Division Multiplexing Optical Networks

- All-optical network architecture.
- Adjacent nodes are connected by two fibers, one for each direction.
- Each fiber carries a fixed number of wavelengths, $\mathcal{C}$.
- For a given source-destination traffic request (s-d pair), define its lightpath to be the route and the wavelength used to support that request.
- There are two constraints:
  - **wavelength-continuity** constraint
  - **distinct wavelength** constraint

$I = \{(1,4), (2,5), (3,6)\}$



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Routing and wavelength assignment (RWA) problem

How to design the **lightpaths** and **wavelengths** evenly to minimize $\mathcal{C}$ for a given set of s-d pairs?

Optical Index

- $G = (V, A)$ is a symmetric digraph: $(x, y) \in A(G) \Leftrightarrow (y, x) \in A(G)$
- Instance I: $I \subseteq \{(x, y) | x, y \in V(G), x \neq y\}$
- Routing $\vec{R}_I$: a set of $|I|$ dipaths specified for all ordered pairs in I
- $\chi(\vec{R}_I)$: minimum k to guarantee the existence of a mapping $\phi : \vec{R}_I \to \{1, 2, \ldots, k\}$ called **path-coloring** such that

$$\phi(P) \neq \phi(P') \quad \text{if} \quad A(P) \cap A(P') = \emptyset$$

Definition

The **optical index** associated with G and I is given by

$$\vec{w}(G, I) := \min_{\vec{R}_I} \chi(\vec{R}_I).$$

Arc-forwarding Index

- load of arc a, $\ell_{\vec{R}_I}(a)$: number of dipaths passing through a in $\vec{R}_I$
- Max. link load: $\ell(\vec{R}_I) = \max_{a \in A(G)} \ell_{\vec{R}_I}(a)$

Definition

The **arc-forwarding index** associated with G and I is given by

$$\vec{\pi}(G,I) := \min_{\vec{R}_I} \ell(\vec{R}_I).$$

Proposition

For any graph G and instance I, one has

$$\vec{w}(G,I) \geq \vec{\pi}(G,I).$$

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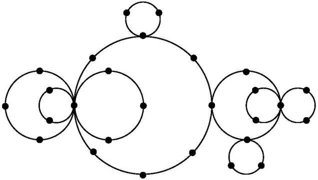
I. All-to-all Instance $I_A = \{(x, y) | x, y \in V(G), x \neq y\}$

$\vec{w}(G, I_A) = \vec{\pi}(G, I_A)$ for G is

- a tree [Gargano *et al.*, 97"]
- a cycle or hypercube [Bermond *et al.*, 00"]
- a tree of cycles [Beauquier *et al.*, 99"]
- $K_n \times K_n \times \cdots \times K_n$ [Beauquier, 99"]
  $C_n \times C_n \times \cdots \times C_n$, n is even
  $P_n \times P_n \times \cdots \times P_n$, n is even



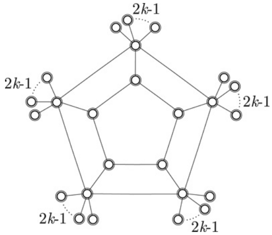
- $C_n \times C_m$ and $P_n \times P_m$ for some particular n, m [Schroer *et al.*, 97"]
- a 4-regular circulant graph $C_n(1, s)$ [Gan *et al.*, 15"]

I. All-to-all Instance $I_A = \{(x,y) \mid x,y \in V(G), x \neq y\}$

$\vec{w}(G, I_A) = \vec{\pi}(G, I_A)$ does NOT always hold. For example,

Theorem (Kosowski, 09")

$\vec{w}(G, I_A) > \vec{\pi}(G, I_A)$ *when G is of the form with* $k \geq 39$.



II. Arbitrary Instance I

Approximation algorithm approaches:

- [Aggarwal *et al.*, 94"] $\vec{w}(G,I) \leq 2\sqrt{|A(G)|}\vec{\pi}(G,I)$ for any graph G
- [Mihail *et al.*, 95"] $\vec{w}(G,I) \leq 2\vec{\pi}(G,I) - 1$ if G is a cycle
- [Beauquier *et al.*, 97"] $\vec{w}(G,I) \leq \frac{5}{3}\vec{\pi}(G,I)$ if G is a tree
- [Bian *et al.*, 09"] $\vec{w}(G,I) \leq 3\vec{\pi}(G,I)$ if G is a tree of cycles

III. Sub-all-to-all Instance

$$I_A(N) = \{(x, y) \mid x, y \in N, x \neq y\}, N \subset V(G)$$

Theorem (–, *IEEE Tran. Inform. Theory, 2017*)

$\vec{w}(G, I_A(N)) = \vec{\pi}(G, I_A(N))$ *for G is a fat-tree, in the case that N is the set of hosts or set of edge-switches.*

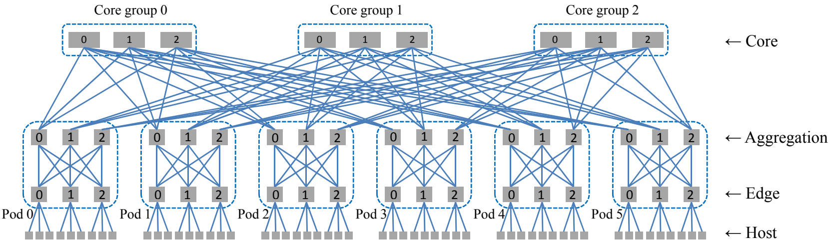


Figure: The 3-ary fat-tree network

	optical index	forwarding index
Symmetric Digraphs	$\vec{w}(G,I)$	$\vec{\pi}(G,I)$
↓	↓	↓
Simple Graphs	$w(G,I)$	$\pi(G,I)$

Undirected Optical Index

- $G = (V, E)$ is a connected simple digraph
- Instance I: $I \subseteq \binom{V(G)}{2} = \{\{x, y\} \mid x, y \in V(G), x \neq y\}$
- Routing R_I: a set of $|I|$ paths specified for all pairs in I
- $\chi(R_I)$: minimum k to guarantee the existence of a mapping $\phi : R_I \to \{1, 2, \ldots, k\}$ called **path-coloring** such that

$$\phi(P) \neq \phi(P') \quad \text{if} \quad E(P) \cap E(P') = \emptyset$$

Definition

The **undirected** optical index associated with G and I is given by

$$w(G, I) := \min_{R_I} \chi(R_I).$$

Edge-forwarding Index

- load of edge e, $\ell_{R_I}(e)$: number of paths passing through e in R_I
- Max. link load: $\ell(R_I) = \max_{e \in E(G)} \ell_{R_I}(e)$

Definition

The **edge-forwarding index** associated with G and I is given by

$$\pi(G, I) := \min_{R_I} \ell(R_I).$$

Proposition

For any graph G and instance I, one has

$$w(G, I) \geq \pi(G, I).$$

Main Results

- All-to-all instance: $I_A = \binom{V(G)}{2}$

Theorem (–, *arXiv*:1509.07029)

$$w(C_n, I_A) = \pi(C_n, I_A) = \begin{cases} \binom{n/2}{2} + \lfloor \frac{n}{4} \rfloor + 1, & \text{if } 2|n; \\ \binom{(n+1)/2}{2}, & \text{if } 2 \nmid n. \end{cases}$$

Main Results

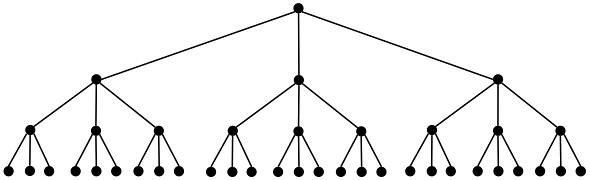
- All-to-all instance: $I_A = \binom{V(G)}{2}$

Theorem (–, *arXiv*:1707.09745)

$$w(T_{m,h}, I_A) = \begin{cases} \pi(T_{m,h}, I_A), & \textit{if } m = 1 \textit{ or } m \geq 4 \textit{ is even;} \\ \pi(T_{m,h}, I_A) + 2^{2h-2} - 2^{h+1} + 1, & \textit{if } m = 2; \\ \pi(T_{m,h}, I_A) + \frac{m(m^h-1)(m^{h-1}-1)}{(m-1)^2}, & \textit{if } m \geq 3 \textit{ is odd;} \end{cases}$$

where $T_{m,h}$ is a complete m-ary tree of height k.

$T_{3,3}$:



Main Results

- $G_{k,t}$: a tree rooted by r, such that $G_{k,t} - r$ has k components, each of which contains t vertices

Lemma (–, *arXiv*:1707.09745)

For any odd integer $k \geq 3$ and positive integer t, one has

$$w(G_{k,t}, I_A) = kt^2 \quad \text{and} \quad \pi(G_{k,t}, I_A) = (k-1)t^2 + t.$$

- $$\frac{w(G_{k,t}, I_A)}{\pi((G_{k,t}, I_A))} = \frac{kt}{(k-1)t+1} \xrightarrow{t\to\infty} \frac{k}{k-1} \leq \frac{3}{2}$$

Theorem (–, *in preparation*)

For any tree T, one has: $1 \leq \dfrac{w(T, I_A)}{\pi(T,_A)} < \dfrac{3}{2}.$

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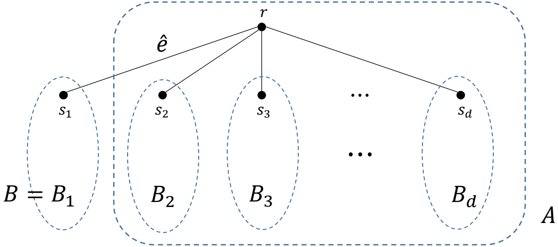
Proof. By induction on n, the order T

$\ell(\hat{e}) = \pi(T, I_A)$ and $|V(A)| \geq |V(B)|$



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Proof. Let $|V(A)| = a$, $|V(B)| = b$ and $|V(B_i)| = b_i$.

$$\begin{cases} b \geq \frac{3}{4}a \\ b < \frac{3}{4}a & \begin{cases} b = 2 \\ b = 3 \\ b = 4 \\ b \geq 5 & \begin{cases} b \geq a/2 \\ b < a/2 & \begin{cases} b_1 = b_2 = \cdots = b_d \\ b_1 = \cdots = b_k > b_{k+1} \geq \cdots \geq b_d \end{cases} \end{cases} \end{cases} \end{cases}$$

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For any tree T, one has: $1 \leq \frac{w(T, I_A)}{\pi(T_{,A})} < \frac{3}{2}$.

Conjecture

For any rational number δ satisfying $1 \leq \delta < \frac{3}{2}$, there exists a tree T such that
$$\frac{w(T, I_A)}{\pi(T, I_A)} = \delta.$$

By the results of $T_{m,h}$ and $G_{k,t}$:

- $\delta = 1 + \frac{2^{2h-2} - 2^{h+1} + 1}{2^{2h} - 2^h}$, $\forall h \geq 1$ ✓
- $\delta = 1 + \frac{t-1}{(k-1)t+1}$, $\forall t \geq 1$ and odd $k \geq 3$ ✓

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Future Works

Future and ongoing works

1. Optical index v.s. arc-forwarding index
   - $C_n \times C_n \times \cdots \times C_n$, $P_n \times P_n \times \cdots \times P_n$, for even n ✓
   - $C_n \times C_m$ for some particular n, m ✓
   - 4-regular circulant graph $C_s(1, r)$ ✓
   - other missing cases?

2. Undirected optical index v.s. edge-forwarding index
   - $1 \leq \frac{w(T,I_A)}{\pi(T,I_A)} < \frac{3}{2}$ ✓
   - for $\delta \in \mathbb{Q}$, $1 < \delta < \frac{3}{2}$, find a suitable tree T such that $\frac{w(T,I_A)}{\pi(T,I_A)} = \delta$
   - what is the ratio $\frac{w(G,I_A)}{\pi(G,I_A)}$ for general simple graphs G?

Thank you for your attention!