Regular unimodular triangulations of dilated empty simplices and Gröbner basis

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This talk is based on the joint work with **Takayuki Hibi** and **Koutarou Yoshida**.

Lattice polytopes and h^*-vectors

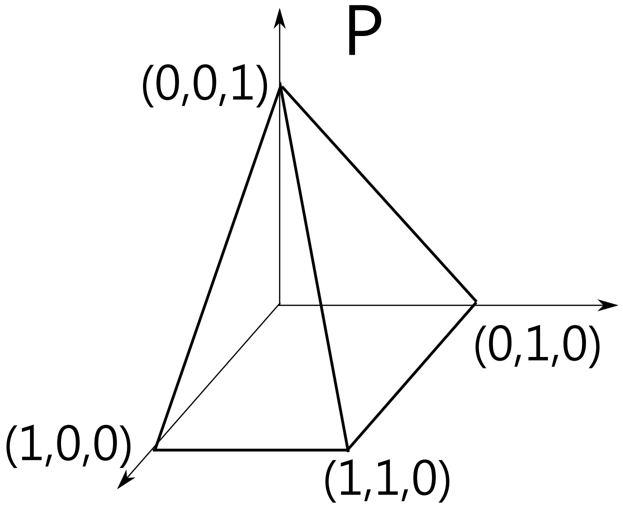
$P \subset \mathbb{R}^d$: lattice polytope $\Longleftrightarrow$ a convex polytope each of whose vertices belongs to $\mathbb{Z}^d$

Consider the **Ehrhart series**:

$$\sum_{n \geq 0} |nP \cap \mathbb{Z}^d| t^n = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where $h_P^*(t)$ is a polynomial in t of degree at most d. We call $h_P^*(t)$ the **h^*-polynomial** of P.

Ex



$$\sum_{n \geq 0} |nP \cap \mathbb{Z}^3| t^n = \frac{1+t}{(1-t)^4}$$

Empty simplices

$P \subset \mathbb{R}^d$: lattice polytope with $h_P^*(t) = 1 + h_1^* t + \cdots\cdots$

$P \subset \mathbb{R}^d$: **empty simplex** $\iff$ a lattice simplex with $P \cap \mathbb{Z}^d = \{\text{vertices of } P\}$

$$\iff h_1^* = 0$$

Remark

- 2-dim empty simplex $\iff \mathrm{conv}((0,0),(1,0),(0,1))$
- 3-dim empty simplex $\iff \mathrm{conv}((0,0,0),(1,0,0),(0,0,1),(p,q,1))$, where p and q are coprime (**White** (1964)).
- 4-dim empty simplex : **Fransisco Santos' talk!** (very recent result)
- ≥ 5-dim empty simplex : **OPEN** (I guess)

For a 3-**dim** lattice polytope P, we know that

$$P \text{ : \textbf{empty simplex}} \iff h_P^*(t) = 1 + (m-1)t^2 \ (m \in \mathbb{Z}_{\geq 1}).$$

On the other hand,

Theorem (Batyrev–Hofscheier, 2010)

$d = 2k - 1 \ (k \in \mathbb{Z}_{\geq 2})$
$P \subset \mathbb{R}^d$ is an **empty simplex** of dim d with $h_P^*(t) = 1 + (m-1)t^k \ (m \in \mathbb{Z}_{\geq 2})$

$$\iff P \cong \mathrm{conv}\left(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_{d-1}, \sum_{i=1}^{k-1} a_i \mathbf{e}_i + \sum_{j=k}^{d-1} (m - a_{d-j}) \mathbf{e}_j + m \mathbf{e}_d\right),$$

where $a_1, \dots, a_{k-1} \in \mathbb{Z}_{>0}$ with $1 \leq a_i \leq m/2$ and $\gcd(a_i, m) = 1$ $(\forall i)$

Ex $d = 3$: $\mathrm{conv}((0,0,0),(1,0,0),(0,1,0),(a_1, m-a_1, m))$

$d = 5$: $\mathrm{conv}(\mathbf{0},(1,0,0,0,0),\dots,(0,0,0,1,0),(a_1,a_2,m-a_2,m-a_1,m))$

IDP and Unimodular Triangulation

$P \subset \mathbb{R}^d$: a lattice polytope

• We say that P has **integer decomposition property (IDP)** if for any $n \in \mathbb{Z}_{>0}$ and $\gamma \in nP \cap \mathbb{Z}^d$, there exist $\gamma^{(1)}, \dots, \gamma^{(n)} \in P \cap \mathbb{Z}^d$ such that $\gamma = \gamma^{(1)} + \cdots + \gamma^{(n)}$.

• We say that a **triangulation (covering)** Δ of P is **unimodular** if $h^*_\sigma(t) = 1$ for $\forall \sigma \in \Delta$.

regular unimodular tri. $\Rightarrow$ unimodular tri. $\Rightarrow$ unimodular covering $\Rightarrow$ IDP

(NOTE: **Regular triangulation** means the triangulation is obtain by "pulling".)

Motivation

OUR GOAL

We want to know the existence of unimodular triangulation of **dilations** of

$$\mathrm{conv}\left(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \sum_{i=1}^{k-1} a_i \mathbf{e}_i + \sum_{j=k}^{d-1} (m - a_{d-j})\mathbf{e}_j + m\mathbf{e}_d\right).$$

For a lattice polytope $P \subset \mathbb{R}^d$ of dim d, the following facts are known:

- nP has IDP for any $n \geq d-1$ (Bruns–Gubeladze–Trung (1997)).
- There exists a constant n_0 such that nP has a unimodular covering for any $n \geq n_0$ (BGT (1997)).
- There exists a constant c such that cP has a unimodular triangulation.

However, the existence of a constant n_0 satisfying the following is widely open:

"nP has a unimodular triangulation for any $n \geq n_0$"

On the other hand,

Theorem (Santos–Ziegler, 2014)

P : 3-dim lattice polytope

$\Longrightarrow$ nP **has a unimodular triangulation for any** $n \geq 6$.

The key of the proof of this theorem is **the existence of regular unimodular triangulation for dilated empty 3-simplices**.

$\longrightarrow$ For the higher-dimesional version of Santos–Ziegler's theorem, the existence of (regular) unimodular triangulations of "dilated empty simplices" might be crucial.

$\longrightarrow$ How about **Batyrev–Hofscheier**'s empty simplices?

Given $k, m \in \mathbb{Z}_{\geq 2}$ and $a_1, \ldots, a_{k-1} \in \mathbb{Z}_{\geq 1}$ with $\gcd(a_i, m) = 1 \ \forall i$, let

$$P(a_1, \ldots, a_{k-1}, m) = \operatorname{conv}\left(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \sum_{i=1}^{k-1} a_i \mathbf{e}_i + \sum_{j=k}^{d-1} (m - a_{d-j})\mathbf{e}_j + m\mathbf{e}_d\right).$$

Proposition

Given k, m, a_i, we have

$$nP(a_1, \ldots, a_{k-1}, m) \text{ has IDP} \iff n \geq k$$

Remark This result is independent of a_i.

Main Theorem

Given k, m, a_i, we have

$kP(a_1, \ldots, a_{k-1}, m)$ has a **regular unimodular triangulation** $\Longleftrightarrow a_1 = \cdots = a_{k-1} = 1$

Remark This is a generalization of (a part of) Santos–Ziegler's lemma.

How can we prove?? $\longrightarrow$ Using **Göbner basis**!!

How to apply Gröbner basis? (We need some algebraic setting.)

K : a field

$K[t_1^{\pm}, \ldots, t_d^{\pm}, s]$: the Laurent polynomial ring with $(d+1)$ variables

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$,
we associate the Laurent monomial $u_\alpha = t_1^{\alpha_1} \cdots t_d^{\alpha_d} \in K[t_1^{\pm}, \ldots, t_d^{\pm}, s]$,

$P \subset \mathbb{R}^d$: a lattice polytope with $P \cap \mathbb{Z}^d = \{v_1, \ldots, v_m\}$

The **toric ring** of P is defined by $K[P] = K[u_\alpha s : \alpha \in P \cap \mathbb{Z}^d]$.

$S = K[x_1, \ldots, x_m]$: polynomial ring with $|P \cap \mathbb{Z}^d|$ variables

Consider the surjective homomorphism $\pi : S \to K[P]$ by setting $\pi(x_i) = u_{v_i} s$.

$I = \ker \pi$: the **toric ideal** of P

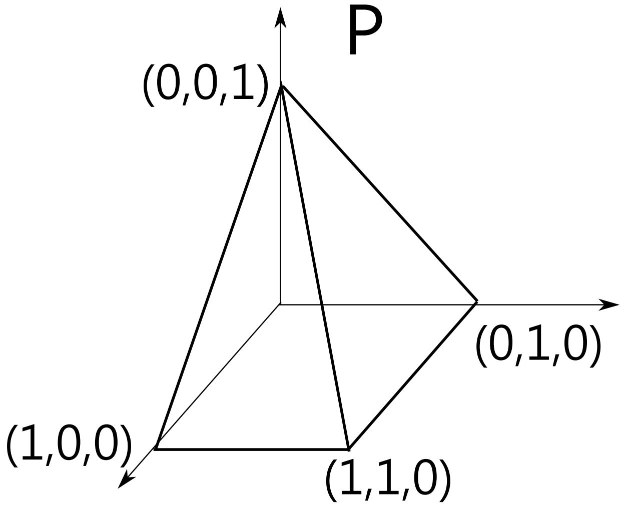
$<$: a monomial order on $S = K[x_1, \ldots, x_m]$ $\quad \mathrm{in}_<(f)$: initial monomial of $f \in S$

$g_1, \ldots, g_\ell \in I \subset S$: a system of generators of I

We say that $g_1, \ldots, g_\ell$ is a **Gröbner basis** of I with respect to $<$

$\Longleftrightarrow \mathrm{in}_<(I) = (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_\ell))$, where $\mathrm{in}_<(I) = (\mathrm{in}_<(f) : f \in I)$.

Ex



$\pi : K[x_1, \ldots, x_5] \to K[s, t_1 s, t_2 s, t_1 t_2 s, t_3 s]$ $\quad I = (x_1 x_4 - x_2 x_3)$: toric ideal of P

$x_1 x_4 - x_2 x_3$ is a Gröbner basis of I (w.r.t. any monomial order)

We say that a Göbner basis $g_1, \ldots, g_\ell$ of I w.r.t. $<$ is **squarefree** if $\mathrm{in}_<(g_i)$ is squarefree.

Theorem (Sturmfels)

$P \subset \mathbb{R}^d$: lattice polytope $\quad I_P$: toric ideal of P

P has a **regular unimodular triangulation** $\iff$ I_P has a **squarefree Gröbner basis**

Strategy of Proof of Main Theorem

$kP(a_1, \ldots, a_{k-1}, m)$ has a reg. uni. tri. $\Longrightarrow a_1 = \cdots = a_{k-1} = 1$

Prove the **non-existence** of $<$ such that $I_{kP(a_1,\ldots,a_{k-1},m)}$ has a squarefree Göbner basis if $(a_1, \ldots, a_{k-1}) \neq (1, \ldots, 1)$.

$a_1 = \cdots = a_{k-1} = 1 \Longrightarrow kP(a_1, \ldots, a_{k-1}, m)$ has a reg. uni. tri.

Construct a monomial order $<$ such that $I_{kP(1,\ldots,1,m)}$ has a squarefree Göbner basis.

ありがとうございます。