Relative t-designs on one shell of Johnson association schemes

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Johnson association scheme and its shell

For positive integers v, k with $v \geq 2k$, let $V = \{1, 2, \cdots, v\}$ and $X = \binom{V}{k}$. Define

$$R_r = \{(x, y) \in X \times X : |x \cap y| = k - r\}.$$

Then $J(v, k) = (X, \{R_r\}_{r=0}^{k})$ is Johnson association scheme.

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(Y, w) is a weighted subset of X on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$ if

1. $Y \subset X$.
2. $w : Y \to \mathbb{R}_{>0}$.
3. $\{r_1, \ldots, r_p\} = \{r \mid Y \cap X_r \neq \emptyset\}$.

Block designs

A t-(v,k,λ) design, or t-design in $J(v,k)$ consists of sets of

- points: V with $|V| = v$,
- blocks: non-empty subset $\mathcal{B}$ of $\binom{V}{k}$,

so that for every $T \in \binom{V}{t}$,

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- $w : \mathcal{B} \to \mathbb{R}_{>0}$.
- Replace $\binom{V}{k}$ by $\binom{V}{k_1} \cup \binom{V}{k_2} \cdots \cup \binom{V}{k_p}$.

$(V, \mathcal{B}, w)$ is called a weighted regular t-wise balanced design if

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$\Longleftrightarrow$ A relative t-design in $H(v,2)$ with w.r.t. $u_0 = (0,0,\cdots,0)$.

Problem

t-designs in $J(v,k)$ $\downarrow$ relative t-designs in $H(v,2)$: $\binom{V}{k_1} \cup \cdots \cup \binom{V}{k_p}$

Question $\uparrow$ relative t-designs in $J(v,k)$: $X_{r_1} \cup \cdots \cup X_{r_p}$

Question: Interpretation of relative t-designs on one shell in $J(v,k)$.

Q-polynomial association scheme

$\mathfrak{X} = (X, \{R_r\}_{r=0}^k)$: a symmetric association scheme (i.e. $R_r = {}^tR_r$).

Bose-Mesner algebra:

$$\mathcal{A} = \langle A_0, A_1, \ldots, A_k \rangle = \langle E_0, E_1, \ldots, E_k \rangle.$$

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It is known that $J(v, k)$ are Q-polynomial association schemes.

Relative t-designs in Q-polynomial A.S.

Definition (Delsarte, 1977)

Let $\mathfrak{X}$ be a Q-polynomial association scheme. A weighted subset (Y, w) of X is called a relative t-design in $\mathfrak{X}$ with respect to u_0 if $E_j\chi_{(Y,w)}$ and $E_j\chi_{\{u_0\}}$ are linearly dependent for all $1 \leq j \leq t$, where

$$\chi_{(Y,w)}(y) = \begin{cases} w(y), & \text{if } y \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

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In particular, if $\mathfrak{X} = J(v, k)$, then the above condition is equivalent that

$$\sum_{z \subset y, y \in Y} w(y) = \lambda_{t,j}$$

depends only on t and $j = |z \cap u_0|$ but not the choice of $z \in \binom{V}{t}$.

Main result

Theorem (Bannai-Z., 2018)

If (Y, w) *is a relative t-design in* $J(v, k)$ *on p shells* $X_{r_1} \cup \cdots \cup X_{r_p}$*, then* $(Y \cap X_{r_i}, w)$ *is a weighted* $(t+1-p)$*-design in* X_{r_i} *as a product association scheme for* $1 \leq i \leq p$*.*

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2. Geometric condition.

Product association schemes

$(Y_\ell, \mathcal{A}_\ell)$: an association scheme of class k_ℓ with Bose-Mesner Algebra $\mathcal{A}_\ell$.
The direct product of some association schemes is

$$(X, \mathcal{A}) = (Y_1, \mathcal{A}_1) \otimes (Y_2, \mathcal{A}_2) \otimes \cdots \otimes (Y_m, \mathcal{A}_m)$$

defined by

$$X = Y_1 \times Y_2 \times \cdots \times Y_m$$

$$\mathcal{A} = \{\otimes_{\ell=1}^{m} B_\ell \mid B_\ell \in \mathcal{A}_\ell, 1 \le \ell \le m\}.$$

An example: structure of one shell of $J(v,k)$.

Recall $u_0 = \{1,2,\ldots,k\}$ and $X_r = \left\{x \in \binom{V}{k} : |x \cap u_0| = k - r\right\}$.

1. If $2 \leq r \leq \frac{k}{2}$, take $X_r := \{(u_0 - x, x - u_0) \mid x \in X_r\}$, i.e.,

$$X_r = J(k,r) \otimes J(v-k,r).$$

2. If $\frac{k}{2} < r \leq \frac{v-k}{2}$, take $X_r := \{(x \cap u_0, x - u_0) \mid x \in X_r\}$, i.e.,

$$X_r = J(k,k-r) \otimes J(v-k,r).$$

3. If $\frac{v-k}{2} < r \leq k-2$, take $X_r := \{(x \cap u_0, (V - u_0) - x \mid x \in X_r\}$, i.e.,

$$X_r = J(k,k-r) \otimes J(v-k,v-k-r).$$

Mixed t-designs

Definition (Martin, 1998)

Let $|V_i| = v_i$ for $i = 1, 2$ and $X = \binom{V_1}{k_1} \times \binom{V_2}{k_2}$. A weighted subset (Y, w) of X is called a weighted t-design in $J(v_1, k_1) \otimes J(v_2, k_2)$ if for $j_1 + j_2 = t$

$$\sum_{\substack{(y_1,y_2)\in Y \\ z_1\subseteq y_1, z_2\subseteq y_2}} w(y_1, y_2) = \lambda_{(j_1,j_2)}$$

is independent on the choice of $(z_1, z_2) \in \binom{V_1}{j_1} \times \binom{V_2}{j_2}$.

- In particular, it is called a mixed t-design if $w = 1$.
- Mixed t-designs $\Longleftrightarrow$ t-design in X_r of $J(v, k)$.

Examples: mixed 2-designs

$(V, \mathcal{B})$ is a symmetric 2-(v, k, λ) design.
Let V_1 be the points set of a block $B \in \mathcal{B}$ and $V_2 = V \setminus V_1$.
$(V, \mathcal{B} \setminus B)$ (with $v - 1$ blocks) forms a mixed 2-design with

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$$\text{blocks} \overset{\text{points}}{\left(\begin{array}{ccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)}$$

$$\qquad V_1 \qquad\qquad V_2$$

2-(7,3,1) design gives a 2-design in $J(3,1) \otimes J(4,2)$.

Lower bound for mixed t-designs

Theorem (Bannai-Bannai-Suda-Tanaka, 2015; Martin, 1998)

Let Y be a t-design in one shell X_r of $J(v,k)$. Then

$$|Y| \geq \begin{cases} \binom{v}{e} - \binom{v}{e-1} & \text{if} \quad t = 2e, \\ 2\left(\binom{v-1}{e} - \binom{v-1}{e-1}\right) & \text{if} \quad t = 2e+1. \end{cases}$$

The design (Y, w) is called tight if the above lower bound is attained.

Tight t-designs in X_r

1. If $t = 2$, then $|Y| = v - 1$.
   - $\frac{k}{2} < r \leq \frac{v-k}{2}$.

   For $v \leq 1000$, all tight mixed 2-designs in X_r come from a symmetric 2-(v, k, λ) with one block removed, except for

v	k	r	$\lambda_{(1,0)}$	$\lambda_{(0,1)}$	$\lambda_{(1,1)}$	$\lambda_{(2,0)}$	$\lambda_{(0,2)}$
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2. If $t = 3$, then $|Y| = 2(v - 2)$.
   Possible parameters of tight 3-designs in X_r for $v \leq 1,000$ are of type:

$$v = 4u, \ k = 2u, \ k_1 = k_2 = r, \quad \text{for } 2 \leq r \leq u.$$

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Question: Is it true for the converse direction?

Further work

1. Relative t-designs on one shell X_r in $J(v,k)$ for P-polynomial structure are the product of a t-(k,k_1,λ_1) design and a t-(k,k_2,λ_2) design.

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2. Existence problem of tight t-designs in X_r.

$$|\mathcal{B}| \geq \begin{cases} \binom{v}{e} & \text{if} \quad t=2e \\ 2\binom{v-1}{e} & \text{if} \quad t=2e+1 \end{cases} \quad \text{for } t\text{-}(v,k,\lambda) \text{ designs.}$$

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- If $t = 2e$, then it is the product of two tight $2e$-designs.

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   - If $t = 2e$, then it is the product of two tight $2e$-designs.
   - Question: How to define tight $(2e + 1)$-designs?
     One possibility is the extension of tight $2e$-designs.

     Result proved by Cameron (1973) for $e = 1$.

     (i) A Hadamard design, i.e., $v = 4\lambda + 3$, $k = 2\lambda + 1$,
     (ii) $v = (\lambda + 2)(\lambda^2 + 4\lambda + 2)$, $k = \lambda^2 + 3\lambda + 1$,
     (iii) $v = 111$, $k = 11$, $\lambda = 1$,
     (iv) $v = 495$, $k = 39$, $\lambda = 3$.

Thank you for your attention!