Polynomial ideals associated to combinatorial objects

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Thanks

To you for inviting me,
to the organizers for such a nice conference,
to Dr. Shoichi Tsuchiya for so much personal help,
to my friends here at Tohoku for making math so interesting!

Date Masamune

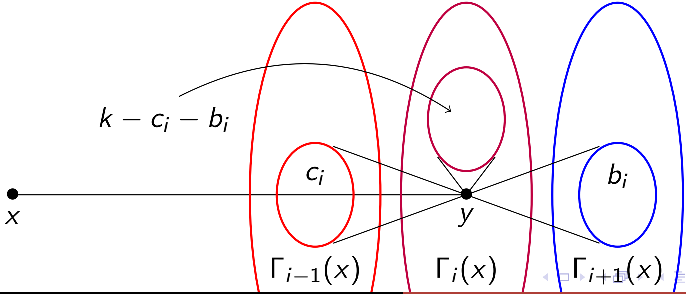
Distance-Regular Graphs (DRGs)

A graph $\Gamma = (X, R)$ of diameter d is *distance-regular* (DRG) if there exist constants

$b_0, b_1, \ldots, b_{d-1};\ c_1, c_2, \ldots, c_d$

such that, whenever x and y are vertices at distance i, there are exactly

- c_i neighbors of y at distance $i-1$ from x, and
- b_i neighbors of y at distance $i+1$ from x.

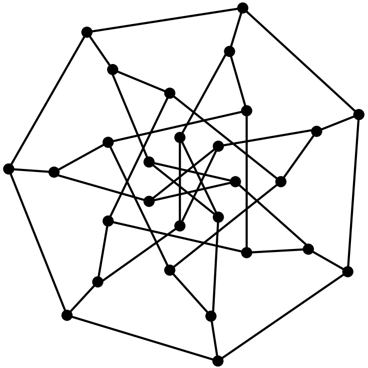


Distance-Regular Graphs (DRGs)

Examples:

- all five Platonic solids
- regular graphs with just three eigenvalues ("strongly regular")
- n-cubes and Hamming graphs
- incidence graphs of symmetric designs
- Moore graphs and generalized polygons
- . . . many other connections!

Coxeter Graph



The Coxeter graph is a cubic distance-regular graph (DRG) of diameter 4 on 28 vertices having girth 7.

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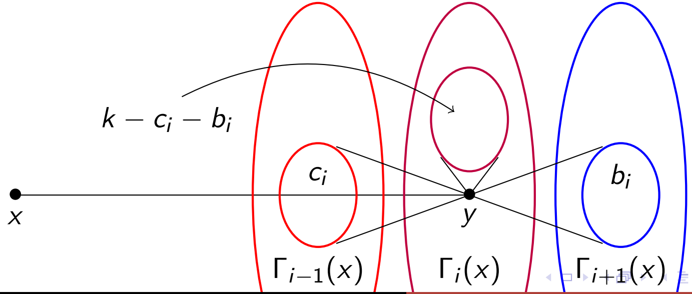
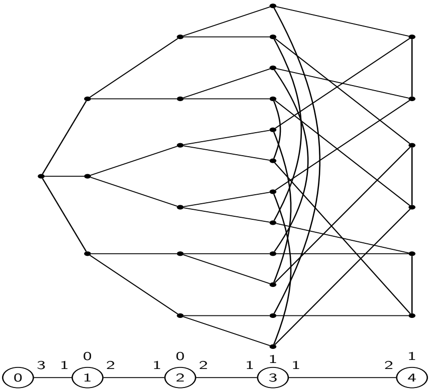


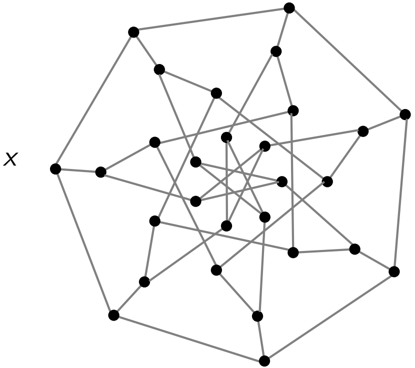
Distance Distribution in the Coxeter Graph



The Coxeter graph is distance-regular:

$b_0 = 3,\ b_1 = b_2 = 2,\ b_3 = 1;\ \ c_1 = c_1 = c_3 = 1,\ c_4 = 2.$

A Closed Walk in the Coxeter Graph



Starting at vertex x, we build a closed walk representing an element of our homotopy group.

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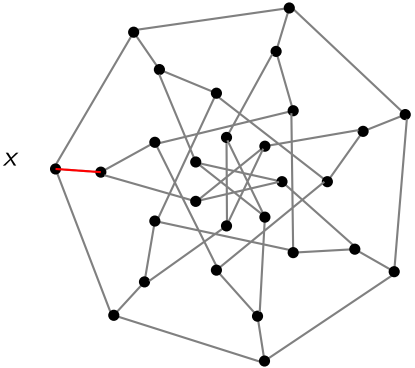

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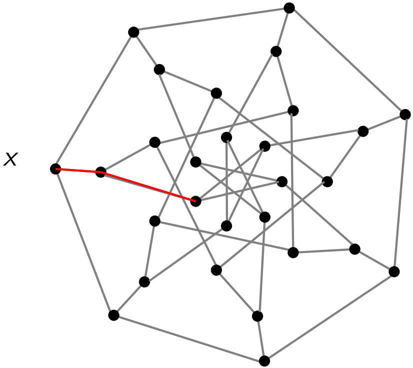

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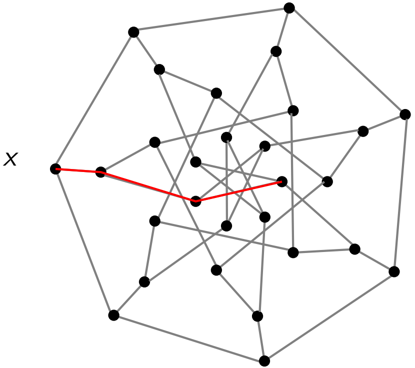

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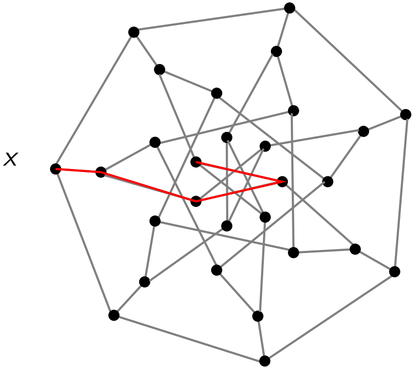

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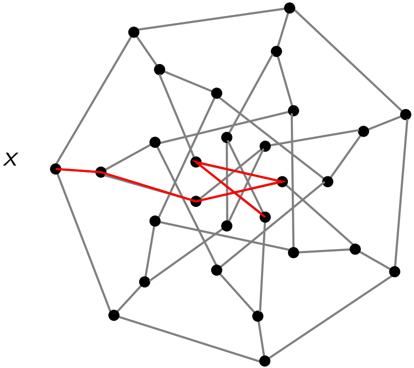

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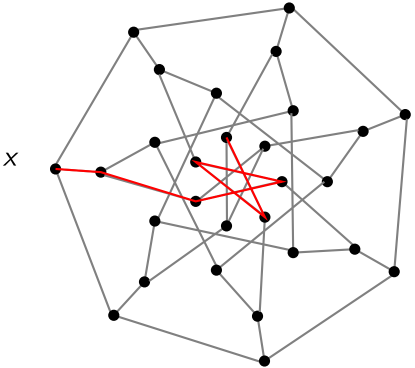

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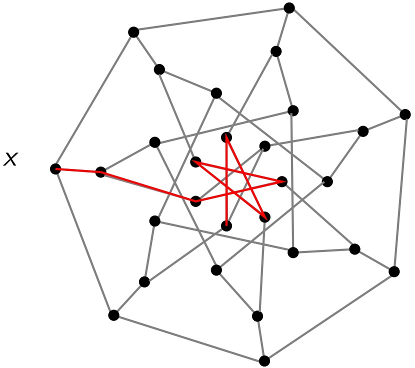

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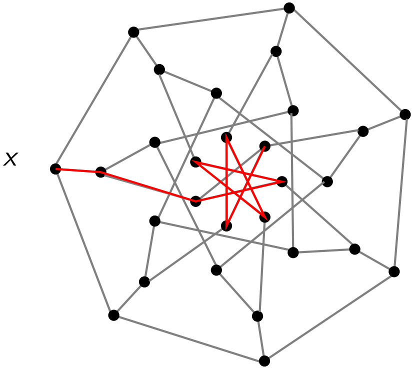

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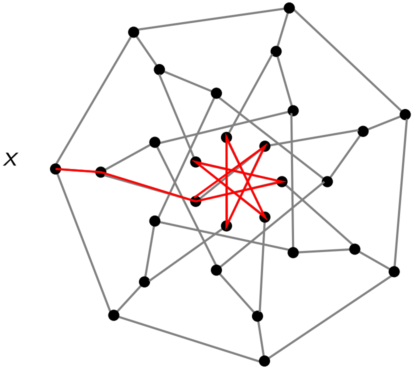

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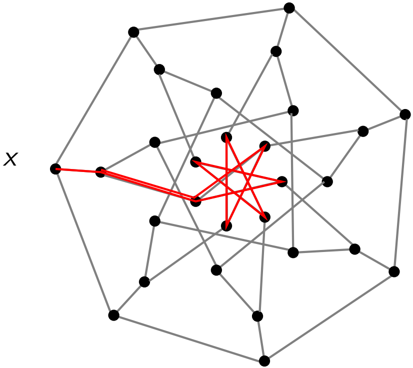

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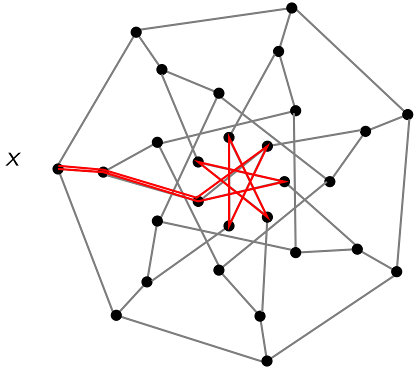
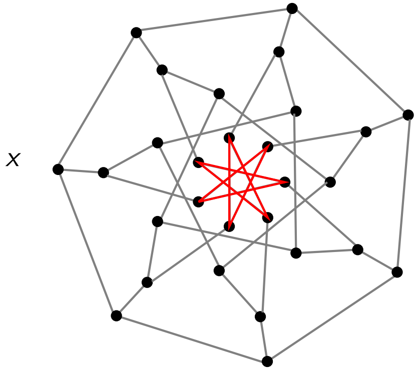


Starting at vertex x, we build a closed walk representing an element of our homotopy group. 11 edges total.

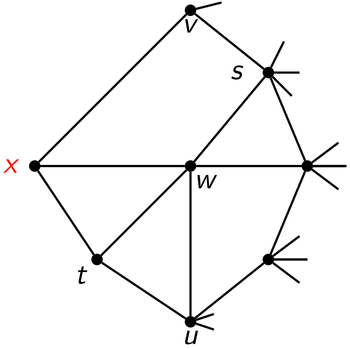
A Closed Walk in the Coxeter Graph



We say this walk (of length 11) has essential length 7.

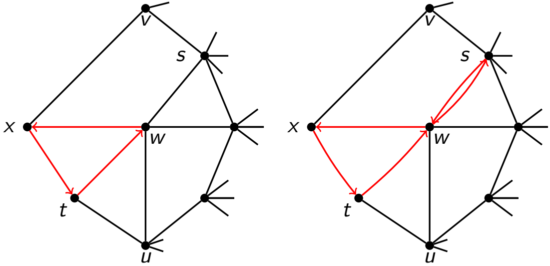
An Excursion into Homotopy

The following idea appears in the thesis work of Heather Lewis (*Discrete Math.* (2000)) under the supervision of Paul Terwilliger.



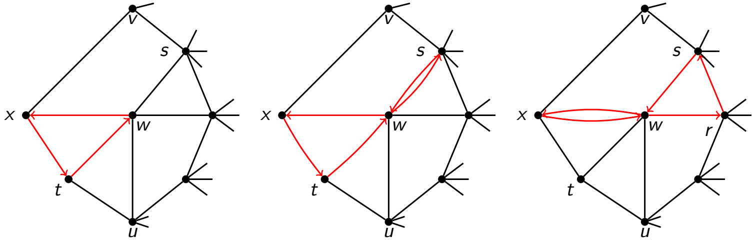
Consider equivalence classes of closed walks in Γ starting and ending at basepoint x.

Discrete Homotopy on a Graph



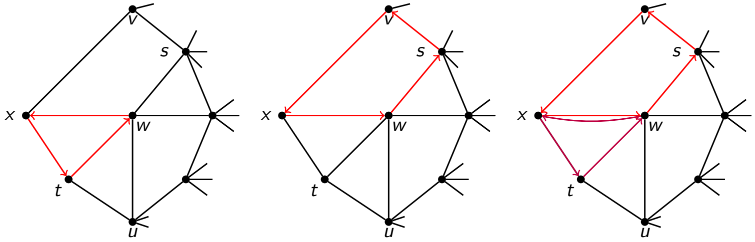
Closed walk *xtwx* is in the same equivalence class as *xtwswx*.
In general, walk $q' = q_1 p p^{-1} q_2$ is equivalent to walk $q = q_1 q_2$: $q' \sim q$

Discrete Homotopy on a Graph



These three walks all have “essential length” 3.

Discrete Homotopy on a Graph



Our group operation is concatenation of walks. Of course, the concatenation of these two walks is represented by another cycle:

$$xtwx \star xwsvx = xtwxwsvx \sim xtwsvx$$

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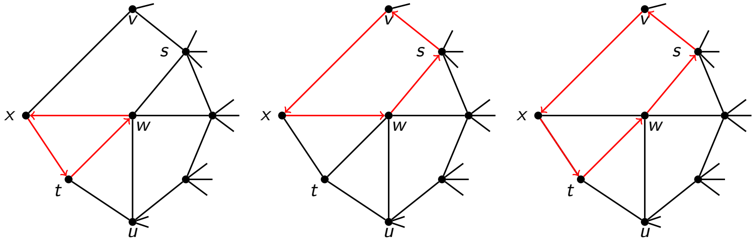

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Subgroups of the Fundamental Group

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For example, if Γ is a simple graph, $\pi_k(\Gamma, x) = 1$ for $k = 0, 1, 2$.

Some results of Heather Lewis

- $\pi_0(\Gamma, x) = \pi_1(\Gamma, x) = \pi_2(\Gamma, x) \subseteq \pi_{2d+1}(\Gamma, x) = \pi(\Gamma, x)$
- a distance-regular graph which is also "DRG dual" has girth at most 6
- For any distance-regular graph which is also "DRG dual", $\pi_6(\Gamma, x) \neq \{e\}$
- and either $\pi_6(\Gamma, x) = \pi(\Gamma, x)$ or
  - Γ is a "pseudoquotient" with $D \in \{2d, 2d+1\}$ and
  - $\pi_6(\Gamma, x) = \pi_{D-1}(\Gamma, x) \neq \pi_D(\Gamma, x) = \pi(\Gamma, x)$

Two Girth Parameters

For a distance-regular graph (DRG) Γ of diameter d, let $g_1(\Gamma)$ denote the **girth** of Γ and let $g_2(\Gamma)$ denote smallest integer ℓ such that

$$\pi_\ell(\Gamma, x) = \pi(\Gamma, x)$$

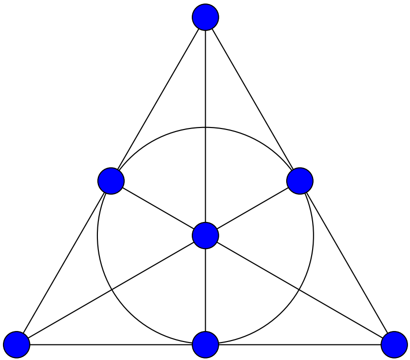
for all vertices x of Γ.

We have

$$3 \leq g_1 \leq g_2 \leq 2d + 1$$

Polynomial functions on the Fano plane

This part is based on joint work with Doug Stinson (Waterloo).



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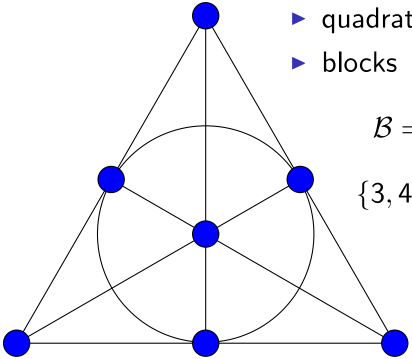

- Point set $X = \mathbb{Z}_7$
- quadratic residues $1, 2, 4$
- blocks

$$\mathcal{B} = \{\{0,1,3\},\{1,2,4\},\{2,3,5\},$$
$$\{3,4,6\},\{4,5,0\},\{5,6,1\},\{6,0,2\}\}$$

Ideal of the Fano plane

$$\mathcal{B} = \{\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,0\},\{5,6,1\},\{6,0,2\}\}$$

with corresponding incidence vectors

$$\mathbf{c} = [1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0] \qquad \textit{etc.}$$

The unique triple in $\mathcal{B}$ containing both 0 and 1 also contains 3.
Including the quadratic polynomial $x_0x_1 - x_0x_3$ in a generating set $\mathcal{G}$ for our ideal also guarantees that any vector $\mathbf{c} \in \mathcal{Z}(\langle \mathcal{G} \rangle)$ with $\mathbf{c}_0 = 1$ and $\mathbf{c}_1 = 1$ must have $\mathbf{c}_3 = 1$ as well.
Up to sign, there are $\binom{7}{2}$ quadratic generators of this form.
If we also include generators to ensure every zero is a 01-vector with entries summing to three, these generate the full ideal.

View design as set of 01-vectors and study ideal

- Consider a k-uniform hypergraph $(X, \mathcal{B})$ with
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- identify each block $B \in \mathcal{B}$ with a 01-vector $\mathbf{c}_B$, with entries indexed by the elements of X, whose i^{th} entry is equal to one if $i \in B$ and equal to zero otherwise

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- **GOAL:** Study $\mathsf{I} = \mathcal{I}(\mathcal{B})$, the ideal of all polynomials that vanish at every point $\mathbf{c}_B$.

Ideal-variety correspondence

For $\mathcal{S} \subseteq \mathbb{C}^v$, we let

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Nullstellensatz: For any ideal J of polynomials,

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where $\mathrm{Rad}(\mathrm{J})$ denotes the *radical* of ideal J, the ideal of all polynomials g such that $g^n \in \mathrm{J}$ for some positive integer n.

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Radical ideal: already closed under this process: $\mathrm{Rad}(\mathrm{J}) = \mathrm{J}$.

The trivial ideal of the complete uniform hypergraph

First example: *complete uniform hypergraph* $(X, \mathcal{K}_k^v)$.

Lemma A: Let X be a finite set of size $v \geq k$ and let $\mathcal{K}_k^v = \binom{X}{k}$ consist of all k-subsets of X. Let

$$\mathcal{G}_0 = \{x_1 + \cdots + x_v - k\} \cup \{x_i^2 - x_i \mid 1 \leq i \leq v\}. \qquad (1)$$

Then $\mathcal{I}(\mathcal{K}_k^v) = \langle \mathcal{G}_0 \rangle$ and $\mathcal{Z}(\langle \mathcal{G}_0 \rangle) = \{\mathbf{c}_B \mid B \in \mathcal{K}_k^v\}$.

Trivial ideal: $\mathcal{T} = \langle x_1 + \cdots + x_v - k,\ x_1^2 - x_1, \ldots, x_v^2 - x_v \rangle$
All of our generating sets will contain $\mathcal{G}_0$.
[There is also a natural notion of "trivial ideal" for spherical codes, binary codes, etc.]

Monomials and t-designs

To each $C \subseteq \{1, 2, \ldots, v\}$ associate the monomial

$$x^C = \prod_{j \in C} x_j$$

For block $B \in \mathcal{B}$, the value of x^C at point $\mathbf{c}_B$ is one if $C \subseteq B$ and zero otherwise.

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A k-uniform hypergraph $(X, \mathcal{B})$ is a t-(v, k, λ) *design* (or a block design of *strength* t) if, for every t-element subset $T \subseteq X$ of points, there are exactly λ blocks $B \in \mathcal{B}$ with $T \subseteq B$

Every t-(v, k, λ) design is an s-(v, k, λ_s) design for each $s \leq t$ where $\lambda_s \binom{k-s}{t-s} = \lambda \binom{v-s}{t-s}$.

Designs approximate a space w.r.t. polynomial test functions

The following characterization of t-designs is well-known.

Lemma B: (Delsarte) Let X be a set of size v and let $(X, \mathcal{B})$ be a k-uniform hypergraph defined on X with corresponding vectors $\mathbf{c}_B$ $(B \in \mathcal{B})$ as defined above. Then $(X, \mathcal{B})$ is a t-design on X if and only if the average over $\mathcal{B}$ of any polynomial $f(\mathbf{x})$ in v variables of total degree at most t is equal to the average of $f(\mathbf{x})$ over the complete uniform hypergraph $\mathcal{K}_k^v$ defined on X.

Two fundamental parameters

Definitions: Let $(X, \mathcal{B})$ be a non-empty, non-complete k-uniform hypergraph on vertex set $X = \{1, \ldots, v\}$ with corresponding ring of polynomials $\mathbb{C}[\mathbf{x}]$. Let $\mathcal{I}(\mathcal{B})$ and $\mathcal{T}$ be defined as above. Define

$$\gamma_1(\mathcal{B}) = \min \{\deg f \mid f \in \mathcal{I}(\mathcal{B}),\ f \notin \mathcal{T}\}$$

and

$$\gamma_2(\mathcal{B}) = \min \{\max\{\deg f : f \in \mathcal{G}\} \mid \mathcal{G} \subseteq \mathbb{C}[\mathbf{x}],\ \langle \mathcal{G} \rangle = \mathcal{I}(\mathcal{B})\}.$$

Two fundamental parameters

So $\gamma_1(\mathcal{B})$ is the smallest possible degree of a non-trivial polynomial that vanishes on each block and $\gamma_2(\mathcal{B})$ is the smallest integer s such that $\mathcal{I}(\mathcal{B})$ admits a generating set all polynomials of which have degree at most s.

Obviously, $\gamma_1(\mathcal{B}) \leq \gamma_2(\mathcal{B})$; designs satisfying equality here are particularly interesting.

Large t implies all low-degree polynomials are trivial

Theorem C: If $(X, \mathcal{B})$ is a t-design ($t \geq 2$) and $f \in \mathcal{I}(\mathcal{B})$ is non-trivial, then $\deg f > t/2$. So, for any non-trivial t-design $(X, \mathcal{B})$, $\gamma_1(\mathcal{B}) \geq (t+1)/2$.

Theorem: For any k-uniform hypergraph $(X, \mathcal{B})$, $\gamma_2(\mathcal{B}) \leq k$.

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Since f^2 is everywhere non-negative, it must evaluate to zero on the incidence vector $\mathbf{c}_B$ of every k-set B. So it belongs to the ideal $\mathcal{I}(\mathcal{K}_k^v)$. Since this ideal is radical and contains f^2, it also contains f. By Lemma A, f must be trivial. The same argument applies to g and, hence, to F.

For Steiner systems, $t/2 < \gamma_1 \leq \gamma_2 \leq t$

A *Steiner system* is a t-(v, k, λ) design with $\lambda = 1$. The question of existence of non-trivial Steiner systems with $t > 5$ has been recently resolved in spectacular fashion by Keevash.

Theorem: Let $(X, \mathcal{B})$ be any t-$(v, k, 1)$ design. For a block $B \in \mathcal{B}$ and any two t-element subsets T, T' contained in B, define

$$g_{T,T'}(\mathbf{x}) = x^T - x^{T'}.$$

where $x^T = \prod_{i \in T} x_i$. Then

(i) $\mathcal{I}(\mathcal{B})$ is generated by $\mathcal{G}_0 \cup \{g_{T,T'}(\mathbf{x}) : B \in \mathcal{B}, T, T' \subseteq B, |T| = |T'| = t\}$;

(ii) $\gamma_2(\mathcal{B}) \leq t$.

Upper Bounds

Theorem: For any partial t-$(v,k,1)$-design $(X,\mathcal{B})$, $\gamma_2(\mathcal{B}) \le t$.

Corollary: Let $(X,\mathcal{B})$ be a t-(v,k,λ) design with $|B \cap B'| < s$ for every pair B, B' of distinct blocks. Then $\gamma_2(\mathcal{B}) \le s$. $\square$

For symmetric 2-designs, all functions are linear

Theorem: Let $(X, \mathcal{B})$ be any non-trivial symmetric 2-(v, k, λ) design. For each pair i, j of distinct points from X define

$$f_{i,j}(\mathbf{x}) = (k - \lambda)x_i x_j - \sum_{i,j \in B \in \mathcal{B}} x^{B,1} + \lambda^2.$$

where $x^{B,1} = \sum_{i \in B} x_i$. Then

(i) $\mathcal{I}(\mathcal{B})$ is generated by $\mathcal{G}_0 \cup \{f_{i,j} \mid i, j \in X\}$;

(ii) $\gamma_1(\mathcal{B}) = \gamma_2(\mathcal{B}) = 2$;

(iii) the coordinate ring $\mathbb{C}[\mathbf{x}]/\mathcal{I}(\mathcal{B})$ admits a basis consisting of cosets $\{x_i + \mathcal{I}(\mathcal{B}) \mid 1 \leq i \leq v\}$.

We have a similar result when $(X, \mathcal{B})$ consists of the points and e-dimensional subspaces of $\mathrm{PG}(d, q)$

Parameter values for Witt designs

Certain orbits of the Mathieu groups provide elegant examples of t-designs.

t-(v,k,λ)	$\gamma_1(\mathcal{B})$	$\gamma_2(\mathcal{B})$
5-$(24,8,1)$	3	3
4-$(23,7,1)$	3	3
3-$(22,6,1)$	2	2
2-$(21,5,1)$	2	2
5-$(12,6,1)$	3	3
4-$(11,5,1)$	3	3
3-$(10,4,1)$	2	2
2-$(9,3,1)$	2	2

The unique 5-$(24, 8, 1)$ design

Theorem: Let $(X, \mathcal{B})$ be the 5-$(24, 8, 1)$ design. For a block $B \in \mathcal{B}$ and points $i, j \in B$, define

$$f_{B,i,j}(\mathbf{x}) = (x_i - x_j)(\mathbf{c}_B \cdot \mathbf{x} - 2)(\mathbf{c}_B \cdot \mathbf{x} - 4).$$

Then

(i) $\mathcal{I}(\mathcal{B})$ is generated by $\mathcal{G}_0 \cup \{f_{B,i,j} \mid i, j \in B \in \mathcal{B}\}$;

(ii) $\gamma_1(\mathcal{B}) = \gamma_2(\mathcal{B}) = 3$.

The Icosahedron

A *spherical code* is a finite subset of the unit sphere S^{m-1} in $\mathbb{R}^m$.

Q: Which polynomials vanish on the 12 vertices of the icosahedron?

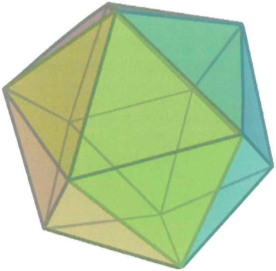

Image Credit:
https://en.wikipedia.org/wiki/Regular_icosahedron

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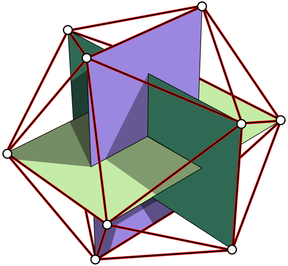

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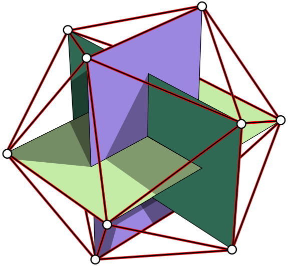

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$F(x_1, x_2, x_3) = x_1 x_2 x_3$ vanishes on all

$$(\pm 1, \pm\phi, 0), \quad (0, \pm 1, \pm\phi), (\pm\phi, 0, \pm 1)$$

Ideals for Spherical Designs

Here, the *trivial ideal* is $\mathcal{T} = \langle x_1^2 + \cdots + x_m^2 - 1 \rangle$ and we define $\gamma_1(X)$ and $\gamma_2(X)$ similarly.

For a spherical t-design, we have $\gamma_1(X) \geq t/2$.

If X is the set of vertices of the icosahedron, then $\mathcal{I}(X)$, the ideal of all polynomials that vanish on X, is generated by the equation of the sphere together with five cubics of the above form.

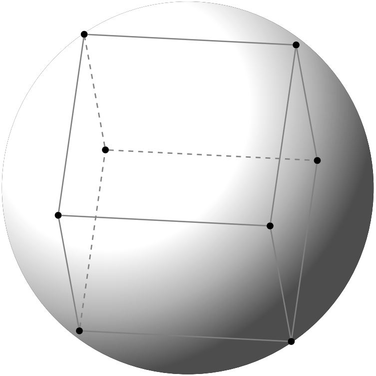
The Icosahedron and Famous Lattices

We can use "sliced zonal polynomials" to generate $\mathcal{I}(X)$ in these cases:

Name	$\|X\|$	Dim	strength	$\gamma_1(X)$	$\gamma_2(X)$
icos.	12	3	5	3	3
E_6	72	6	5	3	3
E_7	126	7	5	3	3
E_8	240	8	7	4	4
Leech	196560	24	11	6	6

(joint with Corre Love Steele arXiv:1310.6626)

Example - the cube



Gram Matrix

These eight vectors

$$\left(\pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}}\right)$$

have Gram matrix (pairwise inner products)

$$G = \frac{1}{3}\left[\begin{array}{rrrr|rrrr} 3 & 1 & 1 & -1 & 1 & -1 & -1 & -3 \\ 1 & 3 & -1 & 1 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & -1 & -3 & 1 & -1 \\ -1 & 1 & 1 & 3 & -3 & -1 & -1 & 1 \\ \hline 1 & -1 & -1 & -3 & 3 & 1 & 1 & -1 \\ -1 & 1 & -3 & -1 & 1 & 3 & -1 & 1 \\ -1 & -3 & 1 & -1 & 1 & -1 & 3 & 1 \\ -3 & -1 & -1 & 1 & -1 & 1 & 1 & 3 \end{array}\right]$$

Gram Matrix

The entrywise square of G

$$G^{\circ 2} = G \circ G = \frac{1}{9} \left[\begin{array}{cccc|cccc} 9 & 1 & 1 & 1 & 1 & 1 & 1 & 9 \\ 1 & 9 & 1 & 1 & 1 & 1 & 9 & 1 \\ 1 & 1 & 9 & 1 & 1 & 9 & 1 & 1 \\ 1 & 1 & 1 & 9 & 9 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 9 & 9 & 1 & 1 & 1 \\ 1 & 1 & 9 & 1 & 1 & 9 & 1 & 1 \\ 1 & 9 & 1 & 1 & 1 & 1 & 9 & 1 \\ 9 & 1 & 1 & 1 & 1 & 1 & 1 & 9 \end{array} \right]$$

is also a Gram matrix (tetrahedron in $\mathbb{R}^4$). And $G(G \circ G) = 0$.

Multiplying Entrywise Powers

For the 3-cube, we have

$$GG = \frac{8}{3}G, \qquad G(G \circ G) = 0, \qquad G(G \circ G \circ G) = \frac{56}{27}G,$$

$$(G \circ G)(G \circ G) = \frac{8}{27}J + \frac{16}{9}G \circ G, \qquad (G \circ G)(G \circ G \circ G) = 0,$$

$$(G \circ G \circ G)(G \circ G \circ G) = \frac{56}{243}G + \frac{16}{9}G \circ G \circ G$$

So the vector space spanned by $J, \; G, \; G \circ G, \; G \circ G \circ G$ is closed under matrix multiplication!
This is special:

DRG Duals

Let $X \subset S^{m-1}$ be a spherical code in $\mathbb{R}^m$ with Gram matrix $G = [\mathbf{x} \cdot \mathbf{y}]_{\mathbf{x},\mathbf{y} \in X}$.

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DRG Duals

Let $X \subset S^{m-1}$ be a spherical code in $\mathbb{R}^m$ with Gram matrix $G = [\mathbf{x} \cdot \mathbf{y}]_{\mathbf{x},\mathbf{y} \in X}$.
Since G is positive semidefinite, $G \circ G \succeq 0$ as well, and $G \circ G \circ G \succeq 0$, etc.
Suppose only s angles occur between pairs of distinct vectors in X.
We say X is a "**DRG dual**" if the vector space

$$\mathrm{span}\left(J, G, G \circ G, \ldots, \underbrace{G \circ \cdots \circ G}_{s \text{ times}} \right)$$

is closed under matrix multiplication.

Association Schemes

- all DRGs are association schemes (P-polynomial a.s.)
- all “DRG duals” are association schemes (Q-polynomial a.s.)

A symmetric association scheme can be thought of as a highly regular coloring of the edges of the complete graph ...

Association Schemes

- all DRGs are association schemes (P-polynomial a.s.)
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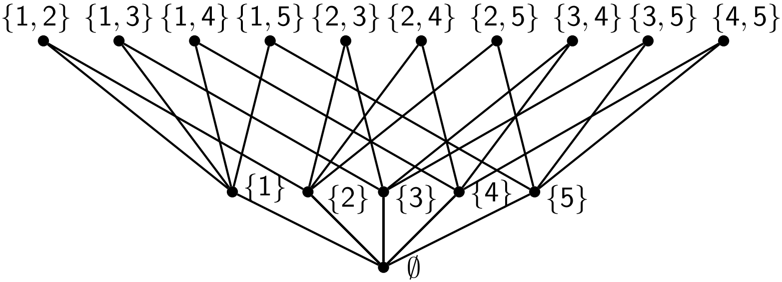
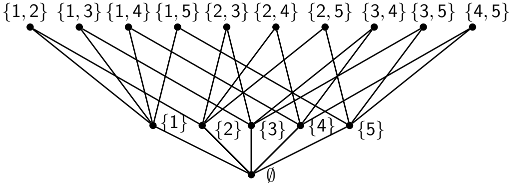
A symmetric association scheme can be thought of as a highly regular coloring of the edges of the complete graph ...
or as a vector space of symmetric matrices closed under both ordinary and entrywise multiplication, and containing the identities, I and J, for both.

Truncated Boolean Lattice (partially ordered set)



For $n = 5$, $\Omega = \{1, 2, 3, 4, 5\}$ and $k = 2$, we take all subsets of Ω of size at most k, ordered by inclusion.

Truncated Boolean Lattice (poset)

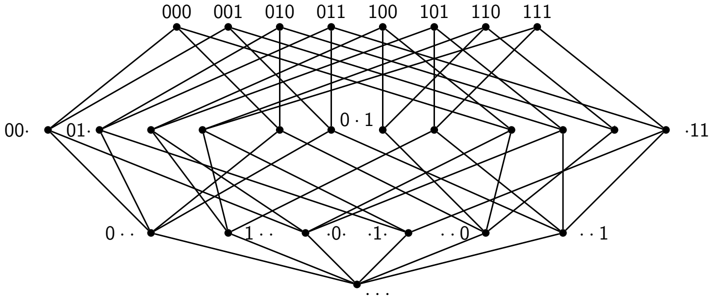


Incidence matrix:

1	1	1	1	0	0	0	0	0	0
1	0	0	0	1	1	1	0	0	0
0	1	0	0	1	0	0	1	1	0
0	0	1	0	0	1	0	1	0	1
0	0	0	1	0	0	1	0	1	1

X consists of 10 points in $\mathbb{R}^5$ and $\mathcal{I}(X)$ is generated by the obvious quadratics (trivial polynomials for designs).

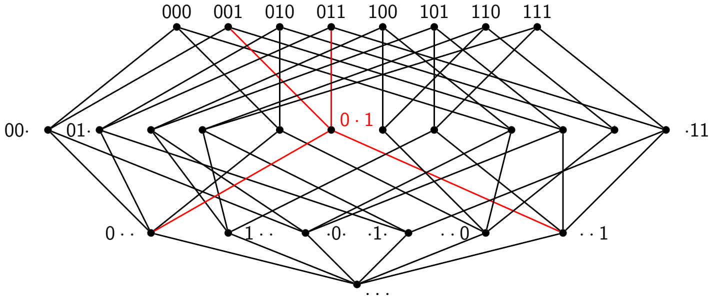
Hamming Lattice (poset)



For $n = 3$ and $q = 2$, we consider all "partial" n-tuples over $\mathbb{Z}_q$, marking unspecified entries with '$\cdot$'. Partial order relation is:

$$a \preceq b \text{ if } a_i = b_i \text{ whenever } a_i \neq \cdot$$

Hamming Lattice (poset)



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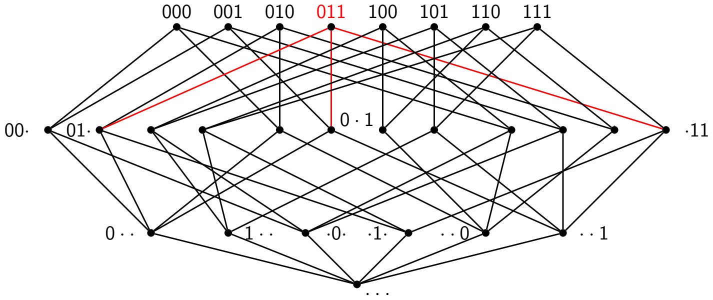

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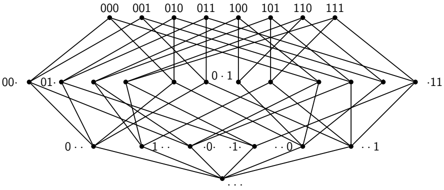


Incidence matrix:

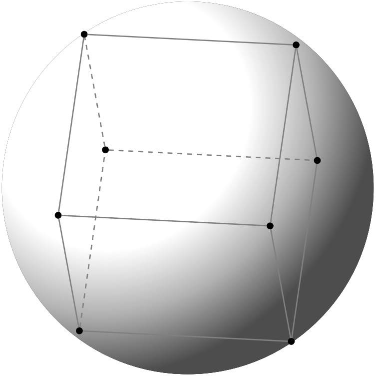
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

X consists of 8 points in $\mathbb{R}^6$ and $\mathcal{I}(X)$ is generated by trivial polynomials together with

$$Y_1 + Y_6 - 1, \quad Y_2 + Y_5 - 1, \quad Y_3 + Y_4 - 1.$$

Similar ideas work for the Grassmann scheme and the bilinear forms scheme.

The Ideal of the Cube



If, instead of looking at the poset, we go back to the Euclidean cube, $\{(\pm 1, \pm 1, \pm 1)\}$, we immediately see that

$$\mathcal{I}(X) = \langle x_1^2 - 1,\ x_2^2 - 1,\ x_3^2 - 1 \rangle$$

Conjectures

Dual DRGs

Thm 1 (Williford & WJM): For fixed $m > 2$, $\gamma_2(E_1)$ is bounded above by a function of m.
Conj 1 (WJM): For $m > 2$, $\gamma_1(E_1) \leq 6$.
Conj 2 (WJM): For $m > 2$, $\gamma_2(E_1) \leq 6$.
Thm 2: If $(X, \mathcal{R})$ is also P-polynomial with $m > 2$, then $\gamma_1(E_1) \leq 3$.
Conj 3: If $(X, \mathcal{R})$ is also P-polynomial with $m > 2$, then $\gamma_2(E_1) \leq 3$, with known exceptions.

distance-regular graphs

Thm 3 (Bang, et al.): For fixed $k > 2$, $g_2(\Gamma)$ is bounded above by a function of k.

Conj 4 (Suzuki): For $k > 2$, $g_1(\Gamma) \leq 12$.

Question: For $k > 2$, is $g_2(\Gamma) \leq 12$?

Thm (Lewis): If Γ is Q-polynomial with $k > 2$, then $g_1(\Gamma) \leq 6$.

Thm (Lewis): If Γ is Q-polynomial with $k > 2$, is $g_2(\Gamma) \leq 6$ or Γ is a pseudoquotient.

Duality

If Γ is a distance-regular graph defined on an abelian group X such that

$$a \sim b \;\Rightarrow\; a + x \sim b + x$$

for all $a, b, x \in X$, then the characters of G give us a DRG dual.

Duality

If Γ is a distance-regular graph defined on an abelian group X such that

$$a \sim b \;\Rightarrow\; a + x \sim b + x$$

for all $a, b, x \in X$, then the characters of G give us a DRG dual. And, in this case, closed walks of length k map to polynomials of degree $\lceil \frac{k}{2} \rceil$ in the ideal of the dual DRG.

Girth $g_1(\Gamma) > 4$ iff $a_1 = 0,\ c_2 = 1$
WHILE
$\gamma_1(X) > 2$ iff $a_1^* = 0,\ c_2^* = 2m_1/(m_1 + 2)$, etc.

The End

Thank you all!

Sparrow Dance, Sendai-shi Festival, Sunday, 20th May, Sun Mall