On enumeration of restricted permutations of genus zero

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hypermaps

A hypermap can be represented by a pair of permutations (σ, α) on $[n] := \{1, 2, \ldots, n\}$ that generate a transitive subgroup of the symmetric group $\mathfrak{S}_n$.

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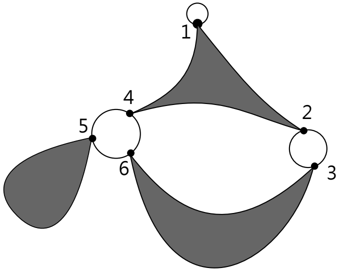

$\sigma = (1)(2,3)(4,5,6)$ (counterclockwise) – vertices
$\alpha = (1,2,4)(3,6)(5)$ (clockwise) – hyperedges
$\alpha^{-1}\sigma = (1,4,5,3)(2,6)$ – facse

genus of a hypermap

The genus of the hypermap (σ, α) is the nonnegative integer $g_{\sigma,\alpha}$ defined by the equation

$$n + 2 - 2g_{\sigma,\alpha} = z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma),$$

where $z(\sigma)$ is the number of cycles of σ.

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permutations		z
σ	$(1)(2,3)(4,5,6)$	3
α	$(1,2,4)(3,6)(5)$	3
$\alpha^{-1}\sigma$	$(1,4,5,3)(2,6)$	2

$$g_{\sigma,\alpha} = \frac{1}{2}(6 + 2 - 3 - 3 - 2) = 0.$$

hypermonopoles

A special case, called the hypermonopole, is a hypermap (σ, α) where σ is the n-cycle $\zeta_n = (1, 2, \ldots, n)$.

The genus g_α of a permutation α is defined as the genus of the hypermonopole (ζ_n, α), i.e.,

$$n + 1 - 2g_\alpha = z(\alpha) + z(\alpha^{-1}\zeta_n).$$

permutations		z
$\sigma = \zeta_7$	$(1, 2, 3, 4, 5, 6, 7)$	1
α	$(1, 2, 7)(3)(4, 5, 6)$	3
$\alpha^{-1}\zeta_7$	$(1)(2, 3, 6)(4)(5)(7)$	5

$$g_\alpha = \frac{1}{2}(7 + 1 - 3 - 5) = 0.$$

genus 0 permutations with restrictions

n	$\mathfrak{S}_n$	Alt_n	Der_n	Inv_n	Bax_n
2	2	1	1	2	2
3	5	1	1	4	5
4	14	3	3	9	14
5	42	3	6	21	42
6	132	11	15	51	132
7	429	11	36	127	429
	↑	↑	↑	↑	↑
	Catalan	Schröder	Riordan	Motzkin	Catalan

genus 1 Baxter permutations enumeration?

n	$\mathfrak{S}_n$	Bax_n
3	1	1
4	10	8
5	70	45
6	420	214
7	2310	941

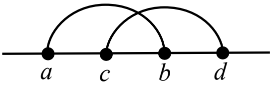
$\mathfrak{S}_n$: ↑ $\dfrac{(2n+3)!}{6(n+1)!n!}$ (Cori-Hetyei)

Bax_n: ↑ unknown

Enumeration of genus zero permutations with restrictions

noncrossing partitions

Two disjoint subsets B and B' of $[n]$ are crossing if there exist $a, b \in B$ and $c, d \in B'$ such that $a < c < b < d$. Otherwise, B and B' are noncrossing.



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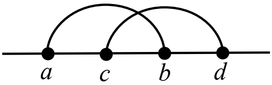

A noncrossing partition of $[n]$ is a set partition of $[n]$, denoted by $\{B_1, B_2, \ldots, B_k\}$, such that the *blocks* B_j are pairwise noncrossing.

a characterization of genus zero perm.

Theorem (Cori 1975)

Let $\alpha \in \mathfrak{S}_n$. Then $g_\alpha = 0$ if and only if the cycle decomposition of σ gives a noncrossing partition of $[n]$, and each cycle of α is increasing.

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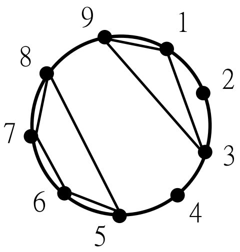
Theorem (Cori 1975)

Let $\alpha \in \mathfrak{S}_n$. Then $g_\alpha = 0$ if and only if the cycle decomposition of σ gives a noncrossing partition of $[n]$, and each cycle of α is increasing.

For example, $\alpha = 3\,2\,9\,4\,6\,7\,8\,5\,1 = (1,3,9)(5,6,7,8)$ is associated with the following noncrossing partition.



sign-balance for genus 0 permutations

Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. The inversion number of σ is defined by

$$\mathsf{inv}(\sigma) := \#\{(\sigma_i, \sigma_j) : \sigma_i > \sigma_j, 1 \leq i < j \leq n\}.$$

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Let $\mathcal{G}_n \subset \mathfrak{S}_n$ be the subset consisting of the permutations of genus zero. We observe that

$$\sum_{\sigma \in \mathcal{G}_n} (-1)^{\mathsf{inv}(\sigma)} = \begin{cases} 0 & n \text{ even} \\ C_{\lfloor \frac{n}{2} \rfloor} & n \text{ odd}, \end{cases}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.

LRMin-distribution for genus 0 permutations

Let LRMin(σ) denote the number of *left-to-right minima* of σ, i.e.,

$$\text{LRMin}(\sigma) := \#\{\sigma_j : \sigma_i > \sigma_j, 1 \leq i < j\}.$$

The distribution of genus zero permutations w.r.t LRMin:

n	$\lvert\mathcal{G}_n\rvert$	1	2	3	4	5	6	7	8
1	1	1							
2	2	1	1						
3	5	2	2	1					
4	14	5	5	3	1				
5	42	14	14	9	4	1			
6	132	42	42	28	14	5	1		
7	429	132	132	90	48	20	6	1	

signed LRMin-distribution for $\mathcal{G}_n$

The sign-balance of LRMin-distribution for $\mathcal{G}_n$:

n	1	2	3	4	5	6	7	8	9
2	1	-1							
3	0	0	-1						
4	-1	1	-1	1					
5	0	0	1	0	1				
6	2	-2	2	-2	1	-1			
7	0	0	-2	0	-2	0	-1		
8	-5	5	-5	5	-3	3	-1	1	
9	0	0	5	0	5	0	3	0	1

a refined sign-balance result

Theorem (Eu-Fu-Pan-Ting 2018)

For all $n \geq 1$, the following identities hold.

1. $$\sum_{\sigma \in \mathcal{G}_{2n+1}} (-1)^{\mathit{inv}(\sigma)} q^{\mathit{LRMin}(\sigma)} = (-1)^n q \sum_{\sigma \in \mathcal{G}_n} q^{2 \cdot \mathit{LRMin}(\sigma)},$$
2. $$\sum_{\sigma \in \mathcal{G}_{2n}} (-1)^{\mathit{inv}(\sigma)} q^{\mathit{LRMin}(\sigma)} = (-1)^n \left(1 - \frac{1}{q}\right) \sum_{\sigma \in \mathcal{G}_n} q^{2 \cdot \mathit{LRMin}(\sigma)},$$

Dyck paths

Let $\mathcal{D}_n$ denote the set of Dyck paths of length n, i.e., the lattice paths from $(0,0)$ to (n,n), using $(0,1)$ step and $(1,0)$ step, that stays weakly above the line $y=x$.

For a Dyck path $\pi \in \mathcal{D}_n$, let

- $\mathrm{area}(\pi)$ = the number of unit squares enclosed by π and the line $y=x$,
- $\mathrm{fpeak}(\pi)$ = the height of the first peak of π.

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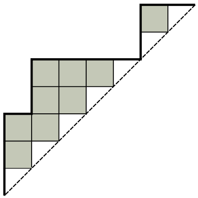


Figure: A Dyck path π with area$(\pi) = 9$ and fpeak$(\pi) = 3$.

a bijection between genus 0 permutations and Dyck paths

Theorem (Stump 2013)

There is a bijection $\phi : \sigma \to \pi$ *of* $\mathcal{G}_n$ *onto* $\mathcal{D}_n$ *such that*

1. $\mathit{area}(\pi) = \mathit{inv}(\sigma)$,
2. $\mathit{fpeak}(\pi) = \mathit{LRMin}(\sigma)$.

a bijection between genus 0 permutations and Dyck paths

Theorem (Stump 2013)

There is a bijection $\phi : \sigma \to \pi$ of $\mathcal{G}_n$ onto $\mathcal{D}_n$ such that

1. *area*$(\pi) =$ *inv*(σ),
2. *fpeak*$(\pi) =$ *LRMin*(σ).

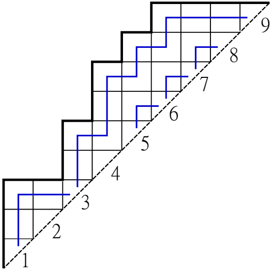


Figure: The corresponding Dyck path of $\sigma = (1, 3, 9)(5, 6, 7, 8)$.

even/odd peaks and valleys on Dyck paths

A peak/valley at (i, j) is said to be even (odd, respectively) if $i + j$ is even (odd, respectively).

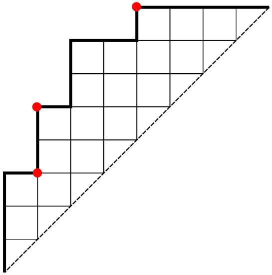


Figure: The red peaks/valleys are even..

a sign-reversing involution on Dyck paths

Establish an involution $\gamma : \pi \to \pi'$ on $\mathcal{D}_n$ by changing the last even peak (or valley) into a valley (or peak). Then

- $|\text{area}(\pi') - \text{area}(\pi)| = 1$,
- $\text{fpeak}(\pi') = \text{fpeak}(\pi)$.

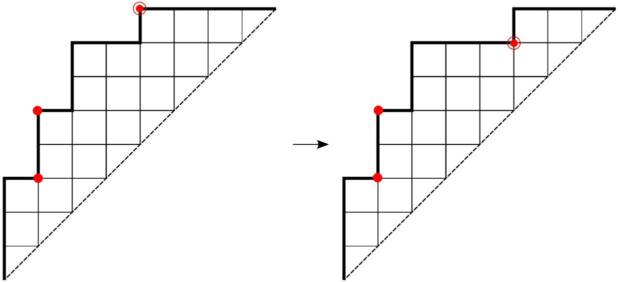
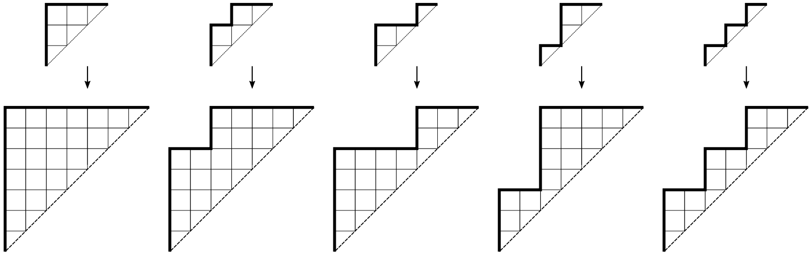


Figure: The map $\gamma : \pi \to \pi'$.

the fixed points of the map γ



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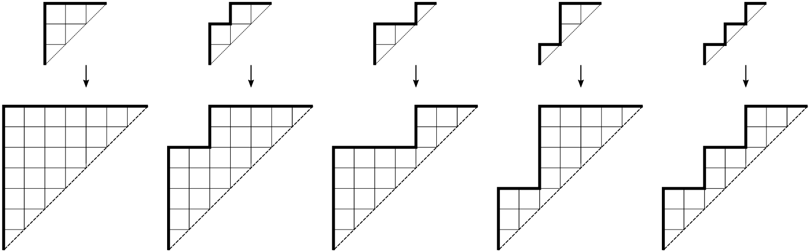

Let ferr(π) denote the number of unit squares above the Dyck path π within the $n \times n$ square, i.e., area$(\pi) = \frac{n(n-1)}{2} - $ ferr(π).

$$\gamma(\pi) = \pi \in \mathcal{D}_{2n+1} \to \text{ferr}(\pi) \text{ is even} \to \text{area}(\pi) \equiv n \pmod 2$$

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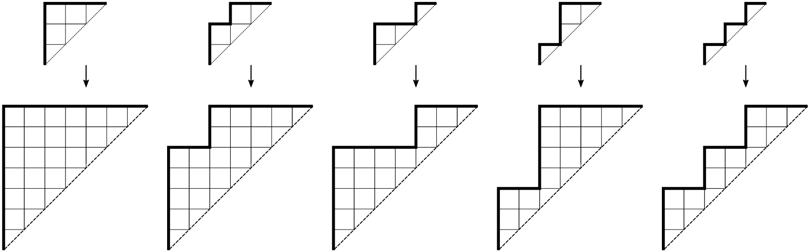
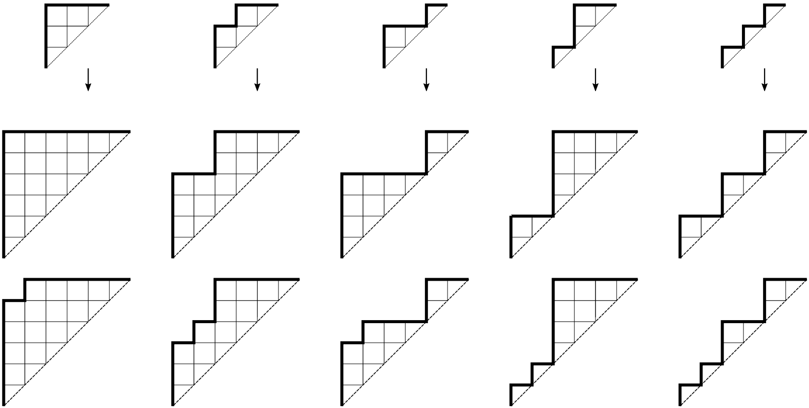
the fixed points of the map γ - odd case



$$\sum_{\sigma \in \mathcal{G}_7} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{LRMin}(\sigma)} = -q^7 - 2q^5 - 2q^3$$

$$= -q \sum_{\sigma \in \mathcal{G}_3} q^{2 \cdot \mathsf{LRMin}(\sigma)}.$$

the fixed points of the map γ - even case



$$\sum_{\sigma \in \mathcal{G}_6} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{LRMin}(\sigma)} = \left(-1 + \frac{1}{q}\right) \sum_{\sigma \in \mathcal{G}_3} q^{2 \cdot \mathsf{LRMin}(\sigma)}.$$

Thanks for your attention.