On enumeration of restricted permutations of genus zero

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Based on joint work with S.-P. Eu, Y.-J. Pan and C.-T. Ting

JCCA 2018, Sendai
A hypermap can be represented by a pair of permutations \((\sigma, \alpha)\) on \([n] := \{1, 2, \ldots, n\}\) that generate a transitive subgroup of the symmetric group \(S_n\).
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\[
\sigma = (1)(2, 3)(4, 5, 6) \quad \text{(counterclockwise)} \quad \text{– vertices}
\]

\[
\alpha = (1, 2, 4)(3, 6)(5) \quad \text{(clockwise)} \quad \text{– hyperedges}
\]

\[
\alpha^{-1}\sigma = (1, 4, 5, 3)(2, 6) \quad \text{– faces}
\]
The genus of the hypermap \((\sigma, \alpha)\) is the nonnegative integer \(g_{\sigma,\alpha}\) defined by the equation

\[
n + 2 - 2g_{\sigma,\alpha} = z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma),
\]

where \(z(\sigma)\) is the number of cycles of \(\sigma\).
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\]

where \(z(\sigma)\) is the number of cycles of \(\sigma\).

<table>
<thead>
<tr>
<th>permutations</th>
<th>(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma)</td>
<td>((1)(2, 3)(4, 5, 6))</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>((1, 2, 4)(3, 6)(5))</td>
</tr>
<tr>
<td>(\alpha^{-1}\sigma)</td>
<td>((1, 4, 5, 3)(2, 6))</td>
</tr>
</tbody>
</table>

\[
g_{\sigma, \alpha} = \frac{1}{2} (6 + 2 - 3 - 3 - 2) = 0.
\]
A special case, called the hypermonopole, is a hypermap \((\sigma, \alpha)\) where \(\sigma\) is the \(n\)-cycle \(\zeta_n = (1, 2, \ldots, n)\).

The genus \(g_\alpha\) of a permutation \(\alpha\) is defined as the genus of the hypermonopole \((\zeta_n, \alpha)\), i.e.,

\[
 n + 1 - 2g_\alpha = z(\alpha) + z(\alpha^{-1}\zeta_n).
\]

<table>
<thead>
<tr>
<th>permutations</th>
<th>(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma = \zeta_7) (1, 2, 3, 4, 5, 6, 7)</td>
<td>1</td>
</tr>
<tr>
<td>(\alpha = (1, 2, 7)(3)(4, 5, 6))</td>
<td>3</td>
</tr>
<tr>
<td>(\alpha^{-1}\zeta_7) (1)(2, 3, 6)(4)(5)(7)</td>
<td>5</td>
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</table>

\[
g_\alpha = \frac{1}{2} (7 + 1 - 3 - 5) = 0.
\]
### genus 0 permutations with restrictions

<table>
<thead>
<tr>
<th>$n$</th>
<th>$S_n$</th>
<th>$\text{Alt}_n$</th>
<th>$\text{Der}_n$</th>
<th>$\text{Inv}_n$</th>
<th>$\text{Bax}_n$</th>
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<td>1</td>
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<tr>
<td>4</td>
<td>14</td>
<td>3</td>
<td>3</td>
<td>9</td>
<td>14</td>
</tr>
<tr>
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<td>6</td>
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</tr>
<tr>
<td>6</td>
<td>132</td>
<td>11</td>
<td>15</td>
<td>51</td>
<td>132</td>
</tr>
<tr>
<td>7</td>
<td>429</td>
<td>11</td>
<td>36</td>
<td>127</td>
<td>429</td>
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</table>

↑↑↑↑↑

Catalan Schröder Riordan Motzkin Catalan
## Genus 1 Baxter Permutations Enumeration

<table>
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<tr>
<th>$n$</th>
<th>$S_n$</th>
<th>$Bax_n$</th>
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<td>1</td>
</tr>
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<td>8</td>
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<td>214</td>
</tr>
<tr>
<td>7</td>
<td>2310</td>
<td>941</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c}
\uparrow \\
\frac{(2n + 3)!}{6(n + 1)!n!} \\
\text{unknown}
\end{array}
\]

(Cori-Hetyei)
Enumeration of genus zero permutations with restrictions
Two disjoint subsets $B$ and $B'$ of $[n]$ are **crossing** if there exist $a, b \in B$ and $c, d \in B'$ such that $a < c < b < d$. Otherwise, $B$ and $B'$ are **noncrossing**.
Two disjoint subsets $B$ and $B'$ of $[n]$ are crossing if there exist $a, b \in B$ and $c, d \in B'$ such that $a < c < b < d$. Otherwise, $B$ and $B'$ are noncrossing.

A noncrossing partition of $[n]$ is a set partition of $[n]$, denoted by $\{B_1, B_2, \ldots, B_k\}$, such that the blocks $B_j$ are pairwise noncrossing.
Theorem (Cori 1975)

Let $\alpha \in S_n$. Then $g_\alpha = 0$ if and only if the cycle decomposition of $\sigma$ gives a noncrossing partition of $[n]$, and each cycle of $\alpha$ is increasing.
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Let $\alpha \in S_n$. Then $g_\alpha = 0$ if and only if the cycle decomposition of $\sigma$ gives a noncrossing partition of $[n]$, and each cycle of $\alpha$ is increasing.

For example, $\alpha = 3 \ 2 \ 9 \ 4 \ 6 \ 7 \ 8 \ 5 \ 1 = (1, 3, 9)(5, 6, 7, 8)$ is associated with the following noncrossing partition.
Let $\sigma = \sigma_1 \cdots \sigma_n \in S_n$. The inversion number of $\sigma$ is defined by

$$\text{inv}(\sigma) := \#\{(\sigma_i, \sigma_j) : \sigma_i > \sigma_j, 1 \leq i < j \leq n\}.$$
Let $\sigma = \sigma_1 \cdots \sigma_n \in S_n$. The inversion number of $\sigma$ is defined by

$$\text{inv}(\sigma) := \# \{(\sigma_i, \sigma_j) : \sigma_i > \sigma_j, 1 \leq i < j \leq n\}.$$ 

Let $\mathcal{G}_n \subset S_n$ be the subset consisting of the permutations of genus zero. We observe that

$$\sum_{\sigma \in \mathcal{G}_n} (-1)^{\text{inv}(\sigma)} = \begin{cases} 0 & \text{ if } n \text{ is even} \\ C_{\lfloor n/2 \rfloor} & \text{ if } n \text{ is odd}, \end{cases}$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$th Catalan number.
LRMin-distribution for genus 0 permutations

Let \( \text{LRMin}(\sigma) \) denote the number of \textit{left-to-right minima} of \( \sigma \), i.e.,

\[
\text{LRMin}(\sigma) := \# \{ \sigma_j : \sigma_i > \sigma_j, 1 \leq i < j \}.
\]

The distribution of genus zero permutations w.r.t LRMin:

| \( n \) | \( |G_n| \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|---------|----------|----|----|----|----|----|----|----|----|
| 1       | 1        | 1  |    |    |    |    |    |    |    |
| 2       | 2        | 1  | 1  |    |    |    |    |    |    |
| 3       | 5        | 2  | 2  | 1  |    |    |    |    |    |
| 4       | 14       | 5  | 5  | 3  | 1  |    |    |    |    |
| 5       | 42       | 14 | 14 | 9  | 4  | 1  |    |    |    |
| 6       | 132      | 42 | 42 | 28 | 14 | 5  | 1  |    |    |
| 7       | 429      | 132| 132| 90 | 48 | 20 | 6  | 1  |    |
The sign-balance of LRMin-distribution for $G_n$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<tbody>
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<td>1</td>
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<td>0</td>
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<td>1</td>
</tr>
</tbody>
</table>
Theorem (Eu-Fu-Pan-Ting 2018)

For all \( n \geq 1 \), the following identities hold.

1. \[
\sum_{\sigma \in G_{2n+1}} (-1)^{\text{inv}(\sigma)} q^{LRMin(\sigma)} = (-1)^n q \sum_{\sigma \in G_n} q^{2 \cdot LRMin(\sigma)},
\]

2. \[
\sum_{\sigma \in G_{2n}} (-1)^{\text{inv}(\sigma)} q^{LRMin(\sigma)} = (-1)^n \left(1 - \frac{1}{q}\right) \sum_{\sigma \in G_n} q^{2 \cdot LRMin(\sigma)},
\]
Let $D_n$ denote the set of Dyck paths of length $n$, i.e., the lattice paths from $(0, 0)$ to $(n, n)$, using $(0, 1)$ step and $(1, 0)$ step, that stays weakly above the line $y = x$.

For a Dyck path $\pi \in D_n$, let

- $\text{area}(\pi) =$ the number of unit squares enclosed by $\pi$ and the line $y = x$,
- $\text{fpeak}(\pi) =$ the height of the first peak of $\pi$. 
Let $\mathcal{D}_n$ denote the set of **Dyck paths** of length $n$, i.e., the lattice paths from $(0, 0)$ to $(n, n)$, using $(0, 1)$ step and $(1, 0)$ step, that stays weakly above the line $y = x$.

For a Dyck path $\pi \in \mathcal{D}_n$, let

- $\text{area}(\pi) =$ the number of unit squares enclosed by $\pi$ and the line $y = x$,
- $f_{\text{peak}}(\pi) =$ the height of the first peak of $\pi$.

**Figure**: A Dyck path $\pi$ with $\text{area}(\pi) = 9$ and $f_{\text{peak}}(\pi) = 3$. 
Theorem (Stump 2013)

There is a bijection $\phi : \sigma \rightarrow \pi$ of $G_n$ onto $D_n$ such that

1. $\text{area}(\pi) = \text{inv}(\sigma)$,
2. $\text{fpeak}(\pi) = \text{LRMin}(\sigma)$. 
There is a bijection \( \phi : \sigma \rightarrow \pi \) of \( G_n \) onto \( D_n \) such that
1. \( \text{area}(\pi) = \text{inv}(\sigma) \),
2. \( \text{fpeak}(\pi) = \text{LRMin}(\sigma) \).

**Figure:** The corresponding Dyck path of \( \sigma = (1, 3, 9)(5, 6, 7, 8) \).
A peak/valley at \((i, j)\) is said to be **even** (**odd**, respectively) if \(i + j\) is even (odd, respectively).

**Figure:** The red peaks/valleys are even..
Establish an involution $\gamma : \pi \to \pi'$ on $\mathcal{D}_n$ by changing the last even peak (or valley) into a valley (or peak). Then

- $|\text{area}(\pi') - \text{area}(\pi)| = 1$,
- $f_{\text{peak}}(\pi') = f_{\text{peak}}(\pi)$.

Figure: The map $\gamma : \pi \to \pi'$. 
the fixed points of the map $\gamma$
Let \( \text{ferr}(\pi) \) denote the number of unit squares above the Dyck path \( \pi \) within the \( n \times n \) square, i.e., \( \text{area}(\pi) = \frac{n(n-1)}{2} - \text{ferr}(\pi) \).

\[
\gamma(\pi) = \pi \in \mathcal{D}_{2n+1} \rightarrow \text{ferr}(\pi) \text{ is even} \rightarrow \text{area}(\pi) \equiv n \pmod{2}
\]
the fixed points of the map $\gamma$ - odd case

$$\sum_{\sigma \in \mathcal{G}_7} (-1)^{\text{inv}(\sigma)} q^{\text{LRMin}(\sigma)} = -q^7 - 2q^5 - 2q^3$$

$$= -q \sum_{\sigma \in \mathcal{G}_3} q^{2 \cdot \text{LRMin}(\sigma)}.$$
the fixed points of the map $\gamma$ - even case

$$\sum_{\sigma \in G_6} (-1)^{\text{inv}(\sigma)} q^{LR\text{Min}(\sigma)} = \left(-1 + \frac{1}{q}\right) \sum_{\sigma \in G_3} q^{2 \cdot LR\text{Min}(\sigma)}.$$
Thanks for your attention.