Degree conditions for partitioning graphs into chorded cycles

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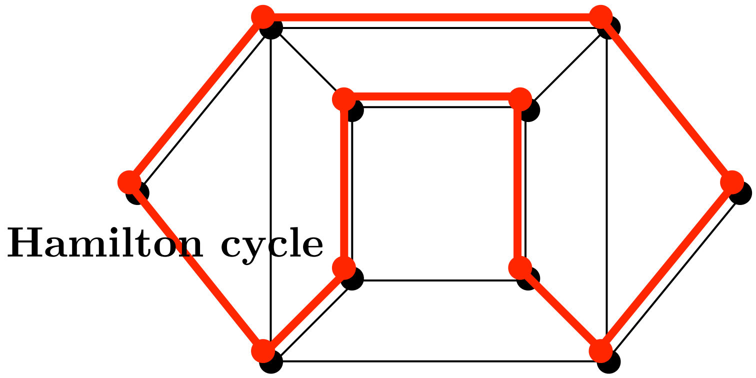
Purpose of this study

We give sharp degree sum conditions for partitioning graphs into a prescribed number of "chorded cycles", and we show the difference between cycles and chorded cycles in terms of sharp degree conditions.

— G : graph of order n —

* **Hamilton cycle** of G

$\overset{\text{def.}}{\Longleftrightarrow}$ cycle of G containing all vertices



Major study :

1. "better" sufficient conditions

2. "relaxed" structures

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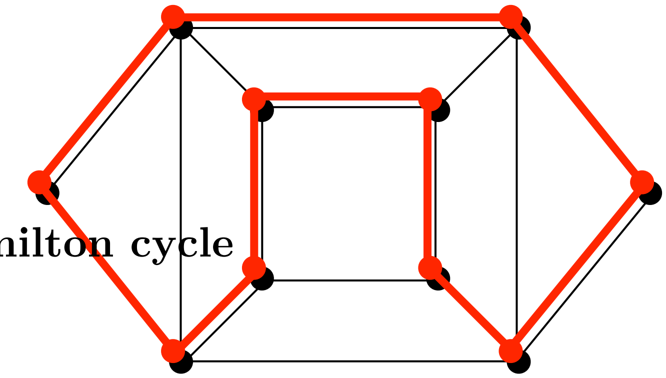

Major study :

1. "better" **degree conditions**
2. "relaxed" structures

(Dirac 1952)

$$n \geq 3, \ \ \delta(G) \geq n/2 \quad \Rightarrow \quad \exists \text{Hamilton cycle}$$

(Ore 1960)

$$n \geq 3, \ \ \sigma_2(G) \geq n \quad \Rightarrow \quad \exists \text{Hamilton cycle}$$

$* \ \delta(G) = \min\{d_G(x) : x \in V(G)\}$

$* \ \sigma_2(G) = \min\{\underline{d_G(x)} + d_G(y) : x \underset{(\neq)}{,} y \in V(G), xy \notin E(G)\}$

degree of x

$(\sigma_2(G) = +\infty$ if G : complete$)$

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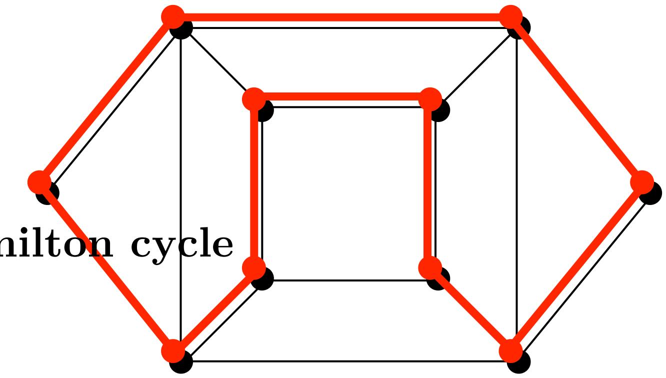

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1. "better" **degree conditions**
2. **partition into cycles**

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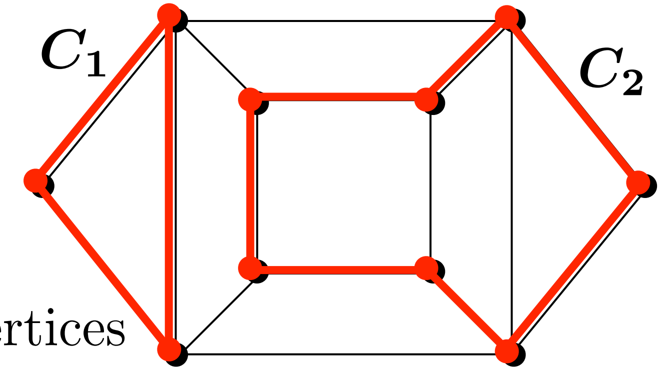

∗ **Partition** of G **into (k) cycles**

$\overset{\text{def.}}{\Longleftrightarrow}$ (k) disjoint cycles in G containing all vertices

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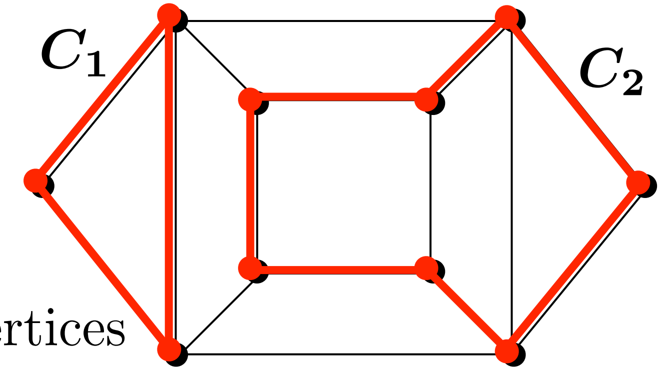

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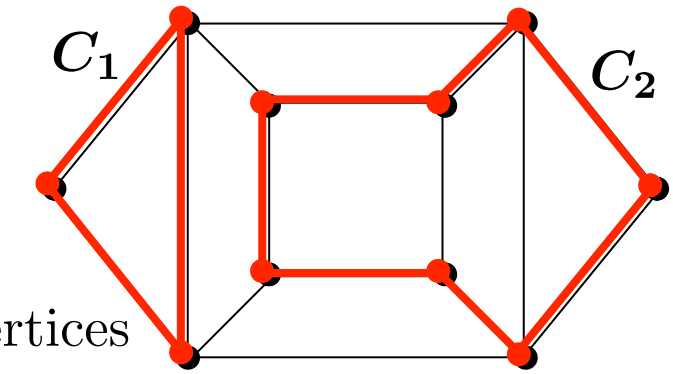

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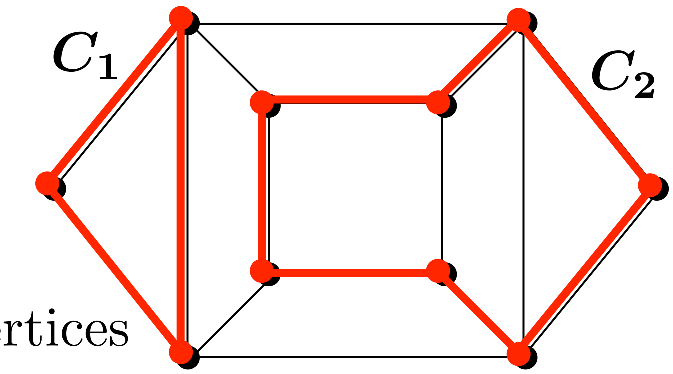

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(Brandt et al. 1997)

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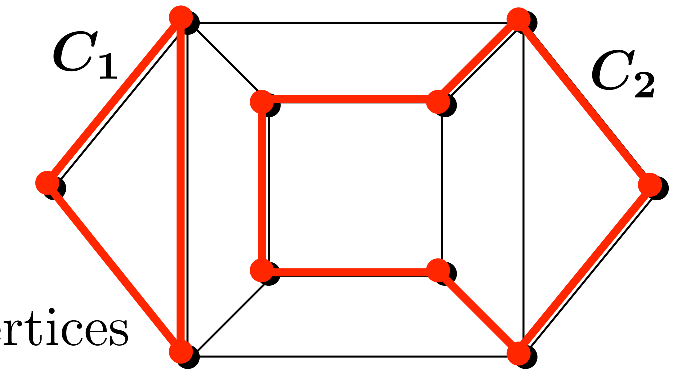

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Direction of study :

partitions into ***k* cycles** with "additional conditions"

$*$ containing **pre-specified edges**

$*$ containing **pre-specified vertices**

$*$ **length** constraints etc.

[CY] S. Chiba, T. Yamashita, *Degree conditions for the existence of vertex-disjoint cycles and paths: a survey*, Graphs Combin. **34** (2018) 1–83.

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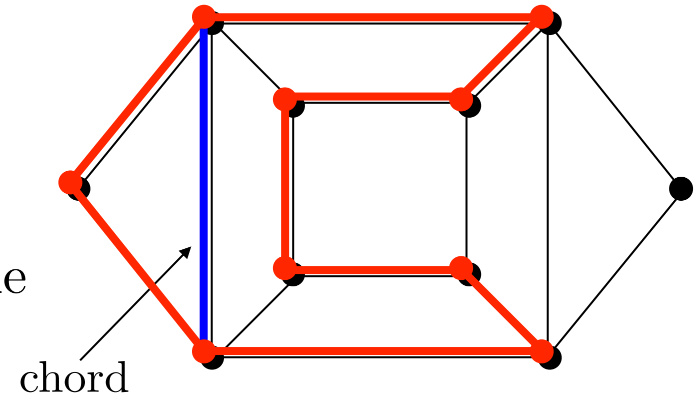
$\overset{\text{def.}}{\Longleftrightarrow}$ subgraph of G consisting of

a **cycle** and

edges joining two vertices of the cycle

↖ not cycle edges



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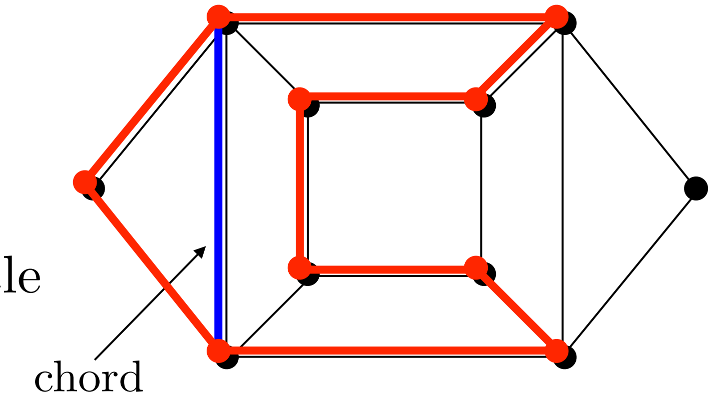

(Remark)

A chorded cycle is a relaxed structure of a complete graph

Complete graph of order t $=$ Chorded cycle of order t with $\frac{t(t-3)}{2}$ chords

$\#$ of chords

Few ⟷ Many

(Hamilton) cycle — complete graph

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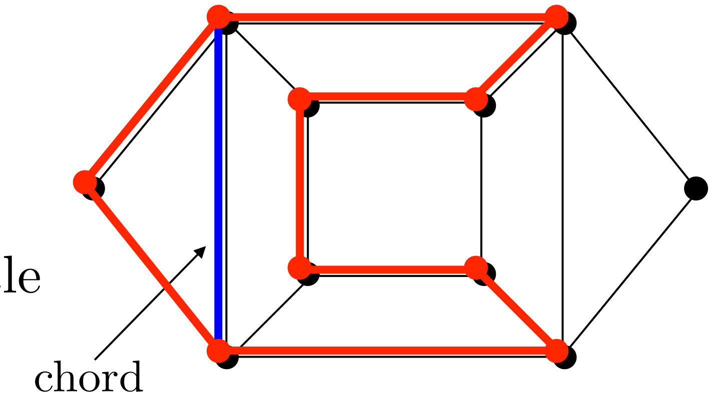

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We first consider the following simple problem

Prob. Determine sharp degree conditions for partitioning graphs into k chorded cycles

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$n \geq 4k-1$, $\sigma_2(G) \geq n$ $\Rightarrow$ $\exists$partition into k cycles

(Qiao, Zhang 2012)

Suppose $\exists s$ chorded cycles and $(k-s)$ cycles, all disjoint, in G

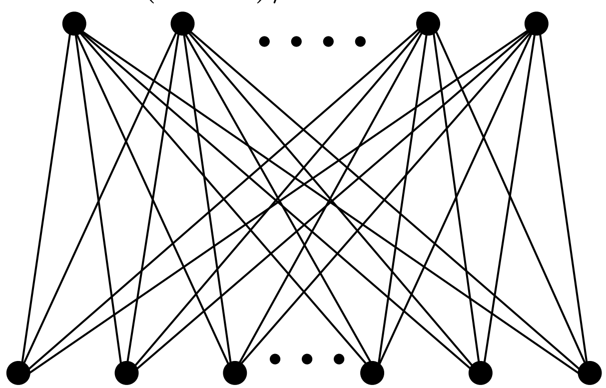
$\delta(G) \geq n/2$ $\Rightarrow$ $\exists$partition into $\underbrace{s \text{ chorded cycles and } (k-s) \text{ cycles}}_{k \text{ cycles}}$

(Remark)

δ condition is sharp

Exm.)

$K_{(n-1)/2,(n+1)/2}$



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Which is better?

conclusion	deg. condition	the rest
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Q2. Can we improve the conclusion in Brandt et al?

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Main Theorem 1

Suppose $\exists s$ chorded cycles and $(k-s)$ cycles, all disjoint, in G

$\boldsymbol{\sigma_2(G) \geq n} \Rightarrow \exists$partition into s chorded cycles and $(k-s)$ cycles

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Q2. Can we improve the conclusion in Brandt et al?

Main Theorem 2

$\boldsymbol{n \geq 4k+2s-1}$, $\sigma_2(G) \geq n$

$\Rightarrow$ **$\exists$partition into s chorded cycles and $(k-s)$ cycles**

(Brandt et al. 1997)

$n \geq 4k-1$, $\sigma_2(G) \geq n \quad \Rightarrow \quad \exists$partition into k cycles

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1. σ_2 condition is sharp
2. $n \geq 4k + 2s - 1$ is necessary

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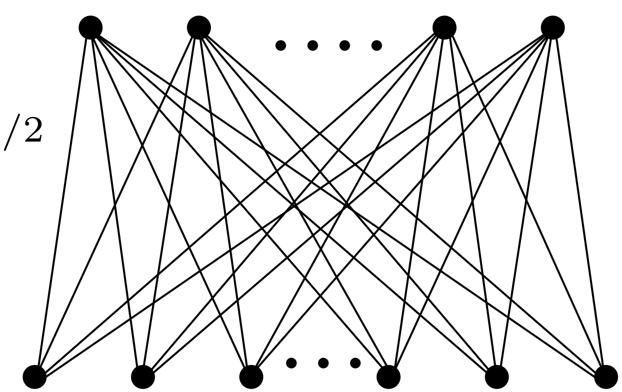

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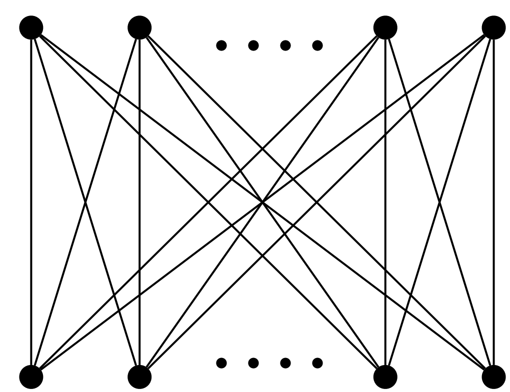

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Main Theorem 2

$\boldsymbol{n \geq 4k+2s-1}$, $\sigma_2(G) \geq n$

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(Remark)

Thm. 2 is obtained from Thm. 1 and the following theorem

(Chiba, Fujita, Gao, Li 2010)

$n \geq 3k+s$, $\sigma_2(G) \geq 4k+2s-1$ $\Rightarrow$ $(*)$

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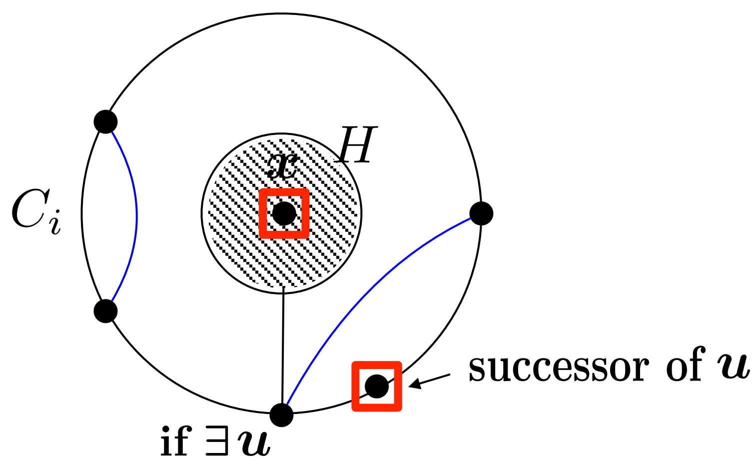

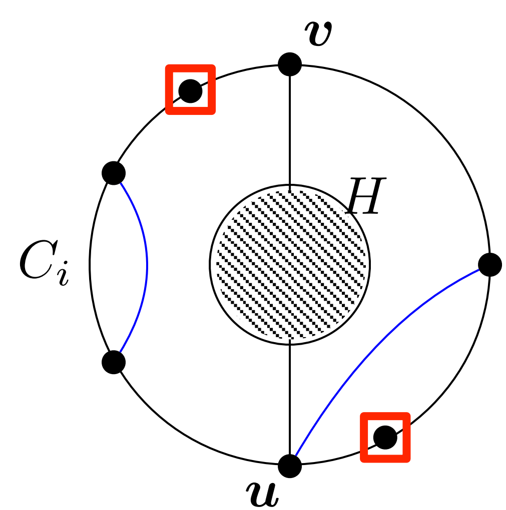

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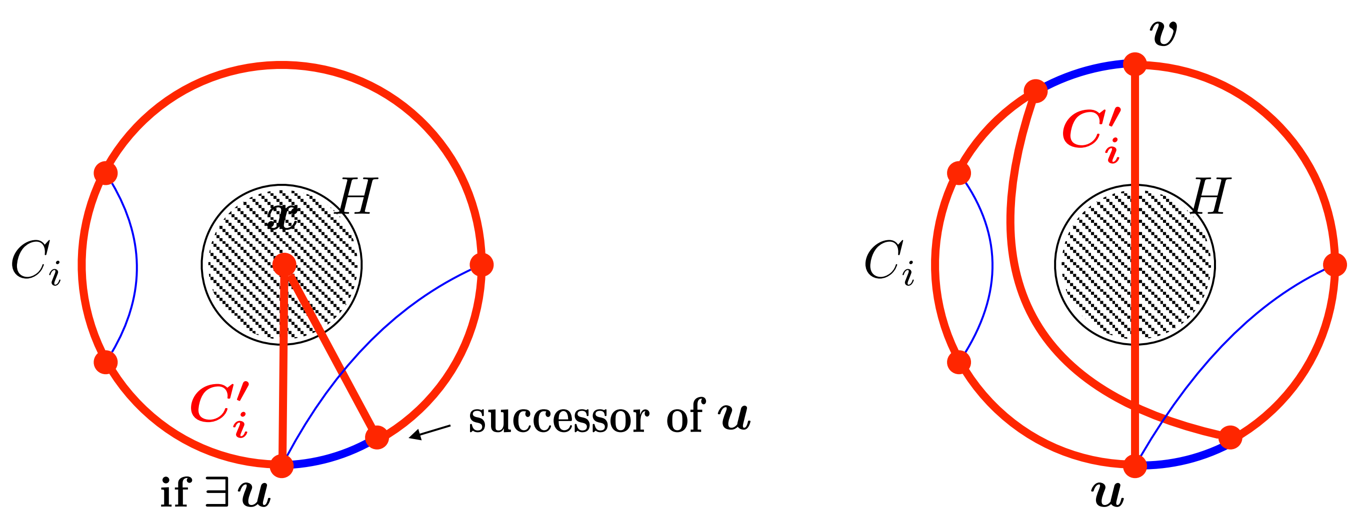

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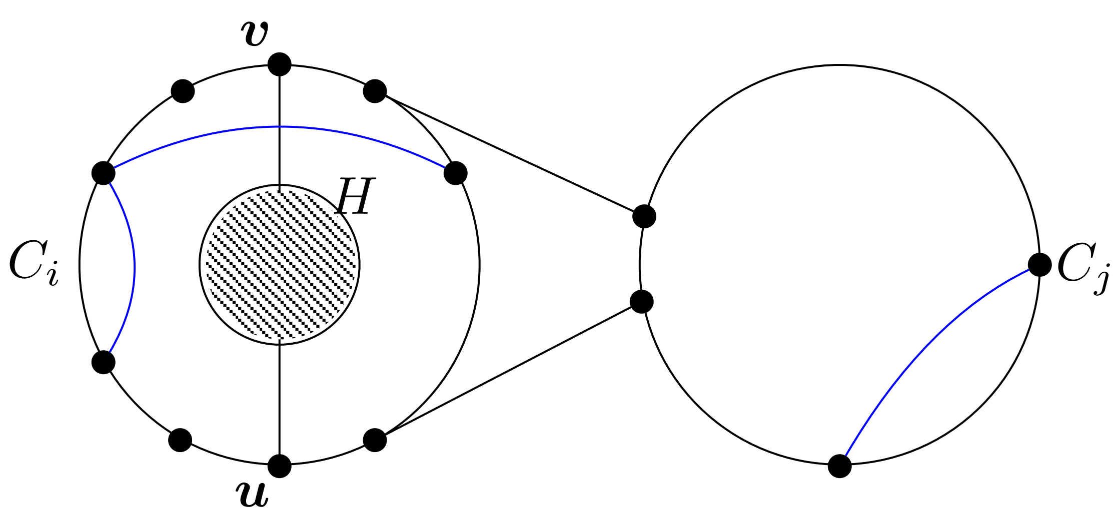

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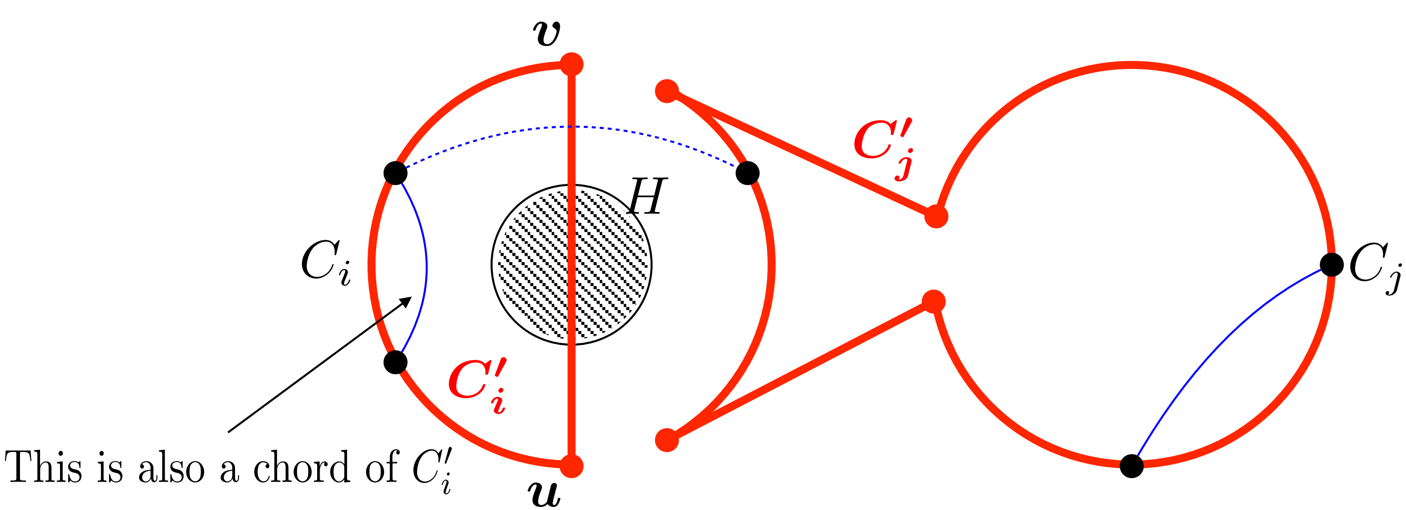

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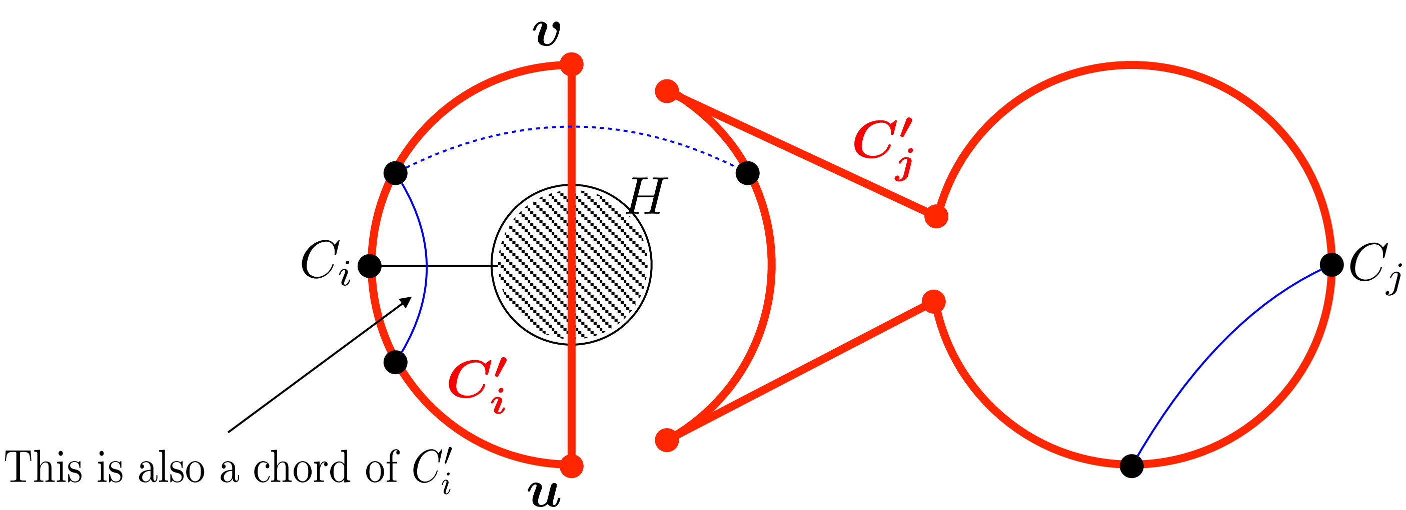

Prob.

Determine sharp deg conditions for partitioning graphs into k chorded cycles with c chords

Few ⟷ Many

# of chords	0	1	2	....	c	....
deg conditions	$\sigma_2 \geq n$ **Brandt et al.**	$\sigma_2 \geq n$ **Main Thms**	???	....	???	....

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⟨Known⟩ degree conditions for $\exists k$ disjoint chorded cycles with c chords

(it may not be a partition)

(Chen et al. 2015)

$$n \gg k, c,\ \ \delta(G) \geq \lceil \sqrt{c+1} + 1 \rceil k + 12 \cdot (9/2)^c \overset{(*)}{\Rightarrow} \exists k \text{ disjoint chorded cycles with } c \text{ chords}$$

(σ_2 condition depending on only k, c also implies $(*)$)

(Chiba, Lichiardopol 2018)

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$$n \gg k, c,\ \ \underline{\delta(G) \geq n/2 \ \ (\text{or } \sigma_2(G) \geq n)} \ \Rightarrow (*)$$

Q. This also implies the following?

The k disjoint chorded cycles can be transformed a partition of G

Thank you for your attention!