A nice partition function for reverse plane partitions derived from a discrete integrable system

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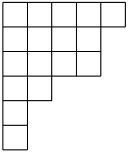
Partitions

A partition is an array $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of nonnegative integers such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell.$$

Example:

$(5, 4, 4, 2, 1, 1, 0) =$ 

Theorem

Let $\mathrm{Par}(a, b)$ denote the set of partitions whose diagrams are contained in an $a \times b$ box. Then

$$\sum_{\lambda \in \mathrm{Par}(a,b)} q^{|\lambda|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1 - q^{i+j}}{1 - q^{i+j-1}}$$

where $|\lambda| := \sum_i \lambda_i$ (size).

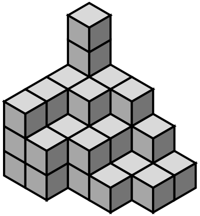
Plane partitions

A plane partition is a 2D array $\pi = (\pi_{i,j})$ of nonnegative integers such that

$$\pi_{i,j} \geq \pi_{i+1,j}, \qquad \pi_{i,j} \geq \pi_{i,j+1} \qquad \text{for } \forall(i,j).$$

Example:

$$\begin{pmatrix} 5 & 3 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 \end{pmatrix} =$$



Theorem (MacMahon)

Let $\mathrm{PP}(a,b,c)$ denote the set of plane partitions whose diagrams are contained in an $a \times b \times c$ box. Then

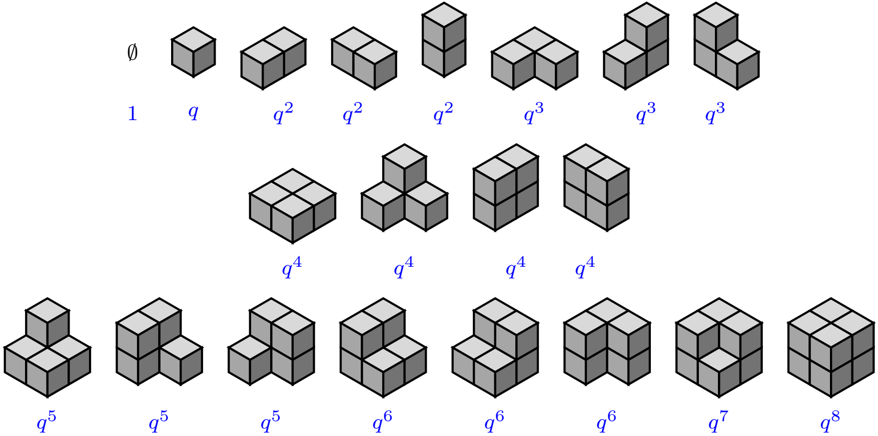
$$\sum_{\pi \in \mathrm{PP}(a,b,c)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

where $|\pi| := \sum_{i,j} \pi_i$ (size).

Example: For $a = b = c = 2$,

$$\sum_{\pi \in \mathrm{PP}(2,2,2)} q^{|\pi|} = 1 + q + 3q^2 + 3q^3 + 4q^4 + 3q^5 + 3q^6 + q^7 + q^8 = \frac{(1-q^4)^2(1-q^5)}{(1-q)(1-q^2)^2}$$

over 20 plane partitions contained in a $2 \times 2 \times 2$ box:



Trace generating function

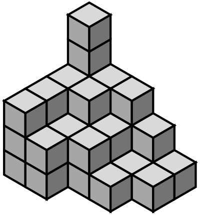
Theorem (Stanley[2])

Let $\mathrm{PP}(a,b,\infty) := \cup_{c=0}^{\infty}\mathrm{PP}(a,b,c)$, the set of plane partitions the bottoms of whose diagrams are contained in $a \times b$ box. Then

$$\sum_{\pi \in \mathrm{PP}(a,b,\infty)} y^{\mathrm{tr}(\pi)} q^{|\pi|} = \prod_{i=1}^{a}\prod_{j=1}^{b}(1 - yq^{i+j-1})^{-1}$$

where $\mathrm{tr}(\pi) = \sum_i \pi_{i,i}$ (trace) and $|\pi| := \sum_{i,j}\pi_{i,j}$ (size).

Example:

tr  $= \mathrm{tr}\begin{pmatrix} 5 & 3 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 \end{pmatrix} = 5 + 3 + 2 + 0 = 10.$

[2] R. P. Stanley, *Theory and application of plane partitions, I, II*, Studies in Appl. Math. **50** (1971), I: 167–188, II; 259–279.

Motivation

- Systematic way to find nice (product) formulas for plane partitions?
- Nice formula generalizing both MacMahon's formula and the Trace GF.?

??

$y = 1$ (→ MacMahon) $\qquad$ $c \to \infty$ (→ Trace GF.)

MacMahon

$$\sum_{\pi \in \mathrm{PP}(a,b,c)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

Trace GF.

$$\sum_{\pi \in \mathrm{PP}(a,b,\infty)} y^{\mathrm{tr}(\pi)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} (1 - y q^{i+j-1})^{-1}$$

Discrete 2D Toda eq.

The discrete two-dimensional (2D) Toda eq.[3]:

$$q_n^{(s,t+1)} + e_n^{(s+1,t)} = q_n^{(s,t)} + e_{n+1}^{(s,t)}, \qquad q_n^{(s,t+1)} e_{n+1}^{(s+1,t)} = q_{n+1}^{(s,t)} e_{n+1}^{(s,t)},$$

$$s, t, n = 0, 1, 2, \ldots, \qquad e_0^{(s,t)} = 0.$$

Theorem (K.)

Let $\mathrm{PP}(a, b, c)$ denote the set of plane partitions whose diagrams are contained in an $a \times b \times c$ box.

1. Assume that $q_n^{(s,t)} \neq 0$ and $e_n^{(s,t)} \neq 0$ solve the discrete 2D Toda eq.
2. Let us determine weight $w(\pi)$ for plane partitions by . . . [a systematic way with $q^{(s,0)}$ and $e^{(0,t)}$].

Then

$$\sum_{\pi \in \mathrm{PP}(a,b,c)} w(\pi) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{q_{k-1}^{(i-1,j)}}{q_{k-1}^{(i-1,j-1)}} = \prod_{i=1}^{a} \prod_{k=1}^{c} \frac{q_{k-1}^{(i-1,b)}}{q_{k-1}^{(i-1,0)}}.$$

[3] R. Hirota, S. Tsujimoto T. Imai, *Difference scheme of soliton equations*, RIMS Kôkyûroku 822, pp. 144–152, 1993.

A nice formula from the discrete 2D Toda eq.

From the solution to the discrete 2D Toda eq.

$$q_n^{(s,t)} = q^n(1 - yq^{s+t+n+1}), \qquad e_{n+1}^{(s,t)} = yq^{s+t+n+1}(1 - q^{n+1})$$

Corollary (K.)

Let $\mathrm{PP}(a,b,c)$ denote the set of plane partitions whose diagrams are contained in an $a \times b \times c$ box. Then

$$\sum_{\pi \in \mathrm{PP}(a,b,c)} y^{\mathrm{tr}(\pi)} q^{|\pi|} \omega(\pi) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - yq^{i+j+k-1}}{1 - yq^{i+j+k-2}},$$

$$\omega(\pi) = \prod_{i=1}^{\min\{a,b\}} \prod_{k=1}^{\pi_{i,i}} \frac{1 - q^{c-k+i}}{1 - yq^{c-k+i}}.$$

This nice formula

- generalizes MacMahon's formula ($y = 1$);
- refines Trace GF ($c \to \infty$).

Another nice formula from the discrete 2D Toda eq.

Assume that (α_j, β_j) for $j = 0, 1, 2, \dots$ satisfies $(\alpha_0, \beta_0) = (0, 0)$ and

$$(\alpha_j, \beta_j) = (\alpha_{j-1} + 1, \beta_{j-1}) \text{ or } (\alpha_{j-1}, \beta_{j-1} + 1) \quad \text{for } j \geq 1.$$

Then

$$q_n^{(s,t)} = q^n (1 - xq^{s+\alpha_t+n+1})(1 - yq^{s+\beta_t+n+1}),$$

$$e_n^{(s,t)} = \begin{cases} xq^{s+\alpha_t+n}(1 - yq^{s+\beta_t+n})(1 - q^n) & \text{if } (\alpha_{t+1}, \beta_{t+1}) = (\alpha_t + 1, \beta_t), \\ yq^{s+\beta_t+n}(1 - xq^{s+\alpha_t+n})(1 - q^n) & \text{if } (\alpha_{t+1}, \beta_{t+1}) = (\alpha_t, \beta_t + 1). \end{cases}$$

solve the discrete 2D Toda eq.

Given a plane partition $\pi \in \mathrm{PP}(a,b,c)$ paint the region $j-1 \le y-x \le j$ in

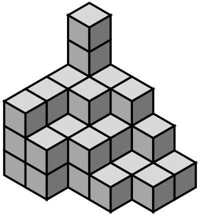
- blue if $(\alpha_j, \beta_j) = (\alpha_{j-1}+1, \beta_{j-1})$,
- red if $(\alpha_j, \beta_j) = (\alpha_{j-1}, \beta_{j-1}+1)$

for $j = 1, 2, 3, \dots$.

Example: When

j	0	1	2	3	4	5	$\cdots$
α_j	0	1	1	1	2	3	$\cdots$
β_j	0	0	1	2	2	2	$\cdots$

a plane partition is painted as



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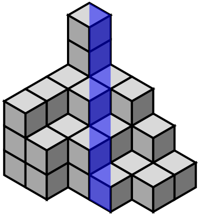

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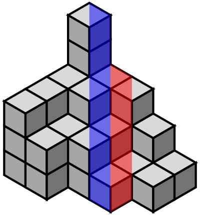

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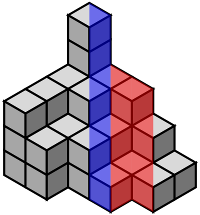

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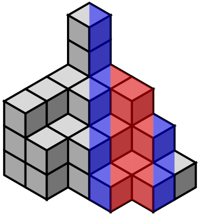

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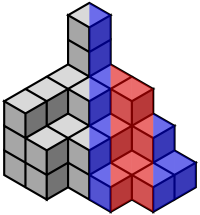
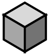
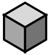
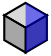
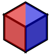
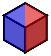
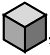
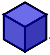
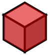
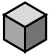


For a cube  at (i,j,k) define its weight by

$v($  $) =$

- $\dfrac{xq(1-q^{c+i-k})}{1-xq^{c+i-k}}$ if painted as 
- $\dfrac{yq(1-q^{c+i-k})}{1-yq^{c+i-k}}$ — 
- $\dfrac{xq^{\alpha_{j-i}+1}(1-yq^{c+i+\beta_{j-i}-k})}{yq^{\beta_{j-i}}(1-xq^{c+i+\beta_{j-i}-k})}$ — 
- $\dfrac{yq^{\beta_{j-i}+1}(1-xq^{c+i+\alpha_{j-i}-k})}{xq^{\alpha_{j-i}}(1-yq^{c+i+\alpha_{j-i}-k})}$ — 
- q — , , 

Then, weight of π by

$v^{\alpha,\beta}_{x,y,a}(\pi) = \prod_{\text{cube} \in \pi} v($  $)$, where the product runs over  $\in \pi$

Corollary (K., in preparation)

Let $\mathrm{PP}(a,b,c)$ denote the set of plane partitions whose diagrams are contained in an $a \times b \times c$ box. Then

$$\sum_{\pi \in \mathrm{PP}(a,b,c)} v_{x,y,q}^{\alpha,\beta}(\pi) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{(1 - xq^{i+\alpha_j+k-1})(1 - yq^{i+\beta_j+k-1})}{(1 - xq^{i+\alpha_{j-1}+k-1})(1 - yq^{i+\beta_{j-1}+k-1})}$$

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - z_j q^{i+\gamma_j+k-1}}{1 - z_j q^{i+\gamma_j+k-2}}$$

where

$$(z_j, \gamma_j) = \begin{cases} (x, \alpha_j) & \text{if } (\alpha_j, \beta_j) = (\alpha_{j-1}+1, \beta_{j-1}), \\ (y, \beta_j) & \text{if } (\alpha_j, \beta_j) = (\alpha_{j-1}, \beta_{j-1}+1). \end{cases}$$

This nice formula reduces into the previous one when $(\alpha_j, \beta_j) = (0, j)$ for $j = 0, 1, 2, \ldots$ so that $(z_j, \gamma_j) = (y, j)$.

Reverse plane partitions

Let λ be a Young diagram.

A reverse plane partition of shape λ is a 2D array $\pi = (\pi_{i,j})_{(i,j)\in\lambda}$ of nonnegative integers such that

$$\pi_{i,j} \le \pi_{i+1,j}, \qquad \pi_{i,j} \le \pi_{i,j+1} \qquad \text{for } \forall (i,j).$$

Example: A reverse plane partition of shape $\lambda = (5,4,4,2)$:

0	0	1	2	4
0	1	2	3	
2	2	4	4	
3	4			

Nice formulas for reverse plane partitions

Theorem (Gansner[4])

Let $\mathrm{RPP}(\lambda, \infty)$ denote the set of reverse plane partitions of shape λ. Then

$$\sum_{\pi \in \mathrm{RPP}(\lambda,\infty)} \prod_{(i,j)\in\lambda} y_{j-i}^{\pi_{i,j}} = \prod_{(i,j)\in\lambda} \left(1 - \prod_{\ell=j-\lambda'_j}^{\lambda_i - i} y_\ell\right)^{-1}.$$

[4] E. R. Gansner, *The Hillman–Grassl correspondence and the enumeration of reverse plane partitions*, J. Combin. Theory Ser. A **30** (1981), 71–89.

For reverse plane partitons a similar systematic way with the discrete 2D Toda eq. works well. From a specific solution

Theorem (K.)

Let $\mathrm{RPP}(\lambda, c)$ denote the set of reverse plane partitions of shape λ with entries $\leq c$. Then

$$\sum_{\pi \in \mathrm{RPP}(\lambda,c)} \prod_{(i,j)\in\lambda} y_{j-i}^{\pi_{i,j}}\, \omega'(\pi) = \prod_{(i,j)\in\lambda} \frac{1 - \prod_{\ell=j-c-\lambda'_{j-c}}^{\lambda_i - i} y_\ell}{1 - \prod_{\ell=j-\lambda'_j}^{\lambda_i - i} y_\ell},$$

$$\omega'(\pi) = \prod_{(i,j)\in\lambda} \prod_{k=1}^{\pi_{i,j}} \frac{1 - \prod_{\ell=j+k-c-1-\lambda'_{j+k-c-1}}^{j-i-1} y_\ell}{1 - \prod_{\ell=j+k-c-\lambda'_{j+k-c}}^{j-i} y_\ell}$$

where λ' is the conjugate of λ, and $\lambda'_j = \lambda'_1$ for $j < 0$.

This nice formula for reverse plane partitions refines Gansner's formula ($c \to \infty$).