Signed Mahonian identities on permutations

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Outline

- Signed Mahonian
- Main results:

1. Caselli's conjecture verified
   - · and explained

2. Various applications
   - · (1) Linear ext. of an injective labeling poset
   - · (2) Restrict pattern avoidance
   - · (3) signed version of Haglund-Loehr-Remmel

- Idea of proof
- Discussions

I

Signed Mahonian

inversion

- For $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathfrak{S}_n$, its inversion statistic inv is

$$\mathsf{inv}(\sigma) := \#\{(i,j) : \sigma(i) > \sigma(j)\}.$$

- For $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathfrak{S}_n$, its descent set Des is

$$\mathsf{Des}(\sigma) := \{i : \sigma(i) > \sigma(i+1)\},$$

and its major statistic maj is

$$\mathsf{maj}(\sigma) := \sum_{i \in \mathsf{Des}} i.$$

- For example
  - $\cdot\ \mathsf{inv}(315426) = \#\{(3,1),(3,2),(5,4),(5,2),(4,2)\} = 5$
  - $\cdot\ \mathsf{Des}(315426) = \{1,3,4\}, \qquad \mathsf{maj}(315426) = 8.$

Mahonian

Theorem [MacMahon]

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)} = [1]_q[2]_q \ldots [n]_q,$$

where $[k]_q := 1 + q + q^2 + \cdots + q^k$.

- For example,

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_3} q^{\mathsf{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)} &= 1 + 2q + 2q^2 + q^3 \\ &= (1)(1+q)(1+q+q^2) \\ &= [1]_q[2]_q[3]_q.\end{aligned}$$

- A statistic (e.g. maj) equidistributed with inv is called a Mahonian statistic.

parabolic quotient

- $\mathfrak{S}_n$ is the Coxeter group of type A_{n-1}.
  - generators are $\{s_1, \ldots, s_{n-1}\}$
  - inv is the length function ℓ.
- Let $J := \{s_{n-k+1}, \ldots, s_n\}$ (the last k generators)
  - W_J be the parabolic subgroup generated by J
  - ${}^J W$ be the left parabolic quotient of W_J, its nothing but...

$$ {}^J W = \{\sigma \in \mathfrak{S}_n : \sigma^{-1}(n-k+1) < \cdots < \sigma^{-1}(n)\} $$

We denote it by $\mathfrak{S}_n^{[n-k+1,n]}$.

Theorem. [Humphreys]

$$ \sum_{\sigma \in \mathfrak{S}_n^{[n-k+1,n]}} q^{\mathsf{inv}(\sigma)} = [k+1]_q [k+2]_q \ldots [n]_q. $$

Major holds too

- Amazingly, major holds also.

Theorem. [Stanley, Foata-Schützenberger]

$$\sum_{\sigma \in \mathfrak{S}_n^{[n-k+1,n]}} q^{\mathsf{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n^{[n-k+1,n]}} q^{\mathsf{maj}(\sigma)} = [k+1]_q [k+2]_q \dots [n]_q.$$

Signed Mahonian

• Gessel and Simion considered the signed Mahonian:

Theorem. [Gessel, Simion]

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = [1]_q [2]_{-q} [3]_q \ldots [n]_{(-1)^{n-1} q},$$

Hence it is natural to consider parabolic quotient $\mathfrak{S}_n(n-k+1:n)$.

Theorem. [Caselli, JCTA 2012]

$$\sum_{\sigma \in \mathfrak{S}_n^{[n-k+1,n]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = [k+1]_{\epsilon^{nk+n+k} q} [k+2]_{\epsilon^{k+1} q} \ldots [n]_{\epsilon^{n-1} q},$$

with $\epsilon := -1$.

Take the first generators

- Now one can take $J =: \{s_1, \ldots, s_k\}$,
  - ... generate the parabolic subgroup W_J,
  - ... and look at the left parabolic quotient ${}^J W$. Now

$$ {}^J W = \mathfrak{S}_n^{[1,k]} := \{\sigma \in \mathfrak{S}_n : \sigma^{-1}(1) < \cdots < \sigma^{-1}(k)\}. $$

  - ... and look at signed Mahonian

Caselli's conjecture

Conjecture [Caselli, JCTA 2012]. If n is even or k is odd, then

$$\sum_{\sigma\in\mathfrak{S}_n^{[1,k]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n^{[n-k+1,n]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}$$

$n\backslash k$	0	1
0	??	??
1	–	??

- It is curious the conjecute holds for $(n,k) \equiv_2 (0,0), (0,1), (1,1),$ but not $(1,0)$

Our work

- We will verify Caselli's conjecture.
  - We will generalize it
  - We can fully explain why it holds only for even n or odd k.

- Our theory will establish many new signed Mahonian identities in...
  - subword preserving permutations,
  - linear extensions of any poset,
  - twisted pattern avoiding permutations,
  - signed version of Haglund-Loehr-Remmel identity,
  - etc, etc.

II

Caselli's conjecture

Relation increment by 2

- Let R be an partial order among numbers $\{1, 2, \ldots, n-2\}$
  - $\cdot$ $R+2$ be the partial order among $\{3, 4, \ldots, n\}$ by a increment of 2.
- Let

$$\mathfrak{S}_n^R := \{\sigma \in \mathfrak{S}_n : \sigma^{-1} \text{ is consistent with all relations in } R\},$$

- For example, let $n = 5$ and $R = \{1 < 3,\ 2 < 3\}$
  - $\cdot$ then $R + 2 = \{3 < 5,\ 4 < 5\}$
  - $\cdot$ $\mathfrak{S}_n^R$ are permutations in which $1, 2$ appear before 3
  - $\cdot$ $\mathfrak{S}_n^{R+2}$ are permutations in which $3, 4$ appear before 5.

descent monomial

- Recall $\mathsf{Des}(\sigma)$ is the descent set of $\sigma \in \mathfrak{S}_n$. Let

$$\mathbf{x}^{\mathsf{Des}(\sigma)} := \prod_{i \in \mathsf{Des}(\sigma)} x_i$$

be its descent monomial.

- For example, $\sigma = 514632$ we have $\mathsf{Des}(\sigma) = \{1, 4, 5\}$,
  - $\mathbf{x}^{\mathsf{Des}(\sigma)} = x_1 x_4 x_5$
  - $\mathsf{maj}(\sigma) = 10$.

Signed descent polymomial

Theorem. [-, 2018] For an arbitrary partial order R on $[n-2]$, we have

$$\sum_{\sigma\in\mathfrak{S}_n^R}(-1)^{\mathsf{inv}(\sigma)}\mathbf{x}^{\mathsf{Des}(\sigma)}=\sum_{\sigma\in\mathfrak{S}_n^{R+2}}(-1)^{\mathsf{inv}(\sigma)}\mathbf{x}^{\mathsf{Des}(\sigma)}.$$

- Substitute x_i by tq^i we have

Theorem. [-, 2018] For an arbitrary partial order R on $[n-2]$, we have

$$\sum_{\sigma\in\mathfrak{S}_n^R}(-1)^{\mathsf{inv}(\sigma)}t^{\mathsf{des}(\sigma)}q^{\mathsf{maj}(\sigma)}=\sum_{\sigma\in\mathfrak{S}_n^{R+2}}(-1)^{\mathsf{inv}(\sigma)}t^{\mathsf{des}(\sigma)}q^{\mathsf{maj}(\sigma)}.$$

Caselli's conj when $(n, k) \equiv_2 (0, 0), (1, 1)$.

- Taking $R = \{1 < 2 < \cdots < k\}$, letting $t = 1$
  - apply above repeatedly
  - When $(n, k) \equiv_2 (0, 0), (1, 1)$, R can be translated to

$$\{n - k + 1 < \cdots < n - 1 < n\}$$

Corollary. Caselli's conj. is verified if $(n, k) \equiv_2 (0, 0)$ or $(1, 1)$.

$$\sum_{\sigma \in \mathfrak{S}_n^{[1,k]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n^{[n-k+1,n]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}$$

$n\backslash k$	0	1
0	OK	??
1	–	OK

Caselli's conj when $k \equiv_2 1$.

- The missing case can be done by

Theorem. [-, 2018] If k is odd, then we have

$$\sum_{\sigma \in \mathfrak{S}_n^{[a,a+k-1]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n^{[a+1,a+k]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}.$$

- By a translation we can fill the missing case $(n,k) \equiv_2 (0,1)$

Corollary. Caselli's conjecture is verified.

$n \backslash k$	0	1
0	OK	OK
1	–	OK

Generating funciton

- We can determine also the generating function.

Theorem. [-, 2018] Let n be arbitrary and k be even. Then

$$\sum_{\mathfrak{S}_n^{[n-k+1,n]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} + \sum_{\mathfrak{S}_n^{[n-k,n-1]}} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}$$
$$= 2[k+2]_{-q}[k+3]_q \dots [n]_{(-1)^{n-k+1}q}.$$

- From this, we can have a nice classification.....

The reason why $(n,k)=(1,0)$ fails

- Let us say $\sum_{\mathfrak{S}_n^{[1,k]}}(-1)^{\mathsf{inv}(\sigma)}q^{\mathsf{maj}(\sigma)}$ is indexed by $[1,k]$.

Corollary. We have:

1. If k is even, then the signed Mahonian polynomials
   - indexed by $[1,k]$, $[3,k+2]$, $[5,k+4]$, $\ldots$ are of the same,
   - indexed by $[2,k+1]$, $[4,k+3]$, $[6,k+5]$, $\ldots$ are of the same,
   - these two polynomials are different.
2. If k is odd, then the signed Mahonian polynomials
   - indexed by $[1,k]$, $[2,k+1]$, $[3,k+2]$, $\ldots$ are all of the same.

- Now we know why $(n,k)=(1,0)$ fails.
  - Because k is even, hence
    - $[n-k+1,n]$ and $[1,k]$ belong to two different families.

III

Applications

· (1) Linear ext. of an injective labeling poset

· (2) Restrict pattern avoidance

· (3) signed version of Haglund-Loehr-Remmel

(1) Linear ext. of an injective labeling poset

- Let P be a poset. Let
  - $\cdot\ f : P \to [n-2]$ be an injective labeling.
  - $\cdot\ f+2$ be the labeling by an increment of 2.
  - $\cdot\ \mathfrak{S}_n^f := \{\sigma \in \mathfrak{S}_n : \sigma^{-1}(f(x)) < \sigma^{-1}(f(y)),$ for $x < y$ in $P\}$
  - $\cdot\ \mathfrak{S}_n^{f+2}$:= similarly.

Theorem. [-, 2018] For any P and any f, we have

$$\sum_{\sigma \in \mathfrak{S}_n^f} (-1)^{\mathsf{inv}(\sigma)} \mathbf{x}^{\mathsf{Des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n^{f+2}} (-1)^{\mathsf{inv}(\sigma)} \mathbf{x}^{\mathsf{Des}(\sigma)}.$$

(2) Restrict pattern avoidance

- We can apply to pattern avoiding permutations.
  - · 'pattern' here is in the classic sense. (231 means $2-3-1$).
- Given a pattern π and a subset $S \subseteq [1, n-2]$
  - · hence $S+2 \subseteq [3, n]$.
- Let

$$\mathsf{AV}'_{n,S}(\pi) := \{\sigma \in \mathfrak{S}_n : \text{ the } \pi\text{-avoidance are only for values in } S\},$$

$$\mathsf{AV}''_{n,S}(\pi) := \{\sigma \in \mathfrak{S}_n : \text{ the } \pi\text{-avoidance are only for values in } S+2\}.$$

· $\mathsf{AV}'_{9,\{1,3,4,6\}}(132)$ are those with no 132 pattern among $\{1, 3, 4, 6\}$.
· $\mathsf{AV}''_{9,\{1,3,4,6\}}(132)$ are those with no 132 pattern among $\{3, 5, 6, 8\}$.

- For a set P of patterns, define $\mathsf{AV}'_{n,S}(P)$, $\mathsf{AV}''_{n,S}(P)$ similarly.

(2) Restrict pattern avoidance

Theorem. [-, 2018] Let P be an arbitrary set of patterns and $S \subseteq [n-2]$. We have

$$\sum_{\sigma \in \mathsf{AV}'_{n,S}(P)} (-1)^{\mathsf{inv}(\sigma)} \mathbf{x}^{\mathsf{Des}(\sigma)} = \sum_{\sigma \in \mathsf{AV}''_{n,S}(P)} (-1)^{\mathsf{inv}(\sigma)} \mathbf{x}^{\mathsf{Des}(\sigma)}$$

• For example,

$$\begin{aligned}\mathsf{AV}'_{4,[2]}(21) = \{&1234, 1243, 1324, 1342, 1423, 1432,\\ &3124, 3142, 3412, 4123, 4132, 4312\}.\end{aligned}$$

$$\begin{aligned}\mathsf{AV}''_{4,[2]}(21) = \{&1234, 1324, 1342, 2134, 2314, 2341,\\ &3124, 3142, 3214, 3241, 3412, 3421\}.\end{aligned}$$

· Same signed descent polynomial $1 + x_2 - x_1x_2 - x_2x_3$.

· hence the same signed Mahonian value.

(3) signed Haglund-Loehr-Remmel Theorem

- Let τ be a permutation of the set $\{1, 2, \ldots, n\} - \{r\}$ for a fixed r.
  - Let $\mathfrak{S}_n^\tau$ be the permutations contains τ as a subseq.

Theorem. [Haglund-Loehr-Remmel] We have

$$\sum_{\sigma \in \mathfrak{S}_n^\tau} q^{\mathsf{maj}(\sigma)} = q^{\mathsf{maj}(\tau)} [n]_q$$

Theorem. [-, 2018] Let τ be a permutation of the set $\{1, 2, \ldots, n\} - \{r, r+1\}$ for a fixed r. We have

$$\sum_{\sigma \in \mathfrak{S}_n^\tau} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = (-1)^{\mathsf{inv}(\tau)} q^{\mathsf{maj}(\tau)} [n-1]_{\epsilon^n q} [n]_{\epsilon^{n-1} q}.$$

IV

Idea of proof

Key lemma

- The key of the proof is to see the cases that permutations by 2.
  - The pivot is the following lemma.

- Let $J \subseteq [n]$ with $j := |J|$, and $\pi \in \mathfrak{S}_J$
  - $\mathfrak{S}_n^{\pi} := \{\sigma \in \mathfrak{S}_n : \sigma^{-1}(\pi(1)) < \cdots < \sigma^{-1}(\pi(j))\}$
  - $\pi + 2 := (\pi(1) + 2) \dots (\pi(j) + 2)$

Key Lemma. For any $\pi \in \mathfrak{S}_{[n-2]}$, there exits a bijection

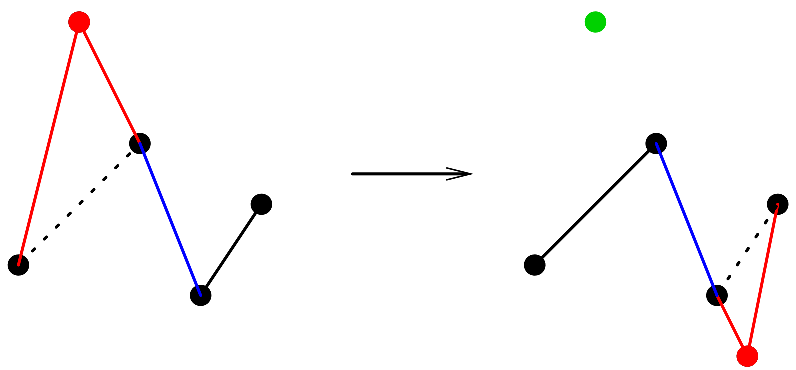
$$\phi : \mathfrak{S}_n^{\pi} \to \mathfrak{S}_n^{\pi+2}$$

such that

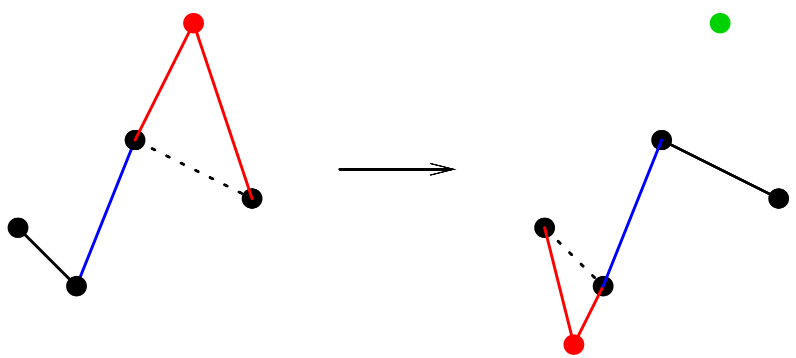
$$\mathsf{inv}(\phi(\sigma)) \equiv_2 \mathsf{inv}(\sigma), \qquad \mathsf{Des}(\phi(\sigma)) = \mathsf{Des}(\sigma).$$

A naïve idea

- The naïve idea:
  - Change $\{n-1, n\}$ to $\{1, 2\}$, keeping the shape and inversions
- The core operations:



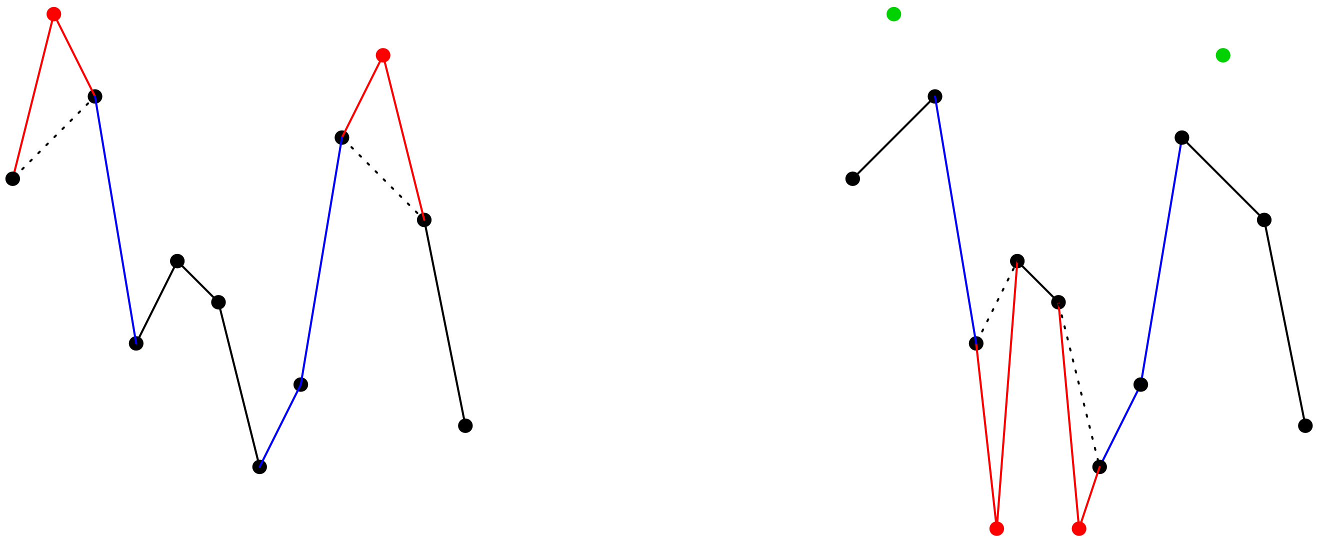
and



For example

- For example,

$$8, {\color{red}12}, 10, 4, 6, 5, 1, 3, 9, {\color{red}11}, 7, 2 \mapsto 10, 12, 6, {\color{red}*}, 8, 7, {\color{red}*}, 3, 5, 11, 9, 4$$



- Now we have
  - same shape (hence same Des)
  - All black numbers $+2$ (hence keeping '$+2$ relation')
  - switch $1, 2$ (or not) such that inv has the correct parity.

Need a detailed analysis

- However there are too many exceptions:
  - If both paths collide?
  - If both paths interwine?
  - If $1, 2$ cannot be exchanged?

- A very careful analysis is needed.
  - Anyway it can be done, and the theorems can be proved. $\square$

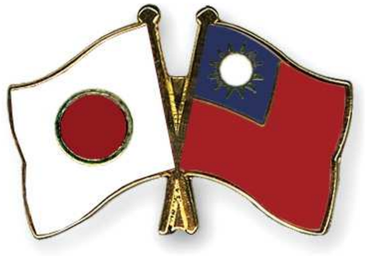
V

Discussions

Discussions

- We believe there is a Coxeter version of the above
  · under investigation, with preliminary results.
- A wreath product version?
- Add “signed" to other settings?
  ·There are plenty of possibilities.

Thanks for your attention



Welcome any discussions and collaborations!