Enumerating partial Latin rectangles

Raúl Fálcon (U. Seville),
Rebecca J. Stones (Nankai U.)

21 May 2018

0	1	2	3	·
·	2	·	1	·
1	3	·	·	0

0	1	2	3	·
·	2	·	1	·
1	3	·	·	0

A **partial Latin rectangle** is...

0	1	2	3	·
·	2	·	1	·
1	3	·	·	0

A **partial Latin rectangle** is...

- an $r \times s$ matrix, [above $r = 3$ and $s = 5$]

0	1	2	3	·
·	2	·	1	·
1	3	·	·	0

A **partial Latin rectangle** is...

- an $r \times s$ matrix, [above $r = 3$ and $s = 5$]
- contains symbols from an n-set, [above $n = 4$]

0	1	2	3	·
·	2	·	1	·
1	3	·	·	0

A **partial Latin rectangle** is...

- an $r \times s$ matrix, [above $r = 3$ and $s = 5$]
- contains symbols from an n-set, [above $n = 4$]
- *Latin*-ness: no repeats in each row or column,

0	1	2	3	·
·	2	·	1	·
1	3	·	·	0

A **partial Latin rectangle** is...

- an $r \times s$ matrix, [above $r = 3$ and $s = 5$]
- contains symbols from an n-set, [above $n = 4$]
- *Latin*-ness: no repeats in each row or column,
- *partial*-ness: we allow empty cells,

0	1	2	3	·
·	2	·	1	·
1	3	·	·	0

A **partial Latin rectangle** is...

- an $r \times s$ matrix, [above $r = 3$ and $s = 5$]
- contains symbols from an n-set, [above $n = 4$]
- *Latin*-ness: no repeats in each row or column,
- *partial*-ness: we allow empty cells,
- with m entries. [above $m = 9$].

0	1	2	3	·
·	2	·	1	·
1	3	·	·	0

A **partial Latin rectangle** is...

- an $r \times s$ matrix, [above $r = 3$ and $s = 5$]
- contains symbols from an n-set, [above $n = 4$]
- *Latin*-ness: no repeats in each row or column,
- *partial*-ness: we allow empty cells,
- with m entries. [above $m = 9$].

This is a member of $\mathrm{PLR}(r, s, n; m) = \mathrm{PLR}(3, 5, 4; 9)$.

Simple question: How many partial Latin rectangles are there?

Simple question: How many partial Latin rectangles are there?

- **Not so easy answer (1)**: Latin squares are not easy to enumerate $\Longrightarrow$ partial Latin rectangles are not easy to enumerate.

Simple question: How many partial Latin rectangles are there?

- **Not so easy answer (1)**: Latin squares are not easy to enumerate $\Longrightarrow$ partial Latin rectangles are not easy to enumerate.
- **Not so easy answer (2)**: What does this even mean?

Simple question: How many partial Latin rectangles are there?

- **Not so easy answer (1)**: Latin squares are not easy to enumerate $\Longrightarrow$ partial Latin rectangles are not easy to enumerate.
- **Not so easy answer (2)**: What does this even mean?
  - We'll talk about four different ways of enumerating partial Latin rectangles.

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles.

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries...

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries... where an entry (x, y, z) implies we put symbol z in cell (x, y).

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries... where an entry (x, y, z) implies we put symbol z in cell (x, y).

The j-th entry clashes with the i-th entry when $j > i$ and:

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries... where an entry (x, y, z) implies we put symbol z in cell (x, y).

The j-th entry clashes with the i-th entry when $j > i$ and:

- **Blue clashes**: they have same column and same symbol.

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries... where an entry (x, y, z) implies we put symbol z in cell (x, y).

The j-th entry clashes with the i-th entry when $j > i$ and:

- **Blue clashes**: they have same column and same symbol.
- **Red clashes**: they have same row and same symbol.

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries... where an entry (x, y, z) implies we put symbol z in cell (x, y).

The j-th entry clashes with the i-th entry when $j > i$ and:

- **Blue clashes**: they have same column and same symbol.
- **Red clashes**: they have same row and same symbol.
- **Green clashes**: they have same row and same column.

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries... where an entry (x, y, z) implies we put symbol z in cell (x, y).

The j-th entry clashes with the i-th entry when $j > i$ and:

- **Blue clashes**: they have same column and same symbol.
- **Red clashes**: they have same row and same symbol.
- **Green clashes**: they have same row and same column.

If C_m denotes the set of possible clashes,

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries... where an entry (x, y, z) implies we put symbol z in cell (x, y).

The j-th entry clashes with the i-th entry when $j > i$ and:

- **Blue clashes**: they have same column and same symbol.
- **Red clashes**: they have same row and same symbol.
- **Green clashes**: they have same row and same column.

If C_m denotes the set of possible clashes, then Inclusion-Exclusion gives

$$m!\,\mathrm{PLR}(r, s, n; m) = \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V|$$

Method 1: Inclusion-Exclusion

We order the entries in m-entry partial Latin rectangles. We thus enumerate length-m non-clashing sequences of entries... where an entry (x, y, z) implies we put symbol z in cell (x, y).

The j-th entry clashes with the i-th entry when $j > i$ and:

- **Blue clashes**: they have same column and same symbol.
- **Red clashes**: they have same row and same symbol.
- **Green clashes**: they have same row and same column.

If C_m denotes the set of possible clashes, then Inclusion-Exclusion gives

$$m!\,\mathrm{PLR}(r, s, n; m) = \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V|$$

where $\mathcal{B}_V$ is the set of length-m sequences of entries with the clashes in V.

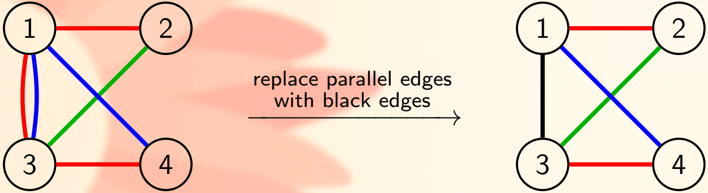
From the previous slide:

$$m!\,\mathrm{PLR}(r, s, n; m) = \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V|.$$

From the previous slide:

$$m! \, \mathrm{PLR}(r, s, n; m) = \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V|.$$

We convert any set of clashes V to an edge-colored graph:

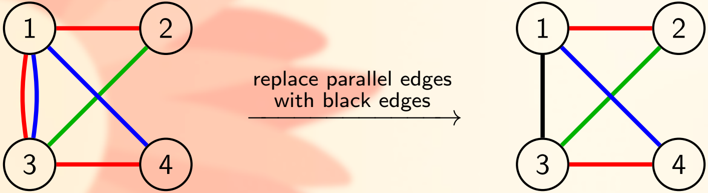


Here, the 1-st entry has a **red clash** with the 2-nd entry. And so on.

From the previous slide:

$$m!\,\mathrm{PLR}(r, s, n; m) = \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V|.$$

We convert any set of clashes V to an edge-colored graph:



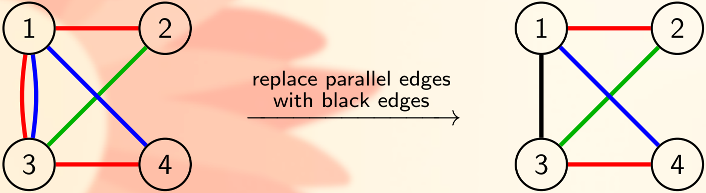
Here, the 1-st entry has a **red clash** with the 2-nd entry. And so on. Then we show

$$|\mathcal{B}_V| = r^{c(\text{delete blue edges})} s^{c(\text{delete red edges})} n^{c(\text{delete green edges})}.$$

From the previous slide:

$$m! \, \mathrm{PLR}(r, s, n; m) = \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V|.$$

We convert any set of clashes V to an edge-colored graph:



Here, the 1-st entry has a **red clash** with the 2-nd entry. And so on. Then we show

$$|\mathcal{B}_V| = r^{c(\text{delete blue edges})} s^{c(\text{delete red edges})} n^{c(\text{delete green edges})}.$$

This shows $m! \, \mathrm{PLR}(r, s, n; m)$ is a 3-variable symmetric polynomial with integer coefficients of degree $3m$, **for fixed** m (i.e., fixed no. entries).

We rearrange and simplify to obtain:

Theorem ("what the paper says"): For all $r, s, n, m \geq 1$, we have

$$m! \, \mathrm{PLR}(r, s, n; m)$$
$$= (rsn)^m + \sum_{v \geq 2} \sum_{e \geq 1} (-1)^e \binom{m}{v} (rsn)^{m-v+1} \sum_{G \in \Gamma_{e,v}} \frac{v!}{|\mathrm{Aut}(G)|} P(G)$$

where $\Gamma_{e,v}$ is the set of unlabeled e-edge v-vertex graphs without isolated vertices, and

$$P(G) = \sum_{\delta} (-2)^{\#\text{black}} r^{c(\cancel{blue})-1} s^{c(\cancel{red})-1} n^{c(\cancel{green})-1}$$

where the sum is over all red/blue/green/black edge colorings δ of G.

We rearrange and simplify to obtain:

Theorem ("what the paper says"): For all $r, s, n, m \geq 1$, we have

$$m!\,\mathrm{PLR}(r,s,n;m) = (rsn)^m + \sum_{v \geq 2} \sum_{e \geq 1} (-1)^e \binom{m}{v} (rsn)^{m-v+1} \sum_{G \in \Gamma_{e,v}} \frac{v!}{|\mathrm{Aut}(G)|} P(G)$$

where $\Gamma_{e,v}$ is the set of unlabeled e-edge v-vertex graphs without isolated vertices, and

$$P(G) = \sum_{\delta} (-2)^{\#\text{black}} r^{c(\cancel{blue})-1} s^{c(\cancel{red})-1} n^{c(\cancel{green})-1}$$

where the sum is over all red/blue/green/black edge colorings δ of G.

What's important here:

- **We compute** $\mathrm{PLR}(r,s,n;m)$ **by computing** $|\mathrm{Aut}(G)|$ **and** $P(G)$ **for small graphs.**

We rearrange and simplify to obtain:

Theorem ("what the paper says"): For all $r, s, n, m \geq 1$, we have

$$m!\,\mathrm{PLR}(r,s,n;m) = (rsn)^m + \sum_{v\geq 2}\sum_{e\geq 1}(-1)^e\binom{m}{v}(rsn)^{m-v+1}\sum_{G\in\Gamma_{e,v}}\frac{v!}{|\mathrm{Aut}(G)|}P(G)$$

where $\Gamma_{e,v}$ is the set of unlabeled e-edge v-vertex graphs without isolated vertices, and

$$P(G) = \sum_{\delta}(-2)^{\#\text{black}}\, r^{c(\cancel{blue})-1}\, s^{c(\cancel{red})-1}\, n^{c(\cancel{green})-1}$$

where the sum is over all red/blue/green/black edge colorings δ of G.

What's important here:

- **We compute $\mathrm{PLR}(r,s,n;m)$ by computing $|\mathrm{Aut}(G)|$ and $P(G)$ for small graphs.** The rest is arithmetic.

So we do that...

G	v	e	$c(G)$	$\lvert\mathrm{Aut}(G)\rvert$	$P(G) = P(G; r, s, n)$
	2	1	1	2	$\overline{100} - 2$
	3	2	1	2	$P($$)^2$
	3	3	1	6	$\overline{200} - 2$
	4	2	2	8	$\overline{111}\, P($$)^2$
	4	3	1	6	$P($$)^3$
	4	3	1	2	$P($$)^3$
	4	1	4	2	$P($$)P($$)$
	4	4	1	8	$\overline{300} + 6\,\overline{110} - 12\,\overline{100} + 16$
	4	5	1	4	$\overline{300} + 2\,\overline{110} - 4\,\overline{100} + 4$
	4	6	1	24	$\overline{300} - 2$
	5	3	2	4	$\overline{111}\, P($$)^3$
	5	4	2	12	$\overline{111}\, P($$)P($$)$
	6	3	3	48	$\overline{222}\, P($$)^3$

Etc. Here, we use shorthand $\overline{110} = rn + rs + sn$.

And by putting those values into the equation, we get...

Theorem ("what the paper says"): Let m be a positive integer. Then,

$$m!\,\mathrm{PLR}(r,s,n;m) = (rsn)^m + \binom{m}{2}(rsn)^{m-1}(2-\overline{100}) + \binom{m}{3}(rsn)^{m-2}(14 - 12\,\overline{100} + 6\,\overline{110} + 2\,\overline{200}) + \binom{m}{4}(rsn)^{m-3}(198 - 228\,\overline{100} + 198\,\overline{110} - 84\,\overline{111} + 72\,\overline{200} - 36\,\overline{210} - 12\,\overline{211} + 6\,\overline{221} - 6\,\overline{300} + 3\,\overline{311}) + \binom{m}{5}(rsn)^{m-4}(-6360\,\overline{100} + 7440\,\overline{110} - 6080\,\overline{111} + 2880\,\overline{200} - 2520\,\overline{210} + 820\,\overline{211} + 480\,\overline{220} + 360\,\overline{221} - 180\,\overline{222} - 480\,\overline{300} + 240\,\overline{310} + 160\,\overline{311} - 80\,\overline{321} + 24\,\overline{400} - 20\,\overline{411}) + \binom{m}{6}(rsn)^{m-5}(-13170\,\overline{211} + 17340\,\overline{221} - 15990\,\overline{222} + 7580\,\overline{311} - 7050\,\overline{321} + 3300\,\overline{322} + 1520\,\overline{331} + 180\,\overline{332} - 90\,\overline{333} - 1740\,\overline{411} + 870\,\overline{421} + 90\,\overline{422} - 45\,\overline{432} + 130\,\overline{511} - 15\,\overline{522}) + \binom{m}{7}(rsn)^{m-6}(-10920\,\overline{322} + 15540\,\overline{332} - 15120\,\overline{333} + 7350\,\overline{422} - 7140\,\overline{432} + 3570\,\overline{433} + 1680\,\overline{442} - 2100\,\overline{522} + 1050\,\overline{532} + 210\,\overline{622}) + \binom{m}{8}(rsn)^{m-7}(-3360\,\overline{433} + 5040\,\overline{443} - 5040\,\overline{444} + 2520\,\overline{533} - 2520\,\overline{543} + 1260\,\overline{544} + 630\,\overline{553} - 840\,\overline{633} + 420\,\overline{643} + 105\,\overline{733}) +$$

some polynomial of degree $\leq 3m - 10$.

And by putting those values into the equation, we get...

Theorem ("what the paper says"): Let m be a positive integer. Then, $m!\,\mathrm{PLR}(r,s,n;m) = (rsn)^m + \binom{m}{2}(rsn)^{m-1}(2-\overline{100}) + \binom{m}{3}(rsn)^{m-2}(14 - 12\,\overline{100} + 6\,\overline{110} + 2\,\overline{200}) + \binom{m}{4}(rsn)^{m-3}(198 - 228\,\overline{100} + 198\,\overline{110} - 84\,\overline{111} + 72\,\overline{200} - 36\,\overline{210} - 12\,\overline{211} + 6\,\overline{221} - 6\,\overline{300} + 3\,\overline{311}) + \binom{m}{5}(rsn)^{m-4}(-6360\,\overline{100} + 7440\,\overline{110} - 6080\,\overline{111} + 2880\,\overline{200} - 2520\,\overline{210} + 820\,\overline{211} + 480\,\overline{220} + 360\,\overline{221} - 180\,\overline{222} - 480\,\overline{300} + 240\,\overline{310} + 160\,\overline{311} - 80\,\overline{321} + 24\,\overline{400} - 20\,\overline{411}) + \binom{m}{6}(rsn)^{m-5}(-13170\,\overline{211} + 17340\,\overline{221} - 15990\,\overline{222} + 7580\,\overline{311} - 7050\,\overline{321} + 3300\,\overline{322} + 1520\,\overline{331} + 180\,\overline{332} - 90\,\overline{333} - 1740\,\overline{411} + 870\,\overline{421} + 90\,\overline{422} - 45\,\overline{432} + 130\,\overline{511} - 15\,\overline{522}) + \binom{m}{7}(rsn)^{m-6}(-10920\,\overline{322} + 15540\,\overline{332} - 15120\,\overline{333} + 7350\,\overline{422} - 7140\,\overline{432} + 3570\,\overline{433} + 1680\,\overline{442} - 2100\,\overline{522} + 1050\,\overline{532} + 210\,\overline{622}) + \binom{m}{8}(rsn)^{m-7}(-3360\,\overline{433} + 5040\,\overline{443} - 5040\,\overline{444} + 2520\,\overline{533} - 2520\,\overline{543} + 1260\,\overline{544} + 630\,\overline{553} - 840\,\overline{633} + 420\,\overline{643} + 105\,\overline{733}) +$ some polynomial of degree $\leq 3m-10$.

What's important here:

- **We computed many leading terms for $m!\,\mathrm{PLR}(r,s,n;m)$ for fixed m.**

And by putting those values into the equation, we get...

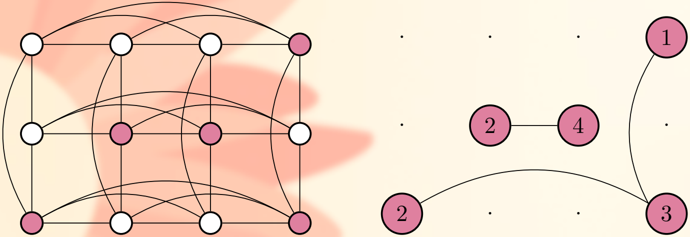
Theorem ("what the paper says"): Let m be a positive integer. Then, $m!\,\mathrm{PLR}(r,s,n;m) = (rsn)^m + \binom{m}{2}(rsn)^{m-1}(2-\overline{100}) + \binom{m}{3}(rsn)^{m-2}(14 - 12\,\overline{100} + 6\,\overline{110} + 2\,\overline{200}) + \binom{m}{4}(rsn)^{m-3}(198 - 228\,\overline{100} + 198\,\overline{110} - 84\,\overline{111} + 72\,\overline{200} - 36\,\overline{210} - 12\,\overline{211} + 6\,\overline{221} - 6\,\overline{300} + 3\,\overline{311}) + \binom{m}{5}(rsn)^{m-4}(-6360\,\overline{100} + 7440\,\overline{110} - 6080\,\overline{111} + 2880\,\overline{200} - 2520\,\overline{210} + 820\,\overline{211} + 480\,\overline{220} + 360\,\overline{221} - 180\,\overline{222} - 480\,\overline{300} + 240\,\overline{310} + 160\,\overline{311} - 80\,\overline{321} + 24\,\overline{400} - 20\,\overline{411}) + \binom{m}{6}(rsn)^{m-5}(-13170\,\overline{211} + 17340\,\overline{221} - 15990\,\overline{222} + 7580\,\overline{311} - 7050\,\overline{321} + 3300\,\overline{322} + 1520\,\overline{331} + 180\,\overline{332} - 90\,\overline{333} - 1740\,\overline{411} + 870\,\overline{421} + 90\,\overline{422} - 45\,\overline{432} + 130\,\overline{511} - 15\,\overline{522}) + \binom{m}{7}(rsn)^{m-6}(-10920\,\overline{322} + 15540\,\overline{332} - 15120\,\overline{333} + 7350\,\overline{422} - 7140\,\overline{432} + 3570\,\overline{433} + 1680\,\overline{442} - 2100\,\overline{522} + 1050\,\overline{532} + 210\,\overline{622}) + \binom{m}{8}(rsn)^{m-7}(-3360\,\overline{433} + 5040\,\overline{443} - 5040\,\overline{444} + 2520\,\overline{533} - 2520\,\overline{543} + 1260\,\overline{544} + 630\,\overline{553} - 840\,\overline{633} + 420\,\overline{643} + 105\,\overline{733}) +$ some polynomial of degree $\le 3m-10$.

What's important here:

- **We computed many leading terms for $m!\,\mathrm{PLR}(r,s,n;m)$ for fixed m.**
- This is exact for $m \le 5$.

Method 2: Chromatic polynomial method

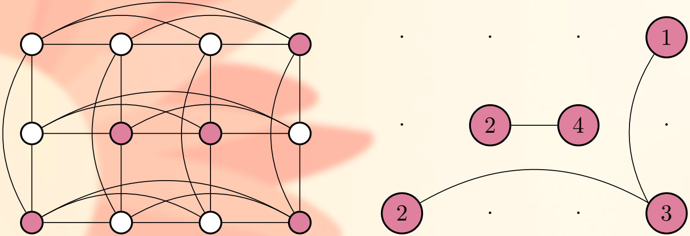
$\mathrm{PLR}(r, s, n; m)$'s are equivalent to proper n-colorings of m-entry induced subgraphs of $K_r \square K_s$.



Example: a proper 4-coloring of an induced subgraph $K_3 \square K_4$.

Method 2: Chromatic polynomial method

$\mathrm{PLR}(r, s, n; m)$'s are equivalent to proper n-colorings of m-entry induced subgraphs of $K_r \square K_s$.



Example: a proper 4-coloring of an induced subgraph $K_3 \square K_4$.

So we get

$$\#\mathrm{PLR}(r, s, n; m) = \sum_{M} \Pi(M; n)$$

where Π is the chromatic polynomial, and the sum is over all induced subgraphs M.

We split up the equation for implementation:

We split up the equation for implementation: we group into isomorphism classes for each component, etc.

We split up the equation for implementation: we group into isomorphism classes for each component, etc.

Theorem ("what the paper says"):

$$\#\mathrm{PLR}(r,s,n;m) = \sum_{k\geq 0} \sum_{\mathbf{K}\in\mathcal{K}_{r,s,m,k}} \sum_{\substack{(t_i)_{i=1}^k \\ \text{good}}} [r]_{e_{\mathrm{row}}} [s]_{e_{\mathrm{col}}} \frac{\prod_{i=1}^k \Pi(\overline{K_i};n)}{\left(\prod_{i=1}^k |\mathrm{Aut}(G_{K_i})|\right)\left(\prod_{i=1}^{\ell} k_i!\right)}$$

where $[r]_{e_{\mathrm{row}}} = r!/(r-e_{\mathrm{row}})!$ and $[s]_{e_{\mathrm{col}}} = s!/(s-e_{\mathrm{col}})!$, (and a bunch of undefined things).

We split up the equation for implementation: we group into isomorphism classes for each component, etc.

Theorem ("what the paper says"):

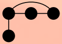
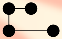
$$\#\mathrm{PLR}(r,s,n;m) = \sum_{k\geq 0} \sum_{\mathbf{K}\in\mathcal{K}_{r,s,m,k}} \sum_{\substack{(t_i)_{i=1}^k \\ \text{good}}} [r]_{e_{\mathrm{row}}} [s]_{e_{\mathrm{col}}} \frac{\prod_{i=1}^k \Pi(\overline{K_i};n)}{\left(\prod_{i=1}^k |\mathrm{Aut}(G_{K_i})|\right)\left(\prod_{i=1}^{\ell} k_i!\right)}$$

where $[r]_{e_{\mathrm{row}}} = r!/(r-e_{\mathrm{row}})!$ and $[s]_{e_{\mathrm{col}}} = s!/(s-e_{\mathrm{col}})!$, (and a bunch of undefined things).

What's important here:

- **We compute $\#\mathrm{PLR}(r,s,n;m)$ by computing $|\mathrm{Aut}(G)|$ and $\Pi(G)$ for small induced subgraphs of $K_r \square K_s$.**

So we do that...

block K	induced subgraph	$\lvert\mathrm{Aut}(G_K)\rvert$	$\Pi(K; n)$
$\boxed{1}$		1	n
$\boxed{1 \quad 1}$		2	$n(n-1)$
$\boxed{1 \quad 1 \quad 1}$		6	$n(n-1)(n-2)$
$\begin{array}{\|cc\|}\hline 1 & 1 \\ 1 & 0 \\ \hline\end{array}$		1	$n(n-1)^2$
$\boxed{1 \quad 1 \quad 1 \quad 1}$		24	$n(n-1)(n-2)(n-3)$
$\begin{array}{\|ccc\|}\hline 1 & 1 & 1 \\ 1 & 0 & 0 \\ \hline\end{array}$		2	$n(n-1)^2(n-2)$
$\begin{array}{\|ccc\|}\hline 1 & 1 & 0 \\ 1 & 0 & 1 \\ \hline\end{array}$		2	$n(n-1)^3$
$\begin{array}{\|cc\|}\hline 1 & 1 \\ 1 & 1 \\ \hline\end{array}$		4	$n(n-1)(n^2-3n+3)$

Etc.

And we get **exact formulas** for small fixed m:

- $1!\#\mathrm{PLR}(r,s,n;1) = \overline{111}$.
- $2!\#\mathrm{PLR}(r,s,n;2) = \overline{222} - \overline{211} + 2\,\overline{111}$.
- $3!\#\mathrm{PLR}(r,s,n;3) = \overline{333} - 3\,\overline{322} + 6\,\overline{222} + 2\,\overline{311} + 6\,\overline{221} - 12\,\overline{211} + 14\,\overline{111}$.
- $4!\#\mathrm{PLR}(r,s,n;4) = \overline{444} - 6\,\overline{433} + 12\,\overline{333} + 11\,\overline{422} + 30\,\overline{332} - 60\,\overline{322} - 6\,\overline{411} - 36\,\overline{321} - 28\,\overline{222} + 72\,\overline{311} + 198\,\overline{221} - 228\,\overline{211} + 198\,\overline{111}$.
- $5!\#\mathrm{PLR}(r,s,n;5) = \overline{555} - 10\,\overline{544} + 20\,\overline{444} + 35\,\overline{533} + 90\,\overline{443} - 180\,\overline{433} - 50\,\overline{522} - 260\,\overline{432} - 460\,\overline{333} + 520\,\overline{422} + 1350\,\overline{332} + 24\,\overline{511} + 240\,\overline{421} - 320\,\overline{322} + 480\,\overline{331} - 480\,\overline{411} - 2520\,\overline{321} - 5090\,\overline{222} + 2880\,\overline{311} + 7440\,\overline{221} - 6360\,\overline{211} + 4512\,\overline{111}$.

and so on up to 13 entries.

Method 3: Sade's Method

Sade's method is the best for exact enumeration of Latin squares.

Method 3: Sade's Method

Sade's method is the best for exact enumeration of Latin squares.

We adapt Sade's method to partial Latin rectangles (if you're familiar with Sade's method, it's what you expect).

Method 3: Sade's Method

Sade's method is the best for exact enumeration of Latin squares.

We adapt Sade's method to partial Latin rectangles (if you're familiar with Sade's method, it's what you expect).

We compute a bunch of numbers, like

$$\mathrm{PLR}(6,6,8;20) = 29211196831079424553728\,00.$$

Method 3: Sade's Method

Sade's method is the best for exact enumeration of Latin squares.

We adapt Sade's method to partial Latin rectangles (if you're familiar with Sade's method, it's what you expect).

We compute a bunch of numbers, like

$$\mathrm{PLR}(6,6,8;20) = 29211196831079424553728000.$$

- **We compute** $\#\mathrm{PLR}(r,s,n;m)$ **when** $r \le s \le n \le 7$**.**

Method 3: Sade's Method

Sade's method is the best for exact enumeration of Latin squares.

We adapt Sade's method to partial Latin rectangles (if you're familiar with Sade's method, it's what you expect).

We compute a bunch of numbers, like

$$\mathrm{PLR}(6,6,8;20) = 2921119683107942455372800.$$

- **We compute** $\#\mathrm{PLR}(r,s,n;m)$ **when** $r \leq s \leq n \leq 7$**.**
- **We compute** $\#\mathrm{PLR}(r,s,n;m)$ **when** $r \leq s \leq 6$ **and** $n = 8$**.**

Method 3: Sade's Method

Sade's method is the best for exact enumeration of Latin squares.

We adapt Sade's method to partial Latin rectangles (if you're familiar with Sade's method, it's what you expect).

We compute a bunch of numbers, like

$$\mathrm{PLR}(6,6,8;20) = 2921119683107942455372800.$$

- **We compute** $\#\mathrm{PLR}(r,s,n;m)$ **when** $r \leq s \leq n \leq 7$**.**
- **We compute** $\#\mathrm{PLR}(r,s,n;m)$ **when** $r \leq s \leq 6$ **and** $n = 8$**.**

(Thanks to Zhuanhao Wu for assistance coding.)

Method 4: Algebraic geometry

We enumerate equivalence classes: (a) paratopism classes, (b) isotopism classes, and (c) isomorphism classes.

Method 4: Algebraic geometry

We enumerate equivalence classes: (a) paratopism classes, (b) isotopism classes, and (c) isomorphism classes.

Burnside's Lemma $\implies$ We need only compute the number of PLRs which are stabilized by each possible symmetry.

Method 4: Algebraic geometry

We enumerate equivalence classes: (a) paratopism classes, (b) isotopism classes, and (c) isomorphism classes.

Burnside's Lemma $\implies$ We need only compute the number of PLRs which are stabilized by each possible symmetry.

Over the polynomial ring $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_{111}, \ldots, x_{rsn}]$, we consider the ideal

$$
\begin{aligned}
I_{r,s,n;m} :=& \langle\, x_{ijk}^2 - x_{ijk} : (i,j,k) \in [r] \times [s] \times [n] \,\rangle \\
&+ \langle\, x_{ijk}x_{i'jk} : (i,j,k) \in [r] \times [s] \times [n],\ i' \in [r],\ i < i' \,\rangle \\
&+ \langle\, x_{ijk}x_{ij'k} : (i,j,k) \in [r] \times [s] \times [n],\ j' \in [s],\ j < j' \,\rangle \\
&+ \langle\, x_{ijk}x_{ijk'} : (i,j,k) \in [r] \times [s] \times [n],\ k' \in [n],\ k < k' \,\rangle \\
&+ \langle\, m - \sum_{i \in [r]} \sum_{j \in [s]} \sum_{k \in [n]} x_{ijk} \,\rangle.
\end{aligned}
$$

Method 4: Algebraic geometry

We enumerate equivalence classes: (a) paratopism classes, (b) isotopism classes, and (c) isomorphism classes.

Burnside's Lemma $\implies$ We need only compute the number of PLRs which are stabilized by each possible symmetry.

Over the polynomial ring $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_{111}, \ldots, x_{rsn}]$, we consider the ideal

$$\begin{aligned} I_{r,s,n;m} := & \langle\, x_{ijk}^2 - x_{ijk} \colon (i,j,k) \in [r] \times [s] \times [n] \,\rangle \\ & + \langle\, x_{ijk} x_{i'jk} \colon (i,j,k) \in [r] \times [s] \times [n],\ i' \in [r],\ i < i' \,\rangle \\ & + \langle\, x_{ijk} x_{ij'k} \colon (i,j,k) \in [r] \times [s] \times [n],\ j' \in [s],\ j < j' \,\rangle \\ & + \langle\, x_{ijk} x_{ijk'} \colon (i,j,k) \in [r] \times [s] \times [n],\ k' \in [n],\ k < k' \,\rangle \\ & + \langle\, m - \sum_{i \in [r]} \sum_{j \in [s]} \sum_{k \in [n]} x_{ijk} \,\rangle. \end{aligned}$$

Zeroes of this ideal correspond to $\mathrm{PLR}(r,s,n;m)$.

We modify the ideal to account for the desired symmetry:

Theorem ("what the paper says"): Let $\Theta = (\delta_1, \delta_2, \delta_3) \in \mathfrak{I}_{r,s,n}$ and $\pi \in S_3$. Define

$$I_{(\Theta,\pi);m} := I_{r,s,n;m} + \langle\, x_{i_1 i_2 i_3} - x_{\delta_{\pi(1)}(i_{\pi(1)})\delta_{\pi(2)}(i_{\pi(2)})\delta_{\pi(3)}(i_{\pi(3)})} : \\ i_1 \in [r],\ i_2 \in [s],\ i_3 \in [n]\,\rangle.$$

Then, the set $\mathrm{PLR}((\Theta,\pi); m)$ has a natural bijection with $\mathcal{V}(I_{(\Theta,\pi);m})$ and

$$\#\mathrm{PLR}((\Theta,\pi); m) = \dim_{\mathbb{Q}}(\mathbb{Q}[\mathbf{x}]/I_{(\Theta,\pi);m}).$$

(from Seidenberg's Lemma.)

We modify the ideal to account for the desired symmetry:

Theorem ("what the paper says"): Let $\Theta = (\delta_1, \delta_2, \delta_3) \in \mathfrak{I}_{r,s,n}$ and $\pi \in S_3$. Define

$$I_{(\Theta,\pi);m} := I_{r,s,n;m} + \langle\, x_{i_1 i_2 i_3} - x_{\delta_{\pi(1)}(i_{\pi(1)})\delta_{\pi(2)}(i_{\pi(2)})\delta_{\pi(3)}(i_{\pi(3)})} : \\ i_1 \in [r],\ i_2 \in [s],\ i_3 \in [n]\,\rangle.$$

Then, the set $\mathrm{PLR}((\Theta,\pi); m)$ has a natural bijection with $\mathcal{V}(I_{(\Theta,\pi);m})$ and

$$\#\mathrm{PLR}((\Theta,\pi); m) = \dim_{\mathbb{Q}}(\mathbb{Q}[\mathbf{x}]/I_{(\Theta,\pi);m}).$$

(from Seidenberg's Lemma.)

- **This is implemented in `Singular` and `Minion` (which implement the appropriate routines).**

We modify the ideal to account for the desired symmetry:

Theorem ("what the paper says"): Let $\Theta = (\delta_1, \delta_2, \delta_3) \in \mathfrak{I}_{r,s,n}$ and $\pi \in S_3$. Define

$$I_{(\Theta,\pi);m} := I_{r,s,n;m} + \langle\, x_{i_1 i_2 i_3} - x_{\delta_{\pi(1)}(i_{\pi(1)})\delta_{\pi(2)}(i_{\pi(2)})\delta_{\pi(3)}(i_{\pi(3)})} : \\ i_1 \in [r],\ i_2 \in [s],\ i_3 \in [n]\,\rangle.$$

Then, the set $\mathrm{PLR}((\Theta, \pi); m)$ has a natural bijection with $\mathcal{V}(I_{(\Theta,\pi);m})$ and

$$\#\mathrm{PLR}((\Theta, \pi); m) = \dim_{\mathbb{Q}}(\mathbb{Q}[\mathbf{x}]/I_{(\Theta,\pi);m}).$$

(from Seidenberg's Lemma.)

- **This is implemented in `Singular` and `Minion` (which implement the appropriate routines).**
- **We compute the size of each equivalence class for $r, s, n \le 6$.**

Thank You