On the Sigma Value and Range of the Join of a Finite Number of Paths

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May 21, 2018

Definition

A vertex coloring of G is a mapping $c : V(G) \longrightarrow \mathbb{N}$.

Basic Concepts

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A vertex coloring of G is a mapping $c : V(G) \longrightarrow \mathbb{N}$.

Definition (Color Sum)

$\forall\ v \in V(G)$,

$$\sigma(v) = \sum_{u \in N_G(v)} c(u).$$

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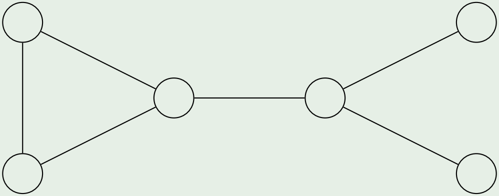
Definition (Sigma Chromatic Number)

$\sigma(G)$ is the minimum number of colors required in a sigma coloring.

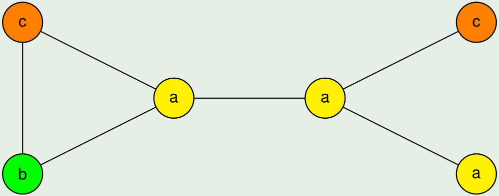
Example

G :



Example

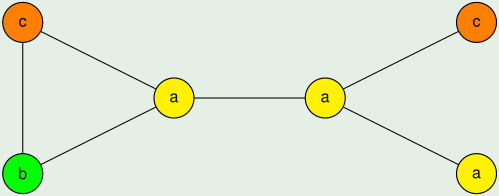
$a, b, c \in \mathbb{N},$ with $a < b < c$



Example

$a, b, c \in \mathbb{N},$ with $a < b < c$

$\sigma = a + b$



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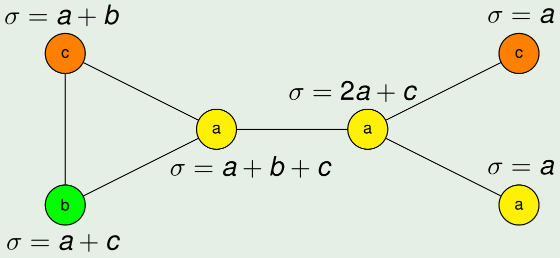
A sigma coloring of G :



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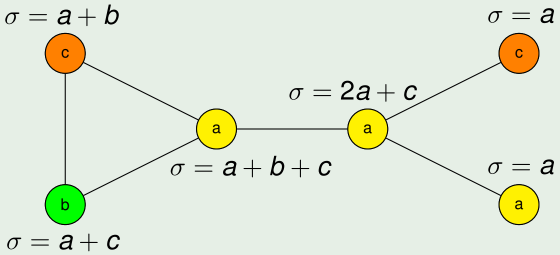
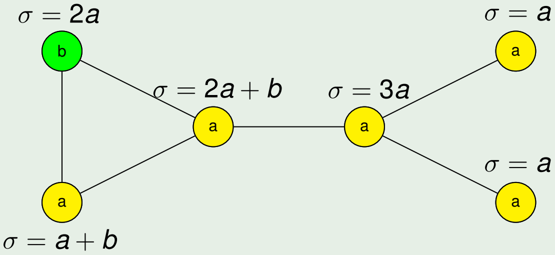


Hence, $\sigma(G) \leq 3$.

Example

$\sigma(G) = 2$:



Let G be a connected graph with $\sigma(G) = k$,

c any vertex coloring of G,

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Definition (Sigma Value)

$\nu(G) = \min\{\max(c) : c \text{ is a sigma } k\text{– coloring of } G\}$

Definition (Sigma Range)

$\rho(G) = \min\{\max(c) : c \text{ is a sigma coloring of } G\}$

Known Results

Theorem 1

$$\sigma(G) \leq \rho(G) \leq \nu(G)$$

Theorem 2

If n is any positive integer, then

$$\rho(P_n) = \nu(P_n) = \begin{cases} 1, & \text{if } n \in \{1,3\} \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 3

For every integer $n \geq 3$,

$$\rho(C_n) = \nu(C_n) = \begin{cases} 2, & \text{if } n \text{ even} \\ 3, & \text{if } n \text{ odd} \end{cases}$$

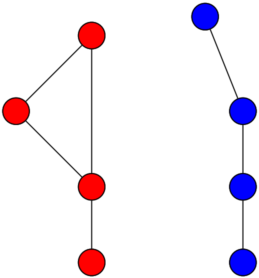
Theorem 4

Let $G = \sum_{i=1}^{k} P_{n_i}$ with $4 \leq n_1 \leq \cdots \leq n_k$. If $n_{i+2} - n_i \geq 2$ for $1 \leq i \leq k-2$ and $(n_1, n_2) \neq (4, 4)$, then

$$\sigma(G) = 2.$$

Definition

The **join** of graphs *G* and *H*,



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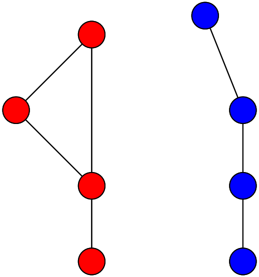

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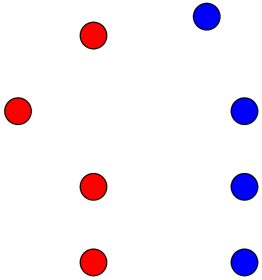

Definition

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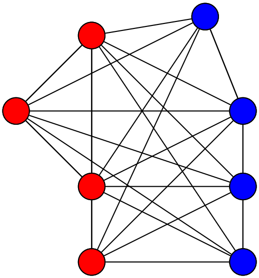
i. $V(G + H) = V(G) \cup V(H)$, and
ii. $E(G + H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$.



Observation 1

Let $G = \sum_{i=1}^{l} P_{n_i}$,

$c : V(G) \to \{1, 2\}$ a coloring of G such that $c_{|P_{n_i}}$ is a sigma 2-coloring, and

$\sigma(u) \neq 1 \ \forall u \in V(P_{n_i})$.

Let $S_i = \sum_{u \in V(P_{n_i})} c(u)$. If $S_j - S_i \geq 3$ for $i < j$, then c is a sigma 2-coloring of G.

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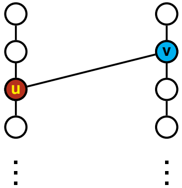

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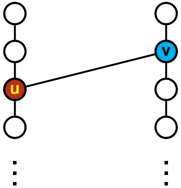
$\sigma(u) \geq 2 + S_j + \lambda$



$\sigma(v) \leq 4 + S_i + \lambda$

$$\lambda = \sum_{u \in V(G) - V(P_{n_i} + P_{n_j})} c(u)$$

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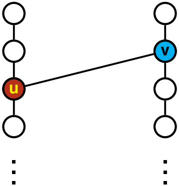

$\sigma(v) \leq 4 + S_i + \lambda$

$$\lambda = \sum_{u \in V(G) - V(P_{n_i} + P_{n_j})} c(u)$$

$\Rightarrow \sigma(v) < \sigma(u)$

$\therefore$ c is a sigma 2-coloring of G

Modular Coloring $c_n^{'}$

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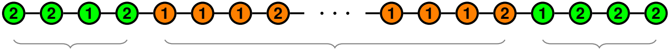

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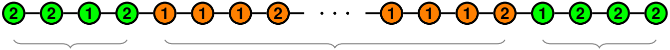

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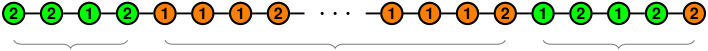

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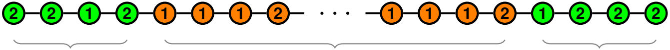

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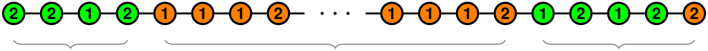

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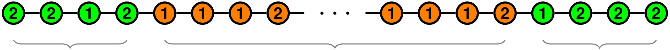

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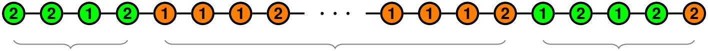

$n \equiv 2 \bmod 4$:

Let $T_n = \sum_{u \in V(P_n)} c_n^{'}(u)$.

If $n \equiv 3 \bmod (4)$, then $T_n = 5(\frac{n-3}{4}) + 2 + 5$.

T_n = sum of the colors of the vertices of P_n using the modular coloring

$$T_n = \begin{cases} 5(\frac{n}{4}) + 4, & \text{if } n \equiv 0 \bmod 4 \\ 5(\frac{n-1}{4}) + 5, & \text{if } n \equiv 1 \bmod 4 \\ 5(\frac{n-2}{4}) + 8, & \text{if } n \equiv 2 \bmod 4 \\ 5(\frac{n-3}{4}) + 7, & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

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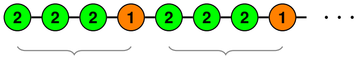
M_n = maximum sum in a sigma 2-coloring (using colors 1 and 2) of P_n



$$M_n = \begin{cases} 7(\frac{n}{4}), & \text{if } n \equiv 0 \bmod 4 \\ 7(\frac{n-1}{4}) + 2, & \text{if } n \equiv 1 \bmod 4 \\ 7(\frac{n-2}{4}) + 4, & \text{if } n \equiv 2 \bmod 4 \\ 7(\frac{n-3}{4}) + 6, & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

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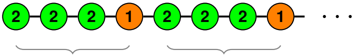

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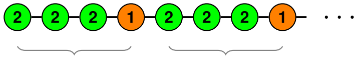

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Note that if $n \geq 13$, then $M_n - T_n \geq 3$.

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For every integer S_n such that $T_n \leq S_n \leq M_n$, there exists a sigma 2-coloring c of the vertices of P_n using the colors 1 and 2 such that $\sum_{v \in V(P_n)} c(v) = S_n$.

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- If $S_n - T_n$ is even and $S_n < M_n$,

$$c(v_k) = \begin{cases} 2, & \text{if } k \equiv 1, 2 \pmod 4), \text{where } 1 \leq k \leq 2s - 2 \\ c_n'(v_k), & \text{otherwise.} \end{cases}$$

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  Then,

$$\sigma_c(v_k) = \begin{cases} 3, & \text{if } 4 \leq k \leq 2s + 2 \text{ is even} \\ 4, & \text{if } 4 \leq k \leq 2s + 3 \text{ is odd} \\ \sigma_{c'_n}(v_k), & \text{otherwise.} \end{cases}$$

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- If $S_n - T_n$ is even and $S_n < M_n$,

$$c(v_k) = \begin{cases} 2, & \text{if } k \equiv 1, 2 \pmod 4, \text{where } 1 \leq k \leq 2s - 2 \\ c'_n(v_k), & \text{otherwise.} \end{cases}$$

- If $S_n - T_n$ is odd and $S_n < M_n$,

$$c(v_k) = \begin{cases} 2, & \text{if } k \equiv 1, 2 \pmod 4, \text{where } 1 \leq k \leq 2s \text{ and } k \neq 2s - 1 \\ c'_n(v_k), & \text{otherwise.} \end{cases}$$

Example:

Suppose $n = 18$.

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- $T_{18} = 27$
- $M_{18} = 32$
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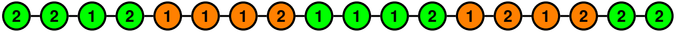

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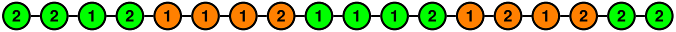
$S_{18} = 27$: 

$S_{18} = 28$:

Example:

Suppose $n = 18$. Then,

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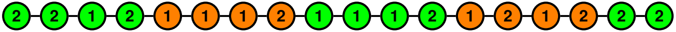
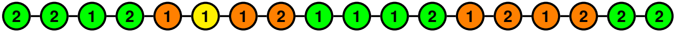
$S_{18} = 27$:


$S_{18} = 28$:


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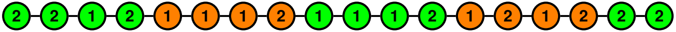
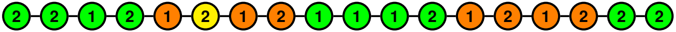
$S_{18} = 27$: 

$S_{18} = 28$: 

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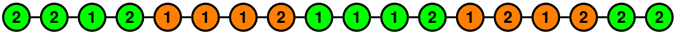

$S_{18} = 29$:

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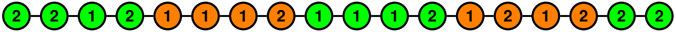
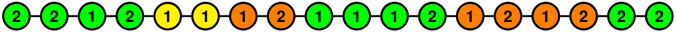
$S_{18} = 27$: 

$S_{18} = 29$: 

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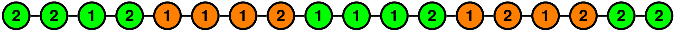

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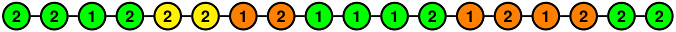

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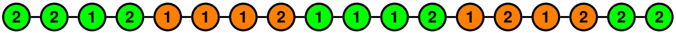
$S_{18} = 27$: 

$S_{18} = 30$:

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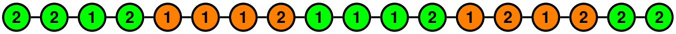
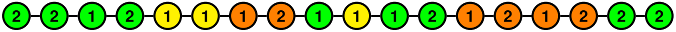
$S_{18} = 27$: 

$S_{18} = 30$: 

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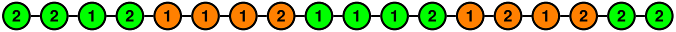
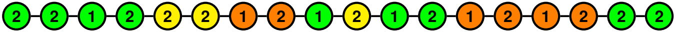


$S_{18} = 30$:



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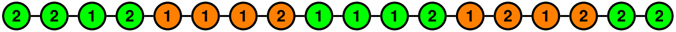
$S_{18} = 27$: 

$S_{18} = 31$:

Example:

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- $M_{18} = 32$
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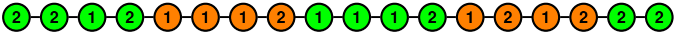
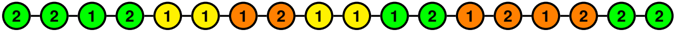
$S_{18} = 27$:


$S_{18} = 31$:


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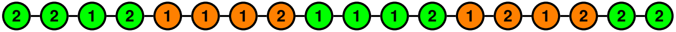
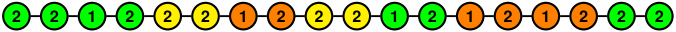


$S_{18} = 31$:



Example:

Suppose $n = 18$. Then,

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- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$S_{18} = 27$: 2–2–1–2–1–1–1–2–1–1–1–2–1–2–1–2–2–2

$S_{18} = 32$: 2–2–2–1–2–2–2–1–2–2–2–1–2–2–2–1–2–2

Main Result

Theorem 3

Let $G = \sum_{i=k}^{l} P_{n_i}$ with $13 \leq n_1 \leq \cdots \leq n_l$. If $n_{i+2} - n_i \geq 4$ for $1 \leq i \leq k - 2$, then

$$\rho(G) = \nu(G) = 2.$$

Outline of Proof:

- Let $G = \sum_{i=1}^{k} P_{n_i}$ be any graph such that $13 \leq n_1 \leq n_2 \leq \cdots \leq n_l$ and $n_{i+2} - n_i \geq 4$ for each $1 \leq i \leq k-2$

Outline of Proof:

- Let $G = \sum_{i=1}^{k} P_{n_i}$ be any graph such that $13 \leq n_1 \leq n_2 \leq \cdots \leq n_l$ and $n_{i+2} - n_i \geq 4$ for each $1 \leq i \leq k - 2$
- **Claim:** There exists $S_{n_1}, S_{n_2}, \cdots, S_{n_k}$ such that
  - $T_{n_i} \leq S_{n_i} \leq M_{n_i}$ for all $1 \leq i \leq k$
  - $S_{n_{i+1}} - S_{n_i} \geq 3$ for all $1 \leq i \leq k - 1$

Outline of Proof:

- Let $G = \sum_{i=1}^{k} P_{n_i}$ be any graph such that $13 \leq n_1 \leq n_2 \leq \cdots \leq n_l$ and $n_{i+2} - n_i \geq 4$ for each $1 \leq i \leq k-2$
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  - $S_{n_{i+1}} - S_{n_i} \geq 3$ for all $1 \leq i \leq k-1$
- By Lemma 1, there exists a sigma 2-coloring $c_i : V(P_{n_i}) \to \{1,2\}$ such that $S_{n_i} = \sum_{v \in V(P_{n_i})} c_i(v)$ and $\sigma(v) \neq 1 \; \forall v \in V(P_{n_i})$.

Outline of Proof:

- Let $c : V(G) \rightarrow \{1, 2\}$ be the coloring determined by $c_1, c_2, \cdots, c_k$.

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- Let $c : V(G) \rightarrow \{1, 2\}$ be the coloring determined by $c_1, c_2, \cdots, c_k$.
- By Observation 1, it follows that c is a sigma 2-coloring of G using the colors 1 and 2.
- Therefore, $\rho(G) = \sigma(G) = 2$.

Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

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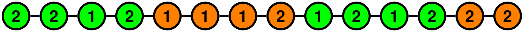
P_{14} :

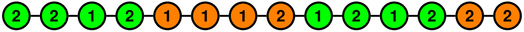


$T_{14} = 22, \quad M_{14} = 25,$

Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

P_{14} :



$T_{14} = 22, \quad M_{14} = 25,$

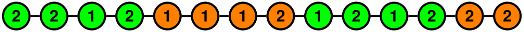
P_{15} :

$T_{15} = 22, \quad M_{15} = 27,$

Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

P_{14} :



$T_{14} = 22, \quad M_{14} = 25,$

P_{15} :

$T_{15} = 22, \quad M_{15} = 27,$

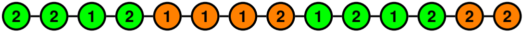
P_{19} :

$T_{19} = 27, \quad M_{19} = 34,$

Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

P_{14} :



$T_{14} = 22, \quad M_{14} = 25,$

P_{15} :

$T_{15} = 22, \quad M_{15} = 27,$

P_{19} :

$T_{19} = 27, \quad M_{19} = 34,$

P_{19} :

$T_{19} = 27, \quad T_{19} = 34,$

Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

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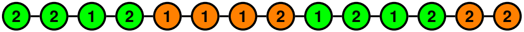


$T_{14} = 22$, $M_{14} = 25$, $S_{14} = 22$

P_{15} :

$T_{15} = 22$, $M_{15} = 27$,

P_{19} :

$T_{19} = 27$, $M_{19} = 34$,

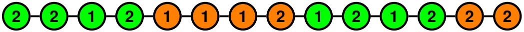
P_{19} :

$T_{19} = 27$, $T_{19} = 34$,

Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

P_{14} :



$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$

P_{15} :

$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$

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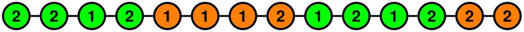
P_{19} :

$T_{19} = 27, \quad T_{19} = 34,$

Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

P_{14} :



$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$

P_{15} :

$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$

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$T_{19} = 27, \quad M_{19} = 34,$

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Example:

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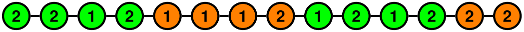

$T_{14} = 22$, $M_{14} = 25$, $S_{14} = 22$

P_{15} :

$T_{15} = 22$, $M_{15} = 27$, $S_{15} = 25$

P_{19} :

$T_{19} = 27$, $M_{19} = 34$,

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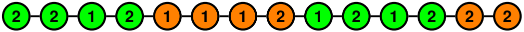

$T_{14} = 22$, $M_{14} = 25$, $S_{14} = 22$

P_{15} :

$T_{15} = 22$, $M_{15} = 27$, $S_{15} = 25$

P_{19} :

$T_{19} = 27$, $M_{19} = 34$, $S_{19} = 28$

P_{19} :

$T_{19} = 27$, $T_{19} = 34$,

Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

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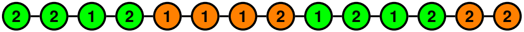

$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$

P_{15} :

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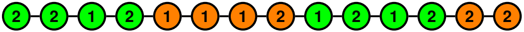

$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$

P_{15} :

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$T_{19} = 27, \quad M_{19} = 34, \quad S_{19} = 28$

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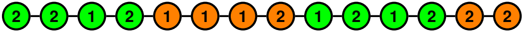
Example:

$G = P_{14} + P_{15} + P_{19} + P_{19}$:

P_{14} : 

$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$

P_{15} : 

$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$

P_{19} : 

$T_{19} = 27, \quad M_{19} = 34, \quad S_{19} = 28$

P_{19} : 

$T_{19} = 27, \quad T_{19} = 34, \quad S_{19} = 31$

Example:

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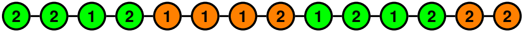

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$T_{19} = 27, \quad T_{19} = 34, \quad S_{19} = 31$

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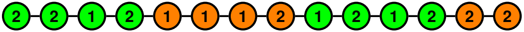

$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$

P_{15} :

$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$

P_{19} :

$T_{19} = 27, \quad M_{19} = 34, \quad S_{19} = 28$

P_{19} :

$T_{19} = 27, \quad T_{19} = 34, \quad S_{19} = 31$

Other Results:

Theorem 4

Let $G = \sum_{i=1}^{k} P_{n_i}$ with $1 \leq n_1 < \cdots < n_k$. If $n_{i+1} - n_i \geq 2$ for $1 \leq i \leq k-1$, then

$$\rho(G) = \nu(G) = 2.$$

Other Results:

Theorem 5

Let $G = \sum_{i=1}^{k} C_{n_i}$, where n_i is even and $4 \leq n_1 < n_2 < \cdots < n_k$. Then,

$$\rho(G) = \nu(G) = 2.$$

Theorem 6

Let $G = \sum_{i=1}^{k} C_{n_i}$, where n_i is even and $4 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. If $n_{i+2} - n_i \geq 4$ for each $1 \leq i \leq k-2$ and $(n_1, n_2) \notin \{(4,4),(6,6)\}$, then

$$\rho(G) = \nu(G) = 2.$$

Ongoing Study:

Let $G = \sum_{i=1}^{k} C_{n_i}$, where $3 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and at least one C_{n_i} is odd. If $n_{i+2} - n_i \geq 3$ for each $1 \leq i \leq k-2$ and $(n_1, n_2) \notin \{(3,3),(5,5)\}$, then,

$$\rho(G) = \nu(G) = 3.$$

References

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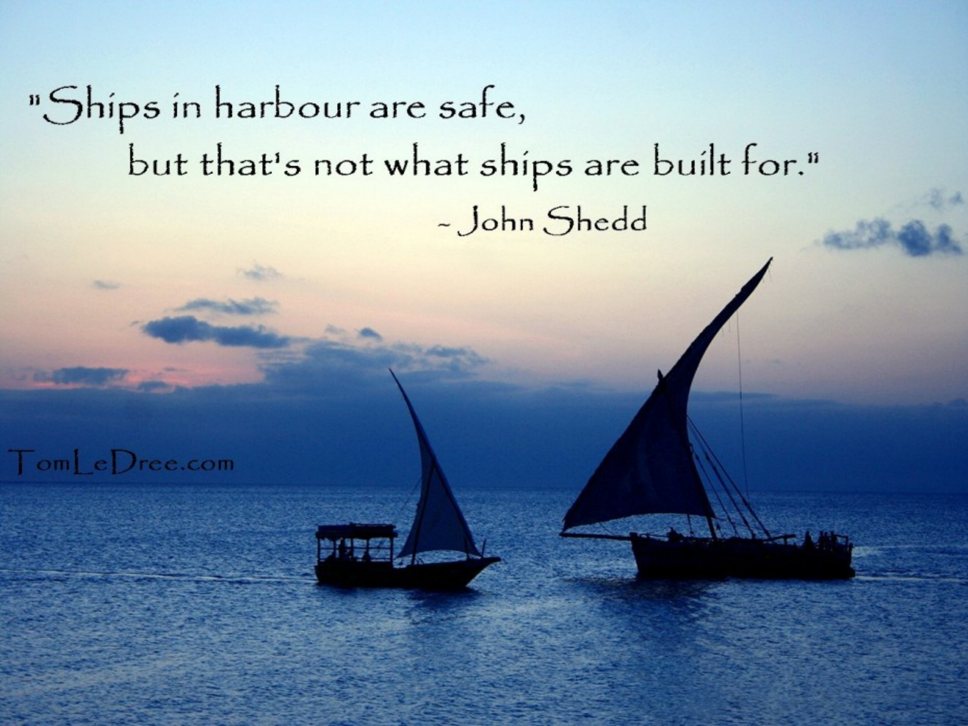
"Ships in harbour are safe,
but that's not what ships are built for."
~ John Shedd
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