

On the Sigma Value and Range of the Join of a Finite Number of Paths

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Basic Concepts

Definition

A **vertex coloring** of G is a mapping $c : V(G) \rightarrow \mathbb{N}$.

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Definition (Color Sum)

$\forall v \in V(G)$,

$$\sigma(v) = \sum_{u \in N_G(v)} c(u).$$

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c is a **sigma coloring** if $\sigma(u) \neq \sigma(v)$, $\forall uv \in E(G)$.

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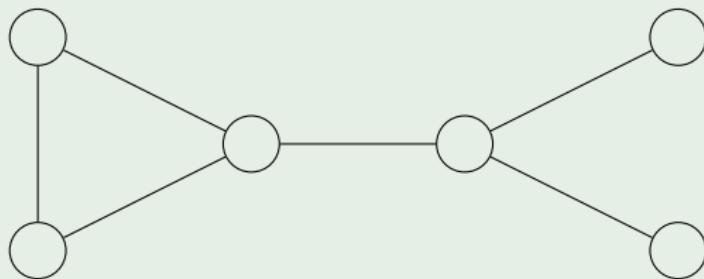
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Definition (Sigma Chromatic Number)

$\sigma(G)$ is the minimum number of colors required in a sigma coloring.

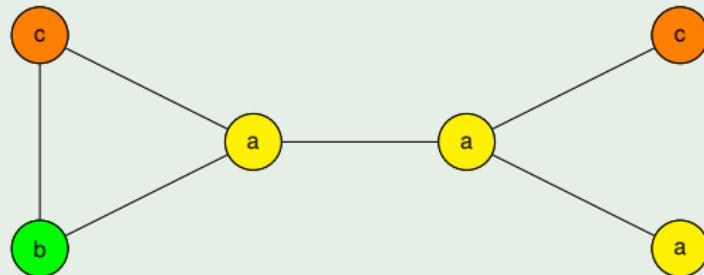
Example

$G :$



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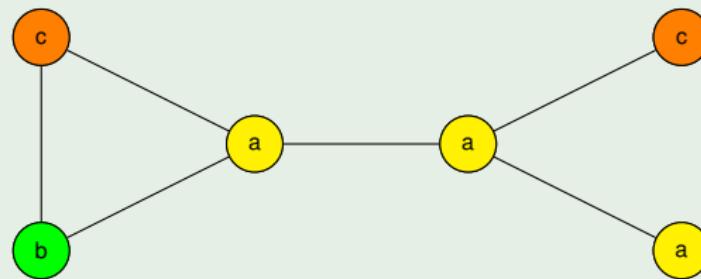
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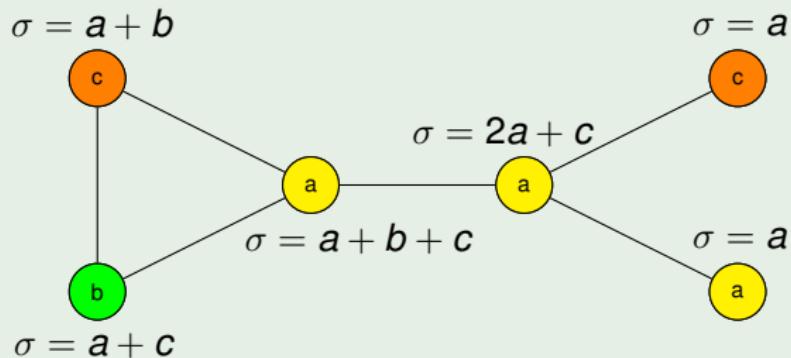
$a, b, c \in \mathbb{N}$, with $a < b < c$

$$\sigma = a + b$$



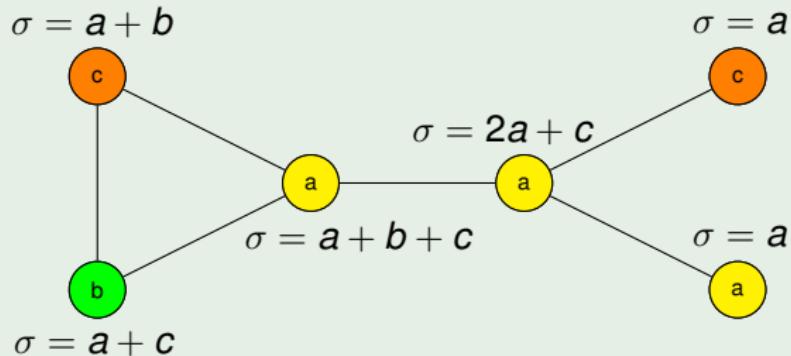
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A sigma coloring of G :



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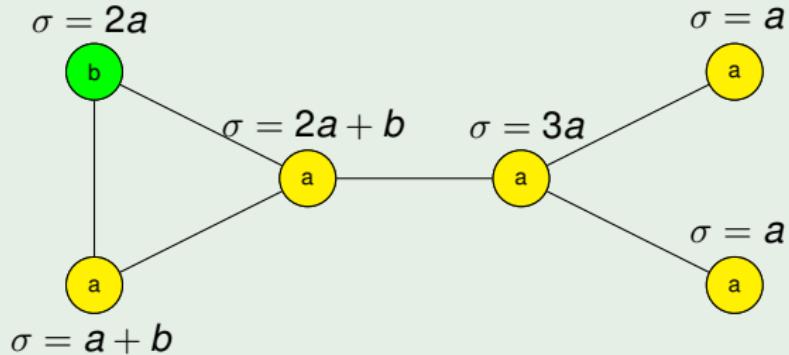
A sigma coloring of G :



Hence, $\sigma(G) \leq 3$.

Example

$$\sigma(G) = 2 :$$



Let G be a connected graph with $\sigma(G) = k$,

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$$\nu(G) = \min\{\max(c) : c \text{ is a sigma } k-\text{coloring of } G\}$$

Definition (Sigma Range)

$$\rho(G) = \min\{\max(c) : c \text{ is a sigma coloring of } G\}$$

Known Results

Theorem 1

$$\sigma(G) \leq \rho(G) \leq \nu(G)$$

Theorem 2

If n is any positive integer, then

$$\rho(P_n) = \nu(P_n) = \begin{cases} 1, & \text{if } n \in \{1, 3\} \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 3

For every integer $n \geq 3$,

$$\rho(C_n) = \nu(C_n) = \begin{cases} 2, & \text{if } n \text{ even} \\ 3, & \text{if } n \text{ odd} \end{cases}$$

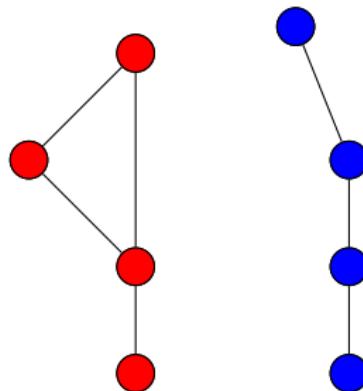
Theorem 4

Let $G = \sum_{i=1}^k P_{n_i}$ with $4 \leq n_1 \leq \dots \leq n_k$. If $n_{i+2} - n_i \geq 2$ for $1 \leq i \leq k-2$ and $(n_1, n_2) \neq (4, 4)$, then

$$\sigma(G) = 2.$$

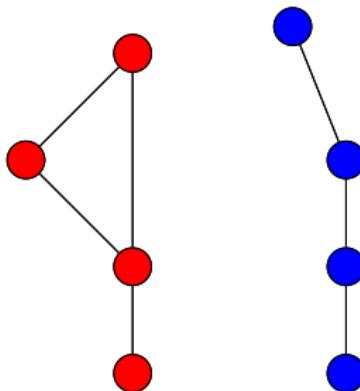
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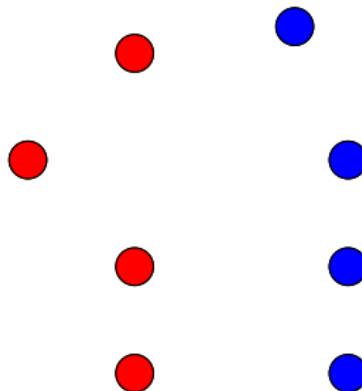
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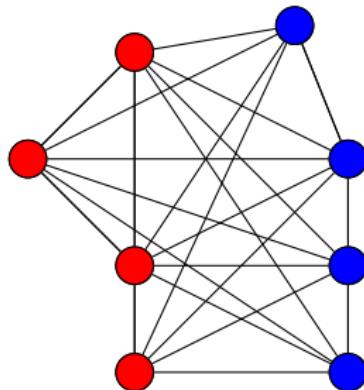
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The **join** of graphs G and H , denoted by $G + H$, with

- i. $V(G + H) = V(G) \cup V(H)$, and
- ii. $E(G + H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$.



Observation 1

Let $G = \sum_{i=1}^l P_{n_i}$,

$c : V(G) \rightarrow \{1, 2\}$ a coloring of G such that $c|_{P_{n_i}}$ is a sigma 2-coloring, and

$$\sigma(u) \neq 1 \quad \forall u \in V(P_{n_i}).$$

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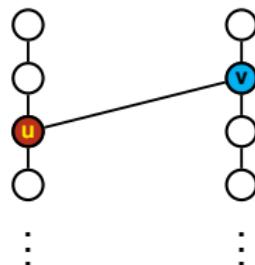
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Proof: If $u \in V(P_{n_i}), v \in V(P_{n_j}), i < j :$

$$G = \dots + P_{n_i} + \dots + P_{n_j} + \dots$$



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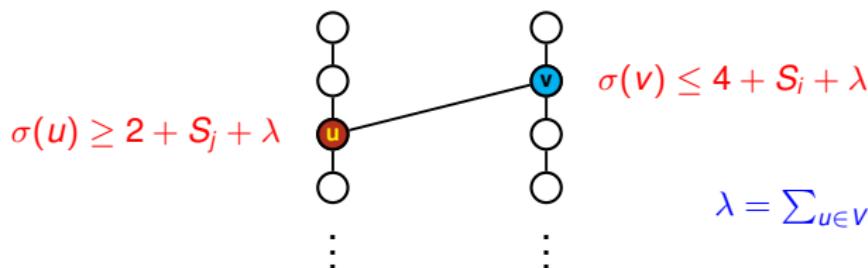
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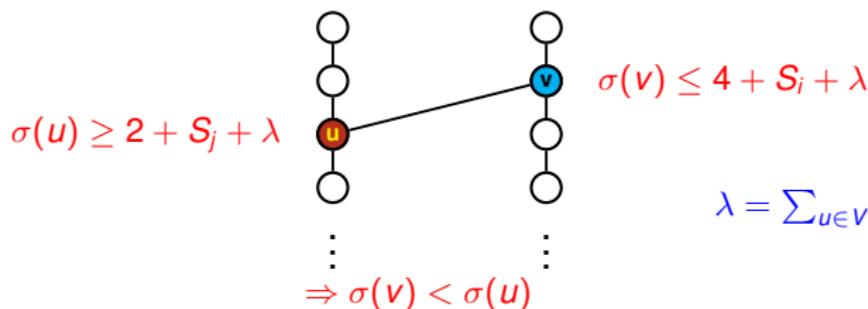
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Proof: If $u \in V(P_{n_i}), v \in V(P_{n_j}), i < j :$

$$G = \dots + P_{n_i} + \dots + P_{n_j} + \dots$$



$$\lambda = \sum_{u \in V(G) - V(P_{n_i} + P_{n_j})} c(u)$$

$\therefore c$ is a sigma 2-coloring of G

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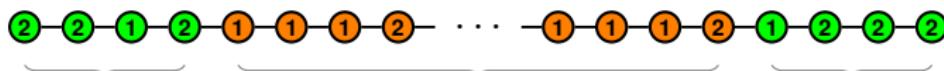


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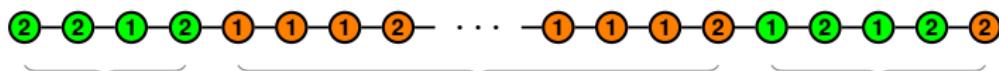
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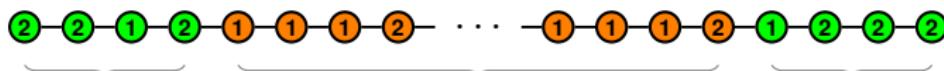


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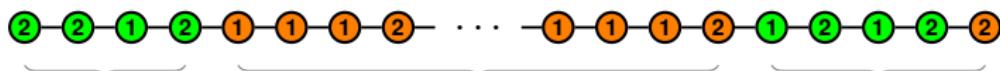
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Let $T_n = \sum_{u \in V(P_n)} c'_n(u)$.

If $n \equiv 3 \pmod{4}$, then $T_n = 5(\frac{n-3}{4}) + 2 + 5$.

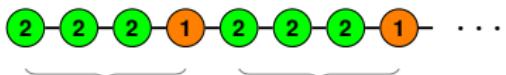
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M_n = maximum sum in a sigma 2-coloring (using colors 1 and 2) of P_n



$$M_n = \begin{cases} 7\left(\frac{n}{4}\right), & \text{if } n \equiv 0 \pmod{4} \\ 7\left(\frac{n-1}{4}\right) + 2, & \text{if } n \equiv 1 \pmod{4} \\ 7\left(\frac{n-2}{4}\right) + 4, & \text{if } n \equiv 2 \pmod{4} \\ 7\left(\frac{n-3}{4}\right) + 6, & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

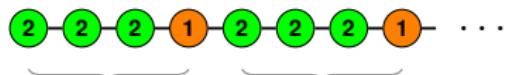
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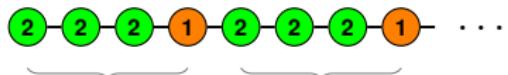
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Note that if $n \geq 13$, then $M_n - T_n \geq 3$.

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For every integer S_n such that $T_n \leq S_n \leq M_n$, there exists a sigma 2-coloring c of the vertices of P_n using the colors 1 and 2 such that $\sum_{v \in V(P_n)} c(v) = S_n$.

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$$S_{18} = 27 : \text{2 green circles} - \text{2 green circles} - \text{1 orange circle} - \text{2 green circles} - \text{1 orange circle} - \text{1 orange circle} - \text{1 orange circle} - \text{2 green circles} - \text{1 orange circle} - \text{2 green circles} - \text{1 orange circle} - \text{2 green circles}$$

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$$S_{18} = 27 : \text{Diagram showing 18 circles in a row. Circles 1 through 18 alternate between green (labeled 2) and orange (labeled 1).}$$

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- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$S_{18} = 27$: A sequence of 18 numbered circles. The sequence starts with a green circle containing the number 2, followed by two orange circles containing the number 2, one green circle containing the number 1, one orange circle containing the number 2, one orange circle containing the number 1, one orange circle containing the number 1, one green circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 1, one green circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 2, and one green circle containing the number 2.

$S_{18} = 28$: A sequence of 18 numbered circles. The sequence starts with a green circle containing the number 2, followed by two orange circles containing the number 2, one green circle containing the number 1, one orange circle containing the number 2, one yellow circle containing the number 2, one orange circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 1, one green circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 1, one orange circle containing the number 2, one green circle containing the number 2, and one green circle containing the number 2.

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$$S_{18} = 27 : \text{ (Diagram showing 18 circles in a row, alternating between green and orange, with values 2 or 1 inside each circle.)}$$

$$S_{18} = 29 :$$

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$S_{18} = 27$: 

$S_{18} = 29$: 

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$S_{18} = 27$: 

$S_{18} = 29$: 

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$$S_{18} = 27 : \text{Diagram showing 18 circles in a row. The sequence starts with two green circles (labeled 2), followed by two orange circles (labeled 1), then two green circles (labeled 2), one orange circle (labeled 1), one green circle (labeled 1), one orange circle (labeled 1), one green circle (labeled 2), one green circle (labeled 1), one orange circle (labeled 1), one green circle (labeled 2), one orange circle (labeled 1), one green circle (labeled 2), one orange circle (labeled 1), one green circle (labeled 2), one orange circle (labeled 1), one green circle (labeled 2), and ends with two green circles (labeled 2).}$$

$$S_{18} = 30 :$$

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$S_{18} = 27$: A sequence of 18 circles representing $S_{18} = 27$. The sequence starts with two green circles (2), followed by two orange circles (2), one green circle (1), one orange circle (2), one orange circle (1), one green circle (1), one orange circle (1), one green circle (2), one green circle (1), one orange circle (2), one green circle (1), one orange circle (2), one green circle (1), one orange circle (2), one green circle (2), one green circle (2).

$S_{18} = 30$: A sequence of 18 circles representing $S_{18} = 30$. The sequence starts with two green circles (2), followed by two orange circles (2), one green circle (1), one yellow circle (2), one yellow circle (1), one orange circle (1), one orange circle (2), one green circle (1), one yellow circle (1), one green circle (1), one orange circle (2), one green circle (1), one orange circle (2), one green circle (1), one orange circle (2), one green circle (2), one green circle (2).

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$S_{18} = 27$:

$S_{18} = 30$:

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$$S_{18} = 27 : \text{ (Diagram showing 18 circles in a row, alternating between green and orange, with values 2 or 1 inside each circle.)}$$

$$S_{18} = 31 :$$

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$S_{18} = 27$:

$S_{18} = 31$:

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
- $M_{18} = 32$
- $27 \leq S_{18} \leq 32$

$S_{18} = 27$:  A horizontal sequence of 18 circles. The colors of the circles follow a repeating pattern: green, green, orange, green, orange, orange, orange, green, green, green, green, orange, green, orange, green, orange, green, green, green.

$S_{18} = 31$:  A horizontal sequence of 18 circles. The colors of the circles follow a repeating pattern: green, green, orange, green, yellow, yellow, orange, green, green, green, orange, green, orange, green, orange, green, green, green, green.

Example:

Suppose $n = 18$. Then,

- $T_{18} = 27$
 - $M_{18} = 32$
 - $27 \leq S_{18} \leq 32$

$$S_{18} = 27 : \textcolor{blue}{2}-2-\textcolor{blue}{1}-\textcolor{blue}{2}-\textcolor{orange}{1}-\textcolor{blue}{1}-\textcolor{blue}{1}-\textcolor{blue}{2}-\textcolor{blue}{1}-\textcolor{blue}{1}-\textcolor{blue}{1}-\textcolor{blue}{2}-\textcolor{orange}{1}-\textcolor{blue}{2}-\textcolor{blue}{1}-\textcolor{blue}{2}-\textcolor{blue}{2}-\textcolor{blue}{2}$$

$$S_{18} = 32 : \textcolor{red}{2} \textcolor{blue}{2} \textcolor{blue}{2} \textcolor{blue}{1} \textcolor{red}{2} \textcolor{blue}{2} \textcolor{blue}{2} \textcolor{blue}{2} \textcolor{blue}{1} \textcolor{red}{2} \textcolor{blue}{2} \textcolor{blue}{2} \textcolor{blue}{2} \textcolor{blue}{1} \textcolor{red}{2} \textcolor{blue}{2} \textcolor{blue}{2} \textcolor{blue}{2} \textcolor{blue}{1} \textcolor{red}{2} \textcolor{blue}{2}$$

Main Result

Theorem 3

Let $G = \sum_{i=k}^l P_{n_i}$ with $13 \leq n_1 \leq \dots \leq n_l$. If $n_{i+2} - n_i \geq 4$ for $1 \leq i \leq k-2$, then

$$\rho(G) = \nu(G) = 2.$$

Outline of Proof:

- Let $G = \sum_{i=1}^k P_{n_i}$ be any graph such that
 $13 \leq n_1 \leq n_2 \leq \dots \leq n_l$ and $n_{i+2} - n_i \geq 4$ for each
 $1 \leq i \leq k-2$

Outline of Proof:

- Let $G = \sum_{i=1}^k P_{n_i}$ be any graph such that
 $13 \leq n_1 \leq n_2 \leq \dots \leq n_l$ and $n_{i+2} - n_i \geq 4$ for each
 $1 \leq i \leq k-2$
- Claim:** There exists $S_{n_1}, S_{n_2}, \dots, S_{n_k}$ such that
 - $T_{n_i} \leq S_{n_i} \leq M_{n_i}$ for all $1 \leq i \leq k$
 - $S_{n_{i+1}} - S_{n_i} \geq 3$ for all $1 \leq i \leq k-1$

Outline of Proof:

- Let $G = \sum_{i=1}^k P_{n_i}$ be any graph such that
 $13 \leq n_1 \leq n_2 \leq \dots \leq n_l$ and $n_{i+2} - n_i \geq 4$ for each
 $1 \leq i \leq k-2$
- Claim:** There exists $S_{n_1}, S_{n_2}, \dots, S_{n_k}$ such that
 - $T_{n_i} \leq S_{n_i} \leq M_{n_i}$ for all $1 \leq i \leq k$
 - $S_{n_{i+1}} - S_{n_i} \geq 3$ for all $1 \leq i \leq k-1$
- By Lemma 1, there exists a sigma 2-coloring
 $c_i : V(P_{n_i}) \rightarrow \{1, 2\}$ such that $S_{n_i} = \sum_{v \in V(P_{n_i})} c_i(v)$ and
 $\sigma(v) \neq 1 \ \forall v \in V(P_{n_i})$.

Outline of Proof:

- Let $c : V(G) \rightarrow \{1, 2\}$ be the coloring determined by c_1, c_2, \dots, c_k .

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- Let $c : V(G) \rightarrow \{1, 2\}$ be the coloring determined by c_1, c_2, \dots, c_k .
- By Observation 1, it follows that c is a sigma 2-coloring of G using the colors 1 and 2.

Outline of Proof:

- Let $c : V(G) \rightarrow \{1, 2\}$ be the coloring determined by c_1, c_2, \dots, c_k .
- By Observation 1, it follows that c is a sigma 2-coloring of G using the colors 1 and 2.
- Therefore, $\rho(G) = \sigma(G) = 2$.

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

$$P_{14} : \quad \begin{array}{ccccccccccccccccc} 2 & - & 2 & - & 1 & - & 2 & - & 1 & - & 1 & - & 1 & - & 2 & - & 1 & - & 2 & - & 2 & - & 2 \\ \textcolor{blue}{T_{14} = 22}, \quad \textcolor{blue}{M_{14} = 25}, \end{array}$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

$$P_{14} : \quad \begin{array}{ccccccccccccccccc} 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 \\ \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} \end{array}$$

$$T_{14} = 22, \quad M_{14} = 25,$$

$$P_{15} : \quad \begin{array}{ccccccccccccccccc} 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 \\ \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} \end{array}$$

$$T_{15} = 22, \quad M_{15} = 27,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

$$P_{14} : \quad \begin{array}{ccccccccccccccccc} 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 \\ \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} \end{array}$$

$$T_{14} = 22, \quad M_{14} = 25,$$

$$P_{15} : \quad \begin{array}{ccccccccccccccccc} 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 \\ \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} \end{array}$$

$$T_{15} = 22, \quad M_{15} = 27,$$

$$P_{19} : \quad \begin{array}{ccccccccccccccccc} 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 \\ \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{orange}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} & \textcolor{green}{\circ} \end{array}$$

$$T_{19} = 27, \quad M_{19} = 34,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

P₁₄:



$$T_{14} = 22, \quad M_{14} = 25,$$

$P_{15} :$



$$T_{15} = 22, \quad M_{15} = 27,$$

P₁₉:



$$T_{19} = 27, \quad M_{19} = 34,$$

P₁₉:



$$T_{19} = 27, \quad T_{19} = 34,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

P₁₄:



$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

P₁₅ :



$$T_{15} = 22, \quad M_{15} = 27.$$

P₁₉:



$$T_{19} = 27, M_{19} = 34,$$

P₁₉:



$$T_{19} = 27, \quad T_{19} = 34,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

P₁₄:



$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

P₁₅ :



$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

P₁₉:



$$T_{19} = 27, M_{19} = 34,$$

P₁₉:



$$T_{19} = 27, \quad T_{19} = 34.$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

P₁₄ :



$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

P₁₅ :



$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

P₁₉:



$$T_{19} = 27, \quad M_{19} = 34,$$

P₁₉:



$$T_{19} = 27, \quad T_{19} = 34,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

$$P_{14} : \quad \text{Diagram showing a sequence of 14 circles. Circles 1, 3, 5, 7, 9, 11, 13 are green (labeled 2). Circles 2, 4, 6, 8, 10, 12, 14 are orange (labeled 1).}$$

$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

$$P_{15} : \quad \text{Diagram showing a sequence of 15 circles. Circles 1, 3, 5, 7, 9, 11, 13, 15 are green (labeled 2). Circles 2, 4, 6, 8, 10, 12, 14 are orange (labeled 1). Circle 10 is yellow (labeled 2).}$$

$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

$$P_{19} : \quad \text{Diagram showing a sequence of 19 circles. Circles 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 are green (labeled 2). Circles 2, 4, 6, 8, 10, 12, 14, 16, 18 are orange (labeled 1). Circles 10 and 12 are yellow (labeled 2).}$$

$$T_{19} = 27, \quad M_{19} = 34,$$

$$P_{19} : \quad \text{Diagram showing a sequence of 19 circles. Circles 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 are green (labeled 2). Circles 2, 4, 6, 8, 10, 12, 14, 16, 18 are orange (labeled 1). Circles 10 and 12 are yellow (labeled 2).}$$

$$T_{19} = 27, \quad T_{19} = 34,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

$$P_{14} : \quad \text{Diagram showing 14 nodes in a sequence. Nodes 1 through 13 are green (labeled 2), and node 14 is orange (labeled 1).}$$

$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

$$P_{15} : \quad \text{Diagram showing 15 nodes in a sequence. Nodes 1 through 14 are green (labeled 2), and node 15 is orange (labeled 1).}$$

$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

$$P_{19} : \quad \text{Diagram showing 19 nodes in a sequence. Nodes 1 through 18 are green (labeled 2), and node 19 is orange (labeled 1).}$$

$$T_{19} = 27, \quad M_{19} = 34, \quad S_{19} = 28$$

$$P_{19} : \quad \text{Diagram showing 19 nodes in a sequence. Nodes 1 through 18 are green (labeled 2), and node 19 is orange (labeled 1).}$$

$$T_{19} = 27, \quad T_{19} = 34,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

$$P_{14} : \quad \text{Diagram showing a sequence of 14 circles. Circles 1, 3, 5, 7, 9, 11, 13 are green (labeled 2). Circles 2, 4, 6, 8, 10, 12, 14 are orange (labeled 1).}$$

$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

$$P_{15} : \quad \text{Diagram showing a sequence of 15 circles. Circles 1, 3, 5, 7, 9, 11, 13, 15 are green (labeled 2). Circles 2, 4, 6, 8, 10, 12, 14 are orange (labeled 1). Circle 10 is yellow (labeled 2).}$$

$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

$$P_{19} : \quad \text{Diagram showing a sequence of 19 circles. Circles 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 are green (labeled 2). Circles 2, 4, 6, 8, 10, 12, 14, 16, 18 are orange (labeled 1). Circle 10 is yellow (labeled 1).}$$

$$T_{19} = 27, \quad M_{19} = 34, \quad S_{19} = 28$$

$$P_{19} : \quad \text{Diagram showing a sequence of 19 circles. Circles 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 are green (labeled 2). Circles 2, 4, 6, 8, 10, 12, 14, 16, 18 are orange (labeled 1).}$$

$$T_{19} = 27, \quad T_{19} = 34,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

$$P_{14} : \quad \text{Diagram showing 14 nodes in a sequence. Nodes 1 through 13 are green (labeled 2), and node 14 is orange (labeled 1).}$$

$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

$$P_{15} : \quad \text{Diagram showing 15 nodes in a sequence. Nodes 1 through 14 are green (labeled 2), and node 15 is orange (labeled 1).}$$

$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

$$P_{19} : \quad \text{Diagram showing 19 nodes in a sequence. Nodes 1 through 18 are green (labeled 2), and node 19 is orange (labeled 1).}$$

$$T_{19} = 27, \quad M_{19} = 34, \quad S_{19} = 28$$

$$P_{19} : \quad \text{Diagram showing 19 nodes in a sequence. Nodes 1 through 18 are green (labeled 2), and node 19 is orange (labeled 1).}$$

$$T_{19} = 27, \quad T_{19} = 34,$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

P₁₄:



$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

P₁₅:



$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

P₁₉:



$$T_{19} = 27, \quad M_{19} = 34, \quad S_{19} = 28$$

P₁₉:



$$T_{19} \equiv 27, \quad T_{19} \equiv 34, \quad S_{19} \equiv 31$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

P₁₄:



$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

$P_{15} :$



$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

P₁₉:



$$T_{19} = 27, \quad M_{19} = 34, \quad S_{19} = 28$$

P₁₉:



$$T_{19} = 27, \quad T_{19} = 34, \quad S_{19} = 31$$

Example:

$$G = P_{14} + P_{15} + P_{19} + P_{19} :$$

P₁₄:



$$T_{14} = 22, \quad M_{14} = 25, \quad S_{14} = 22$$

P₁₅:



$$T_{15} = 22, \quad M_{15} = 27, \quad S_{15} = 25$$

P₁₉:



$$T_{18} = 27, \quad M_{18} = 34, \quad S_{18} = 28$$

P₁₉:



$$T_{18} = 27, \quad T_{18} = 34, \quad S_{18} = 31$$

Other Results:

Theorem 4

Let $G = \sum_{i=1}^k P_{n_i}$ with $1 \leq n_1 < \dots < n_k$. If $n_{i+1} - n_i \geq 2$ for $1 \leq i \leq k-1$, then

$$\rho(G) = \nu(G) = 2.$$

Other Results:

Theorem 5

Let $G = \sum_{i=1}^k C_{n_i}$, where n_i is even and $4 \leq n_1 < n_2 < \dots < n_k$.

Then,

$$\rho(G) = \nu(G) = 2.$$

Theorem 6

Let $G = \sum_{i=1}^k C_{n_i}$, where n_i is even and $4 \leq n_1 \leq n_2 \leq \dots \leq n_k$. If

$n_{i+2} - n_i \geq 4$ for each $1 \leq i \leq k-2$ and $(n_1, n_2) \notin \{(4, 4), (6, 6)\}$,
then

$$\rho(G) = \nu(G) = 2.$$

Ongoing Study:

Let $G = \sum_{i=1}^k C_{n_i}$, where $3 \leq n_1 \leq n_2 \leq \dots \leq n_k$ and at least one C_{n_i} is odd. If $n_{i+2} - n_i \geq 3$ for each $1 \leq i \leq k-2$ and $(n_1, n_2) \notin \{(3, 3), (5, 5)\}$, then,

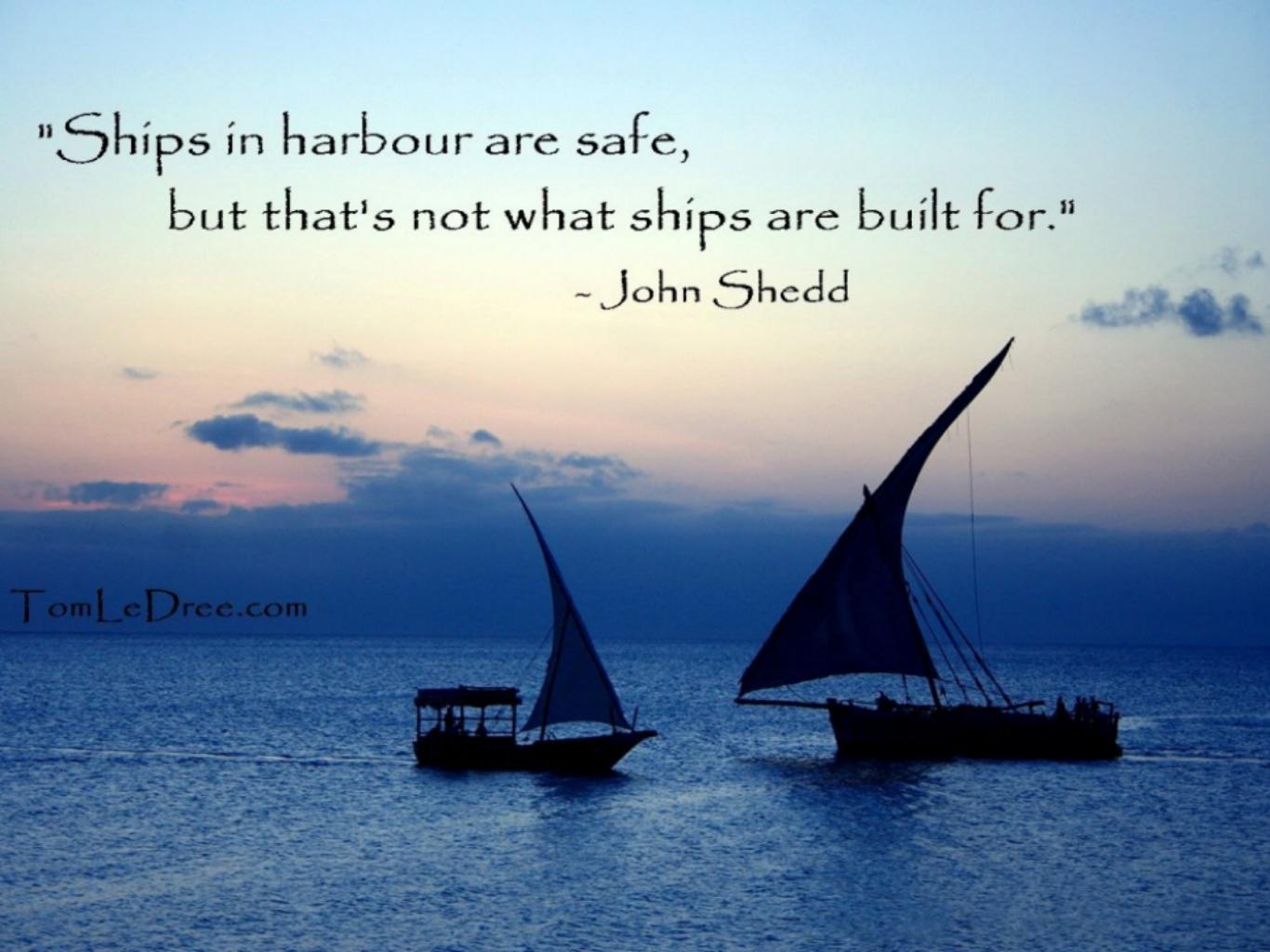
$$\rho(G) = \nu(G) = 3.$$

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"Ships in harbour are safe,
but that's not what ships are built for."

- John Shedd

A photograph of two traditional East African sailboats, known as dhows, silhouetted against a vibrant sunset. The sky is a gradient from deep blue to warm orange and yellow near the horizon, with scattered clouds. The water is a dark blue. The boats have large, triangular sails and are moving across the frame from left to right.

TomLeDree.com