The Sigma Chromatic Number of Corona of Cycles or Paths with Complete Graphs

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Overview

1. Preliminaries
   - Basic Concepts
   - Known Results
2. Corona Graphs $C_m \odot K_n$ and $P_m \odot K_n$
3. Main Results
4. Open Problems
5. References

Basic Concepts

Let G be a simple connected graph.

Definition (Vertex Coloring)

A vertex coloring of G is a mapping $c : V(G) \longrightarrow \mathbb{N}$.

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Definition (Color Sum)

$\forall\ v \in V(G)$,

$$\sigma(v) = \sum_{u \in N_G(v)} c(u).$$

Basic Concepts

Definition (Sigma Coloring)

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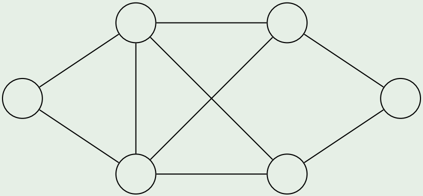
Definition (Sigma Chromatic Number)

$\sigma(G)$ is the least number of colors required in a sigma coloring.

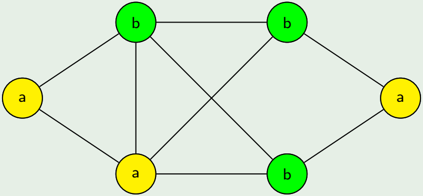
Example

G :



Example

A sigma coloring of G : ($a < b$ and $b \neq 2a$)



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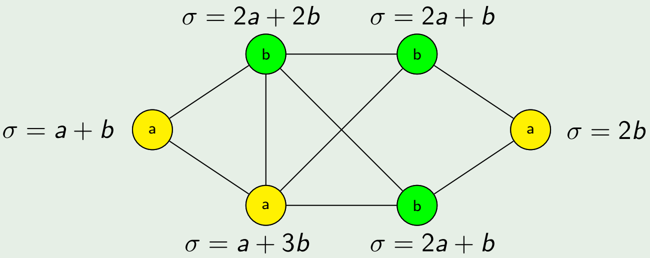

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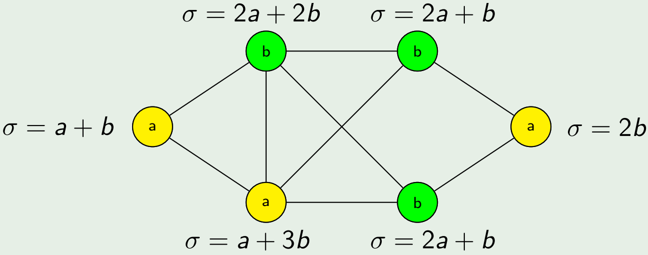
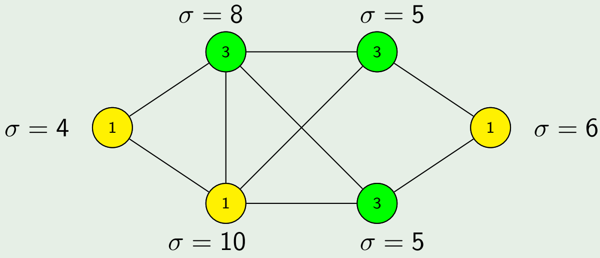


Thus, $\sigma(G) = 2$.

Example

For instance, if $a = 1$ and $b = 3$:

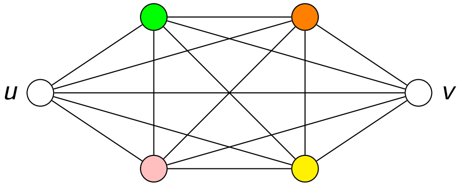


Known Results

Observation 1

If u and v are adjacent vertices where $N(u) - \{v\} = N(v) - \{u\}$, then

$$\sigma(u) \neq \sigma(v) \text{ if and only if } c(u) \neq c(v).$$



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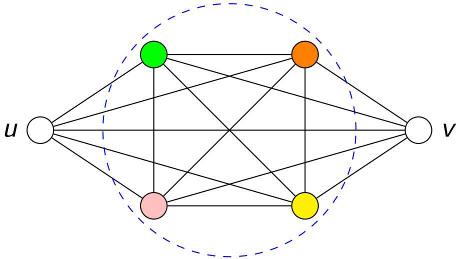

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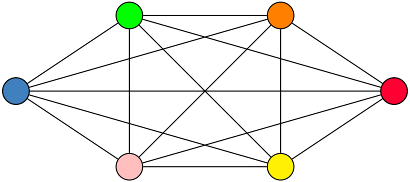

Known Results

Theorem 2 (Chartrand, Okamoto, Zhang [2])

For any graph G,

$$\sigma(G) \leq \chi(G).$$

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Corollary 3

For the complete graph K_n,

$$\sigma(K_n) = n.$$

Known Results

Corollary 4

If m is any positive integer then $\sigma(P_m) = \begin{cases} 1, & \text{if } m = 1 \text{ or } m = 3, \\ 2, & \text{if } m \notin \{1,3\}. \end{cases}$

Known Results

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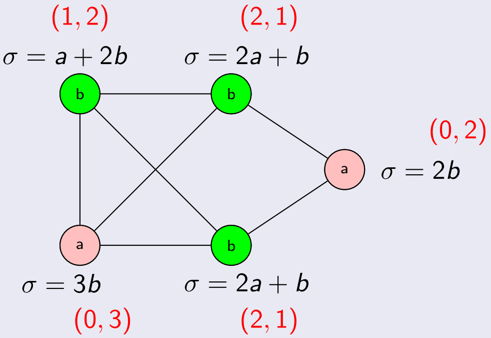
Corollary 5

For every integer $m \geq 3$, $\sigma(C_m) = \begin{cases} 2, & \text{if } m \text{ is even}, \\ 3, & \text{if } m \text{ is odd}. \end{cases}$

Another way to represent color sums:



Known Results

Lemma 6

Let $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)\}$ be a finite set of distinct ordered pairs of nonnegative integers. Then there exist positive integers a and b where $a < b$ such that

$$\alpha_i \cdot a + \beta_i \cdot b \neq \alpha_j \cdot a + \beta_j \cdot b,$$

for $i \neq j$ and $1 \leq i, j \leq r$.

Known Results

Lemma 7

Let $\{(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2), \ldots, (\alpha_r, \beta_r, \gamma_r)\}$ be a finite set of distinct ordered triples of nonnegative integers. Then there exist positive integers a, b and d where $a < b < d$ such that

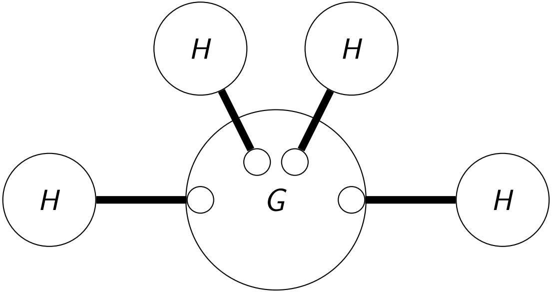
$$\alpha_i \cdot a + \beta_i \cdot b + \gamma_i \cdot d \neq \alpha_j \cdot a + \beta_j \cdot b + \gamma_j \cdot d,$$

for $i \neq j$ and $1 \leq i, j \leq r$.

Definition

The **corona of two graphs** G and H, written as $G \odot H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H.

$G \odot H$:

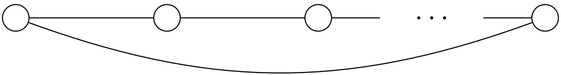


Definition

Corona Graph $C_m \odot K_n$

The corona graph $C_m \odot K_n$ is the graph obtained by taking one copy of C_m

$C_m \odot K_4$:



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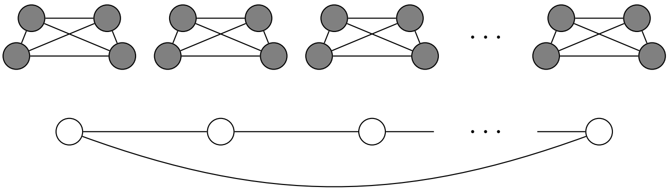

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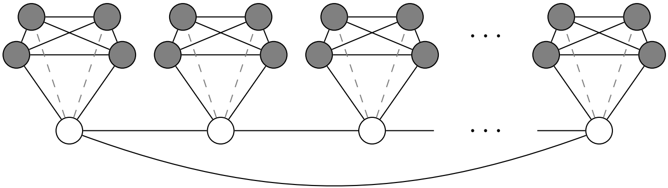
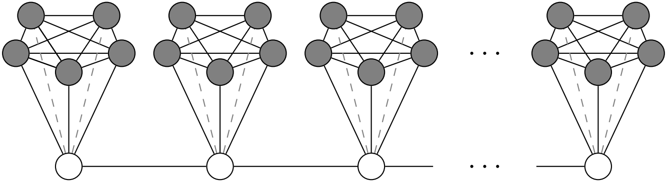
$C_m \odot K_4$:



Corona Graph $P_m \odot K_n$

$P_m \odot K_5$:



Preliminary

Observation 9

Let G and H be disjoint graphs with sigma colorings c_1 and c_2, respectively. Define c as the coloring of $G \odot H$ given by

$$c(v) = \begin{cases} c_1(v), & \text{if } v \in V(G) \\ c_2(v), & \text{if } v \text{ is a vertex in any copy of } H \text{ in } G \odot H. \end{cases}$$

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If u and v are adjacent vertices that are both in G or both in H, then

$$\sigma_c(u) \neq \sigma_c(v).$$

Main Result

Theorem 10

Let m and n be positive integers with $m \geq 2$ and $n \geq 2$. Then,

$$\sigma(P_m \odot K_n) = n.$$

Outline of the Proof:

We want to show that

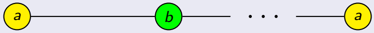
$$\sigma(P_m \odot K_n) = n.$$

Outline of the Proof:

Let

c_1 : a sigma 2-coloring of P_m,

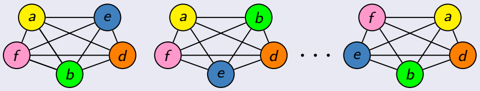
P_m :



Outline of the Proof:

Let
c_1 : a sigma 2-coloring of P_m,
c_2 : a sigma n-coloring of K_n
with $c_1(V(P_m)) \subseteq c_2(V(K_n))$.

K_5 :



Sigma coloring of K_5

Outline of the Proof:

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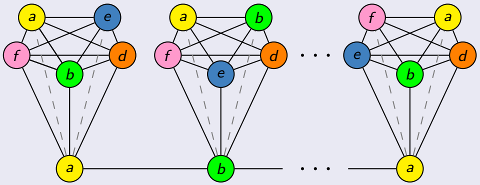
c_2 : a sigma n-coloring of K_n
with $c_1(V(P_m)) \subseteq c_2(V(K_n))$.
Let
c be the coloring of $P_m \odot K_n$
obtained from c_1 and c_2.

$P_m \odot K_5$:



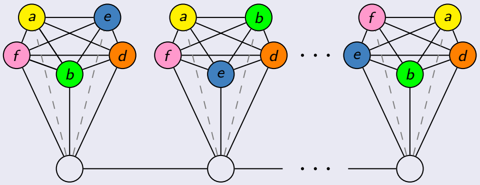
Outline of the Proof:

Note that

$$\sigma(P_m \odot K_n) \geq n$$

since the restriction of a sigma coloring to the subgraph K_n must also be a sigma coloring.

$P_m \odot K_5$:



$$\sigma(P_m \odot K_5) \geq 5.$$

Outline of the Proof:

Note that

$$\sigma(P_m \odot K_n) \geq n$$

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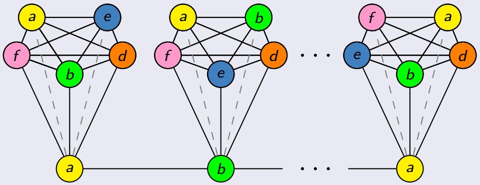
We can show that **c** is a sigma n-coloring of $P_m \odot K_n$.

$P_m \odot K_5$:



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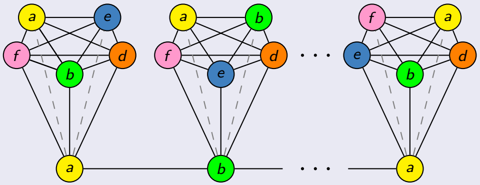
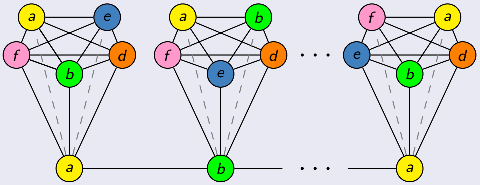


Outline of the Proof:

Let u, v be any two adjacent vertices in $P_m \odot K_n$.

$P_m \odot K_5$:



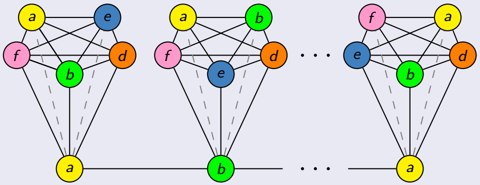
Outline of the Proof:

Let u, v be any two adjacent vertices in $P_m \odot K_n$.

We consider cases. We show that

$\sigma(u) \neq \sigma(v)$.

$P_m \odot K_5$:



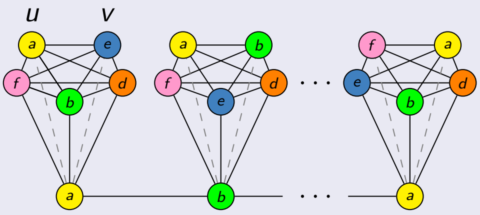
Outline of the Proof:

Case 1: If u and v are both in (a copy of) K_n, then

$$\begin{aligned}\sigma(u) &= \sigma_{c_2}(u) + c(x)\\ &\neq \sigma_{c_2}(v) + c(x)\\ &= \sigma(v)\end{aligned}$$

where $x \in V(P_m) \cap N(u) \cap N(v)$

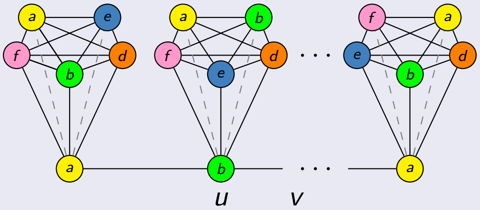
$P_m \odot K_5$:



Outline of the Proof:

Case 2: If u and v are both degree-2 adjacent vertices of P_m then

$P_m \odot K_5$:



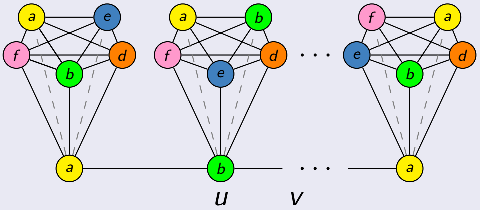
Outline of the Proof:

Case 2: If u and v are both degree-2 adjacent vertices of P_m then

$$\begin{aligned}\sigma(u) &= \sigma_{c_1}(u) + \lambda \\ &\neq \sigma_{c_1}(v) + \lambda \\ &= \sigma(v)\end{aligned}$$

where $\lambda = \sum_{x \in V(K_n)} c_2(x)$.

$P_m \odot K_5$:



Outline of the Proof:

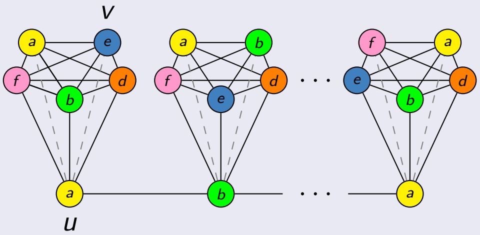
Case 3: If $u \in V(P_m)$ and $v \in V(K_n)$ are adjacent, then

$$\deg(v) = n$$

whereas

$$\deg(u) \geq n + 1.$$

$P_m \odot K_5$:



$$\deg(v) = 5 \text{ whereas } \deg(u) \geq 6$$

Recall Lemma 6:

$\sigma(u) \neq \sigma(v) \Leftrightarrow (\alpha_u, \beta_u) \neq (\alpha_v, \beta_v)$

Outline of the Proof:

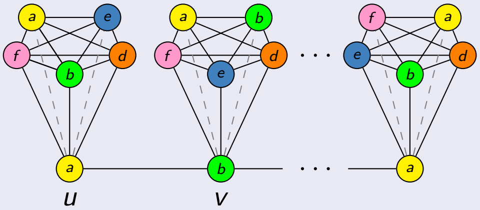
Case 4: If a degree -1 vertex u of P_m and degree-2 vertex v of P_m are adjacent then

$$\deg(u) = n + 1$$

whereas

$$\deg(v) = n + 2.$$

$P_m \odot K_5$:



Recall Lemma 6:
$\sigma(u) \neq \sigma(v) \Leftrightarrow (\alpha_u, \beta_u) \neq (\alpha_v, \beta_v)$

Hence, **c** is a sigma coloring of the corona graph. Thus

$$\sigma(P_m \odot K_n) \leq n.$$

Other Main Results

Theorem 11

Let m be a positive integer with $m \geq 3$. Then,

$$\sigma(C_m \odot K_2) = \sigma(C_m).$$

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Theorem 12

Let m and n be a positive integer with $m, n \geq 3$. Then,

$$\sigma(C_m \odot K_n) = n.$$

Other Main Results

Theorem 13

Let G be a simple connected graph with $|G| \geq 2$. Then,

$$\sigma(G \odot K_n) \leq \max\{\sigma(G), n\},$$

where $n \geq 3$.

Idea of the Proof:

Let $m = \sigma(G)$.

Case 1: $m \geq n$.

Let c_1 be a sigma m-coloring of G.

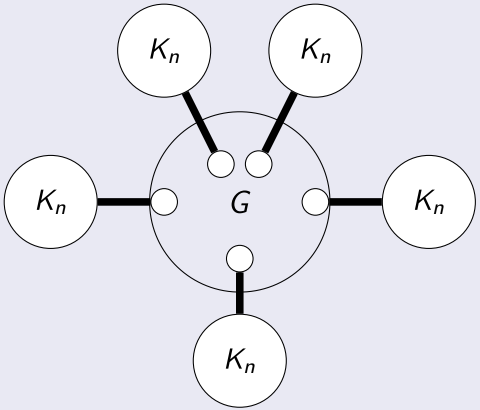
Let c_2 be a sigma n-coloring of each K_n using any n of the m colors.

Enough to compare adjacent vertices with deg(u)=deg(v).

It can be easily shown that $\sigma(u) \neq \sigma(v)$.

Case 2: $m < n$. The proof is similar to case 1.

$G \odot K_n$:



Generalization:

Theorem 14

Let G and H be simple connected graphs with $\sigma(G) = m, \sigma(H) = n$ and $m, n \geq 3$. Then,

$$\sigma(G \odot H) \leq \max\{m, n\}.$$

Generalization:

Theorem 14

Let G and H be simple connected graphs with $\sigma(G) = m, \sigma(H) = n$ and $m, n \geq 3$. Then,

$$\sigma(G \odot H) \leq \max\{m, n\}.$$

If $n \geq m$,

$$\sigma(G \odot H) = \max\{m, n\},$$

Open Problems

1. Describe sigma colorings of corona graphs.
2. Determine the sigma chromatic number of corona of 3 graphs and describe their sigma colorings.
3. Find the sigma chromatic number of corona of a finite number of graphs.

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Arigatou Gozaimasu!

(Maraming Salamat!)