A new Bartholdi zeta function for a graph

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Contents

Ihara zeta function I

Let $G = (V(G), E(G))$ be a finite connected graph, say with n vertices and m undirected edges(multiple edges and loops are permitted). Let $D(G) = \{(u, v), (v, u) | uv \in E(G)\}$ be the arc set of the symmetric digraph corresponding to G. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. And let $e^{-1} = (v, u)$ be the inverse of $e = (u, v)$.

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A path P is a sequence $P = (e_1, \cdots, e_s)$ of n arcs such that $e_i \in D(G)$ and $t(e_i) = o(e_{i+1}), \forall i$. The length of $P = (e_1, \cdots, e_n)$ is $\nu(P) = s$. A path P is said to have a backtrack or a bump if $e_{i+1} = e_i^{-1}$ for some $1 \leq i \leq s - 1$, it is said to have a tail if $e_s = e_1^{-1}$. A cycle means that the starting vertex is the same as the terminal vertex.

Ihara zeta function II

Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a multiple of B. A cycle C is prime if it is not a multiple of a strictly smaller cycle. A cycle C is reduced if it has no backtracks or tails.

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The Ihara (vertex) zeta function of G is defined by

$$\zeta(u, G) = \prod_{[C]} (1 - u^{\nu(C)})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Ihara zeta function III

Theorem 1.1 (Ihara, Bass)

Let G be a connected graph with n vertices and m edges. Then we have the Ihara three term determinant formula

$$\zeta(u, G)^{-1} = (1 - u^2)^{m-n} \det(I_n - Au + Qu^2),$$

where A is the adjacency matrix of graph G and $Q + I_n$ is the diagonal matrix where jth diagonal entry is the degree of the jth vertex of G.

Edge zeta function I

The edge matrix W for graph G is a $2m \times 2m$ matrix with a, b entry corresponding to the oriented edges a and b. This a, b entry is a complex variable ω_{ab} if the terminal vertex of edge a is the same as the initial vertex of edge b with $b \neq a^{-1}$, and is 0 otherwise.

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$$N_E(C) = w_{e_1 e_2} w_{e_2 e_3} \cdots w_{e_{s-1} e_s} w_{e_s e_1}.$$

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The edge zeta function is

$$\zeta_E(W, G) = \prod_{[C]} (1 - N_E(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Edge zeta function II

$$\zeta_E(W, G)^{-1} = \det(I_{2m} - W).$$

If we set all non-zero variables in W equal to $u \in \mathbb{C}$, the edge zeta function specializes to the Ihara (vertex) zeta function.

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$$\zeta(u, G)^{-1} = \det(I_{2m} - W_1 u).$$

How to get the formula $\zeta(u, G)^{-1} = (1 - u^2)^{m-n} \det(I_n - Au + Qu^2)$?

A trick on linear algebra

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Combine them together, we get

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$$\det(I_{2m} - W_1 u) = (1 - u^2)^{m-n}\det(I_n - Au + Qu^2)$$

Proof of Ihara three term determinant formula I

$$S_{n\times 2m}, S_{ve} = \begin{cases} 1 & \text{if v is the starting vertex of edge e,} \\ 0 & \text{otherwise} \end{cases}$$

$$T_{n\times 2m}, t_{ve} = \begin{cases} 1 & \text{if v is the terminal vertex of edge e,} \\ 0 & \text{otherwise.} \end{cases}$$

$$J = \begin{pmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{pmatrix}$$

Then we have

(1) $SJ = T,\ TJ = S$;

(2) $A = ST^t,\ Q + I_n = SS^t = TT^t$.

(3) $W_1 + J = T^t S$.

Proof of Ihara three term determinant formula II

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\begin{aligned}
\det(I_{2m} - W_1 u) &= \det((I_{2m} + uJ) - uT^t S) \\
&= \det(I_{2m} + uJ)\det(I_n - S(I_{2m} + uJ)^{-1} uT^t) \\
&= (1-u^2)^m \det(I_n - S\frac{(I_{2m} - uJ)}{1-u^2} uT^t) \\
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This approach is actually equivalent to Bass's approach, also Watanabe and Fukumizu's appraoch.

H. Bass, The Ihara-Selberg zeta function of a tree lattice. Int J. Math. 1992, 3: 717-797.

Y. Watanabe, K. Fukumizu, Graph zeta function in the Bethe Free energy and Loopy Belief Propagation. Advances in neural information processing systems. MIT Press, 2010. p.2017-2025.

Let G be a connected graph. Then the cyclic bump count $cbc(\pi)$ of a cycle $\pi = (\pi_1, \cdots, \pi_s)$ is

$$cbc(\pi) = |\{i = 1, \cdots, s | \pi_i^{-1} = \pi_{i+1}\}|,$$

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where $\pi_{s+1} = \pi_1$.
The Bartholdi zeta function of G is defined by

$$\zeta(u, t, G) = \prod_{[C]} (1 - t^{cbc(C)} u^{\nu(C)})^{-1},$$

where $[C]$ runs over all equivalence classes of prime (maybe reduced) cycles of G.

$(l+1)$-variable Bartholdi zeta function II

I. Sato and his coauthors defined an $(l+1)$-variable Bartholdi zeta function as follows:
Suppose $V(G) = V_1 \cup \cdots \cup V_l$ is a partition of $V(G)$. For $j = 1, \cdots, l$, the j-th cyclic bump count $cbc_j(\pi)$ of a cycle $\pi = (\pi_1, \cdots, \pi_s)$ is

$$cbc_j(\pi) = |\{i = 1, \cdots, s | \pi_i^{-1} = \pi_{i+1}, t(\pi_i) \in V_j\}|,$$

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The $(l+1)$-variable Bartholdi zeta function of G is defined by

$$\zeta(u, t_1, \cdots, t_l, G) = \prod_{[C]} (1 - t_1^{cbc_1(C)} \cdots t_l^{cbc_l(C)} u^{\nu(C)})^{-1},$$

where $[C]$ runs over all equivalence classes of prime (maybe not reduced) cycles of G.

Main Results I–The Bartholdi edge zeta function

We define the generalized edge matrix W for a graph G by introducing $w_{ab} \neq 0$ where $a = b^{-1}$.

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where $[C]$ runs over all equivalence classes of prime (maybe not reduced) cycles of G.
Similarly, we can get

$$\zeta_E(W', G)^{-1} = \det(I_{2m} - W').$$

Now we specialize the entries in W' as follows:

$$W'_{ab} = \begin{cases} u_b & \text{if } t(a) = o(b) \text{ and } a \neq b^{-1}, \\ u_b t_b & \text{if } a = b^{-1}, \\ 0 & \text{otherwise} \end{cases}$$

Main Results II–A new Bartholdi zeta function

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That is, for every edge and every bump, we assign a weight.
After this specialization, we have the new Bartholdi zeta function as follows:

$$\zeta(u_1, \cdots, u_{2m}, t_1, \cdots, t_{2m}, G) = \det(I_{2m} - W'|_{speicialization})$$

Main Results III–The determinant formula for the new Bartholdi zeta function I

$$
\begin{aligned}
\zeta &= \det(I_{2m} - W'|_{speicialization}) \\
&= \det(I_{2m} - (W_1\Omega + J\Omega\Lambda)) \\
&= \det((I_{2m} + J\Omega(I - \Lambda)) - T^t S\Omega) \\
&= \det(I_{2m} + J\Omega(I - \Lambda)) \det(I_n - S\Omega(I_{2m} + J\Omega(I - \Lambda))^{-1} T^t) \\
&= \prod_{e \in E(G)} (1 - u_e u_{\bar{e}}(1 - t_e)(1 - t_{\bar{e}})) \det(I_n - \hat{A} + \hat{Q}),
\end{aligned}
$$

where $\Omega_{ee} = u_e$, $\Lambda_{ee} = t_e$, $\Omega = diag(\Omega_1, \Omega_2)$, $\Lambda = diag(\Lambda_1, \Lambda_2)$,

Main Results III–The determinant formula for the new Bartholdi zeta function II

$$\zeta(u_1,\cdots,u_{2m},t_1,\cdots,t_{2m},G)=\prod_{e\in E(G)}(1-u_eu_{\bar{e}}(1-t_e)(1-t_{\bar{e}}))\det(I_n-\hat{A}+\hat{Q}),$$

where $\hat{A}=S\Omega((I-\Omega_1\Omega_2(I-\Lambda_1)(1-\Lambda_2))^{-1})^{\oplus 2}T^t$,
$\hat{Q}=S\Omega J\Omega(I-\Lambda)((I-\Omega_1\Omega_2(I-\Lambda_1)(1-\Lambda_2))^{-1})^{\oplus 2}T^t$
One can specialize to I. Sato's $(l+1)$-variable Bartholdi zeta function by setting $u_e=u$ and $t_e=t_j$, where $t(e)\in V_j$.

Main Results IV–A new Bartholdi L-function

Similarly, we can also define a new Bartholdi L-function.

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And we can get a determinant expression of the new Bartholdi L-function and the new Bartholdi zeta function of a regular covering of G.

Open Questions

We can define Bartholdi path zeta function, and we can specialize variables to get path zeta function. But we don't know how to specialize variables to get Bartholdi edge zeta functions, even we don't know how to specialize variables to get Bartholdi zeta functions from Bartholdi path zeta functions.

Main References

H. Bass, The Ihara-Selberg zeta function of a tree lattice. Int J. Math. 1992, 3: 717-797.

I. Sato, H. Mitsuhashi, H, Morita, A Generalized Bartholdi zeta function for a general graph. Linear and Multilinear Algebra 2016, 991-1008.

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Thanks

Thank you for your attention!