Unique realisations of graphs

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Motivation

Motivating question

When can a physical structure only be built in one way?

This question appears in areas such as:

- chemistry (stereoisomers),
- civil engineering and mechanical engineering,
- computer graphics, and computer-aided design (CAD)

amongst others.

Model

What properties do such structures have?

- Consist of parts of fixed size and/or shape.
- Geometric constraints between these parts e.g. fixed distance or angle.
- Sometimes these geometric constraints can be stretched or compressed.

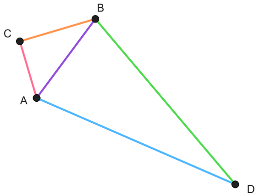
One of the simplest models of such structures are...

Length-frameworks

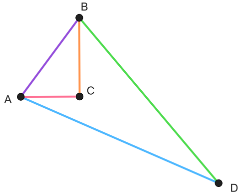
A **length-framework** (G, p) consists of a graph G and a map $p : V \to \mathbb{R}^d$.

- each vertex represents a part of our framework
- an edge represents a fixed distance between the two endvertices
- the **realisation** p assigns a location to each vertex

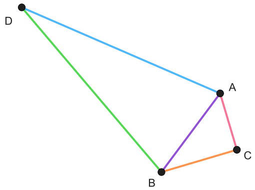
Examples of length-frameworks



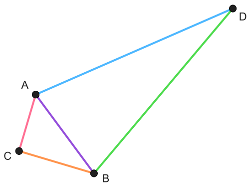
(a) (G, p)



(b) (G, q_1). Reflection in AB.



(c) (G, q_2). Reflection in x-axis.



(d) (G, q_3). Rotation by 180°

Unique realisations

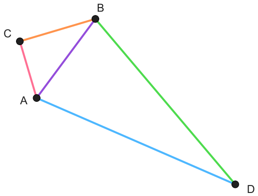
- The length-frameworks (G, p) and (G, q) are **equivalent** if $(p(u) - p(v))^2 = (q(u) - q(v))^2$ for all $uv \in E(G)$.
- (G, p) and (G, q) are **congruent** if (G, q) can be obtained by translating, rotating and/or reflecting (G, p).

Definition

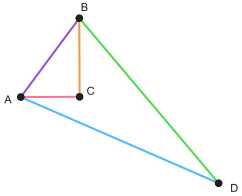
A length-framework (G, p) is **globally rigid** if every framework (G, q) which is equivalent to (G, p) is also congruent to (G, p).

In other words, p is the unique realisation of G (up to translation, rotation and reflection) which satisfies this set of edge length constraints.

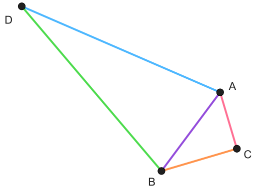
$K_4 - e$ is not globally rigid



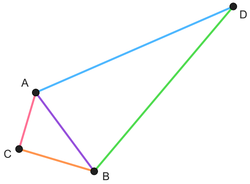
(a) (G, p)



(b) Equivalent but not congruent to (G, p).



(c) Equivalent and congruent



(d) Equivalent and congruent

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Rigidity

Important!

From now on, we only consider frameworks in $\mathbb{R}^2$.

Informally...
A length framework is **flexible** if its vertices can be moved relative to each other whilst preserving the edge lengths. Otherwise it is **rigid**.

Examples

- (K_n, p) is rigid for all n and all p.
- paths on at least 3 vertices are flexible.
- cycles on at least 4 vertices are flexible.

Formally...
We need to define what it means to "move" the framework.

Motions

A **motion** of a length framework (G, p) is a continuous function $p_t = P(t)$ for $0 \leq t \leq 1$ where $p_t : V(G) \to \mathbb{R}^2$ is a realisation of G which satisfies

(M1) $p_0(v) = p(v)$ for all $v \in V(G)$; and

(M2) for all $uv \in E(G)$ and $t \in [0, 1]$, $\|p_t(u) - p_t(v)\| = \|p(u) - p(v)\|$.

A motion is **trivial** if (M2) holds for all $u, v \in V(G)$. In other words, when (G, p_t) can be obtained from (G, p) by a translation and/or rotation.

Definition

A length framework is **rigid** if the only continuous motions which preserve the edge constraints are trivial. A length framework which is not rigid is said to be **flexible**.

Questions in rigidity theory

- Is (G, p) rigid?
- Is (G, p) globally rigid?
- Is rigidity a combinatorial property (determined by G)? Or a geometric property (determined by p)?

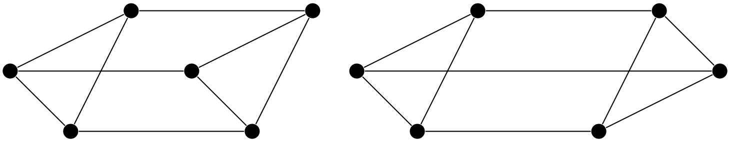
Key question 1

When can we G determine whether (G, p) is rigid (or globally rigid) by only considering the structure of G?

Key question 2

What structure of G guarantees characterises rigidity (or global rigidity) in these cases?

When does the choice of realisation affect rigidity?



(a) A flexible framework (G, p_1). (b) A rigid framework (G, p_2).

Figure: Non-generic length frameworks.

A realisation p or framework (G, p) is **generic** if the coordinates in p are algebraically independent over $\mathbb{Q}$.

For generic length-frameworks, rigidity is a combinatorial property.

i.e. the structure of G determines when (G, p) is rigid.

Rigidity results

Theorem (Laman, 1970)

Let (G, p) be a generic length framework. Then (G, p) is rigid if and only if it has a spanning subgraph H which has

- $|E(H)| = 2|V(H)| - 3$, *and*
- $|F| \leq 2|V(F)| - 3$ *for all* $\emptyset \neq F \subseteq E(H)$.

The rigidity matroid

Laman's result characterises the independent sets of a matroid, $\mathcal{R}(G)$:

Theorem

A set of edges $E' \subseteq E(G)$ is independent in the **rigidity matroid** $\mathcal{R}(G)$ *if and only if $|F| \leq 2|V(F)| - 3$ for all $\emptyset \neq F \subseteq E'$.*

This leads to the alternative statement of Laman's result in terms of matroid rank:

Theorem

Let (G, p) be a generic length framework, and let $|V(G)| = n$. Then (G, p) is rigid if and only if $\text{rank}(\mathcal{R}(G)) = \text{rank}(\mathcal{R}(K_n))$.

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3-connectivity and global rigidity

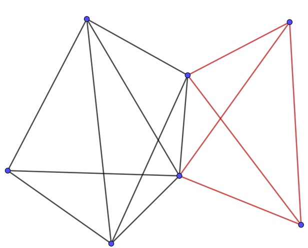
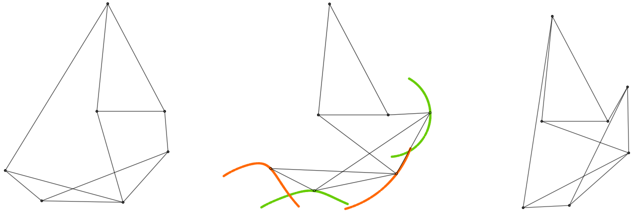


Figure: If the underlying graph G of a length framework (G, p) is not 3-connected, then G is not globally rigid.

Rigidity and global rigidity

- If a framework is flexible, then it cannot be globally rigid.
- If a framework is rigid, it may not be globally rigid:



(a) (G, p) (b) $(G - e, p_t)$ (c) (G, q)

Figure: If $(G - e, p)$ is not rigid, then we may be able to find an equivalent but non-congruent realisation (G, q) to (G, p).

We say a framework (G, p) is **redundantly rigid** if $(G - e, p)$ is rigid for all $e \in E(G)$.

Global rigidity results

Theorem (Hendrickson, 1992)

If a generic framework (G, p) is globally rigid in $\mathbb{R}^d$ then either G is a complete graph with at most $d + 1$ vertices, or the following conditions hold:

1. *G is $(d + 1)$-connected, and*
2. *G is redundantly rigid.*

Theorem (Jackson and Jordán, 2005)

A generic length framework (G, p) is globally rigid (in $\mathbb{R}^2$) if and only if either G is a complete graph on at most 3 vertices, or G is 3-connected and redundantly rigid.

Matroid connectivity

In the rigidity matroid $\mathcal{R}(G)$, an edge set $C \neq \emptyset$ is a **circuit** if and only if

(i) $|C| = 2|V(C)| - 2$, and

(ii) $|F| \leq 2|V(F)| - 3$, for all $\emptyset \neq F \subset C$.

The rigidity matroid $\mathcal{R}(G)$ of a graph G is **connected** if and only if for all $e, f \in E(G)$ either

- $e = f$, or
- there exists a circuit C of $\mathcal{M}$ such that $e, f \in C$.

Theorem (Jackson and Jordán, 2005)

A generic length framework (G, p) is globally rigid (in $\mathbb{R}^2$) if and only if either G is a complete graph on at most 3 vertices, or both of the following hold:

(i) *G is 3-connected, and*

(ii) *$\mathcal{R}(G)$ is connected.*

Contents

Other models

Motivating question

Can we extend the results for length-frameworks to models which have both length and angle constraints?

- It is difficult to model angles combinatorially
- We can capture some of the behaviour of angle constraints using simpler models, such as:
  - direction-length frameworks (easy to work with, but not very realistic)
  - point-line frameworks (more realistic, but much more difficult to work with)

Direction-length frameworks

A **direction-length graph** is a loop-free multi-graph $G = (V; D, L)$ with two types of edges:

- **Length edges** $uv \in L$ represent distance constraints between their endvertices
- **Direction edges** $uv \in D$ represent slope constraints: u and v must stay on a line of fixed slope.

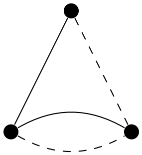


Figure: Solid lines depict length edges, and dashed lines depict direction edges.

A **direction-length framework** is a pair (G, p) where G is a direction-length graph, and $p : V(G) \to \mathbb{R}^2$.

DL-global rigidity

Given two direction-length frameworks (G, p) and (G, q),

- (G, p) and (G, q) are **equivalent** if
  - for all $uv \in L$: $(p(u) - p(v))^2 = (q(u) - q(v))^2$
  - for each $uv \in D$, there exists some $\lambda \in \mathbb{R}$ such that $q(u) - q(v) = \lambda(p(u) - p(v))$.
- (G, p) and (G, q) are **congruent** if (G, q) can be obtained from (G, p) by translating and/or rotating by 180°.

Definition

A DL-framework (G, p) is **DL-globally rigid** if every framework (G, q) which is equivalent to (G, p) is also congruent to (G, p).

In other words, p is the unique realisation of G (up to translation and rotation by 180°) which satisfies this set of edge constraints.

Example

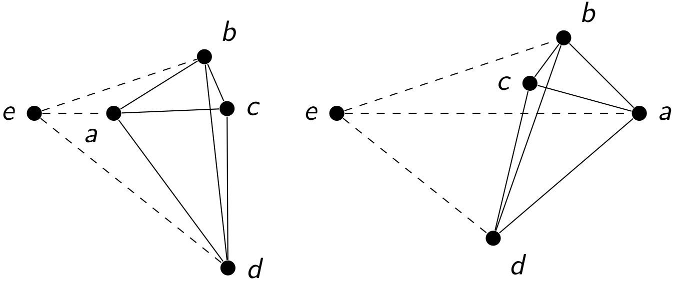


Figure: Two equivalent but non-congruent realisations of a direction-length framework.

DL-rigidity

Informally...
a direction-length framework is **DL-rigid** if its only continuous motions are translations.

Theorem (Servatius and Whiteley, 1999)

A generic DL-framework (G, p) is DL-rigid if and only if it has a spanning subgraph H which has

- $|E(H)| = 2|V(H)| - 2$,
- $|F| \leq 2|V(F)| - 2$ *for all* $\emptyset \neq F \subseteq E(H)$, *and*
- $|F| \leq 2|V(F)| - 3$ *for all* $\emptyset \neq F \subseteq L$ *and all* $\emptyset \neq F \subseteq D$.

These last two conditions characterise independence in the **DL-rigidity matroid** $\mathcal{R}_{DL}$.

Vertex connectivity and matroid connectivity

Given a DL-graph G:

- An edge set $C \subseteq E(G)$ is a **circuit** in the rigidity matroid $\mathcal{R}_{DL}(G)$ if C is dependent in $\mathcal{R}_{DL}(G)$, but every proper subset of C is independent in $\mathcal{R}_{DL}(G)$.
- The matroid $\mathcal{R}_{DL}(G)$ is **connected** is for all distinct $e, f \in E(G)$ there exists a circuit C of $\mathcal{R}_{DL}(G)$ such that $e, f \in C$.
- G is **direction-balanced** if whenever G has a 2-vertex-cut, both sides of the cut contain a direction edge.

Main result

Theorem (Jackson, Keevash and C., 2018+)

Let $G = (V; D, L)$ be a direction-length graph. Then (G, p) is DL-globally rigid for all generic realisations p if and only if

- *G is DL-rigid, and*
- *either $|L| = 1$; or G has a subgraph H such that*
    - *$L \subseteq E(H)$,*
    - *$D \cap E(H) \neq \emptyset$,*
    - *H is direction-balanced, and*
    - *$\mathcal{R}_{DL}(H)$ is connected.*

This statement builds on partial results by Servatius and Whiteley (1999), Jackson and Jordán (2010), Jackson and Keevash (2011) and C. (2018+).