Great antipodal sets on unitary groups and Hamming graphs

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Remark and Notation

Usually, when we study design theory on a certain space M, for a fixed subspace $\mathcal{H} \subset C(M)$, we find suitable subsets $X \subset M$ as $\mathcal{H}$-design. But, in this talk, for a fixed subset $X \subset M$, we find suitable subspaces $\mathcal{H} \subset C(M)$ such that X is $\mathcal{H}$-design.

- n: integer with $n \geq 2$
- $[n] := \{1, 2, \ldots, n\}$
- $2^{[n]}$: the power set of $[n]$ i.e., $2^{[n]} := \{\alpha \,|\, \alpha \subset [n]\}$
- For a set X, $\binom{X}{2} := \{\{x, y\} \,|\, x, y \in X,\ x \neq y\}$

Hamming cube Q_n and $C(Q_n)$

- $X := \{1, -1\}^n$
- $E := \{\{\boldsymbol{a}, \boldsymbol{b}\} \in \binom{X}{2} \mid \#\{i \mid a_i \neq b_i\} = 1\}$, where $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$
- Hamming cube graph $Q_n = (X, E)$ $(= H(n, 2))$
- $C(Q_n)$: the space of $\mathbb{C}$-valued functions on X
- The inner product $(\cdot, \cdot)$ on $C(Q_n)$:
  $(f, g) := \frac{1}{2^n} \sum_{\boldsymbol{a} \in X} \overline{f(\boldsymbol{a})} g(\boldsymbol{a})$ for $f, g \in C(Q_n)$
- For $i \in [n]$, define $\varepsilon_i \in C(Q_n)$:

$$\varepsilon_i(\boldsymbol{a}) = \varepsilon_i(a_1, a_2, \ldots, a_n) := a_i$$

- For $\alpha \in 2^{[n]}$, $\varepsilon_\alpha := \prod_{i \in \alpha} \varepsilon_i$.

Remark 1

$\{\varepsilon_\alpha\}_{\alpha \in 2^{[n]}}$ is an orthonormal basis of $C(Q_n)$.

Reproducing kernels on $C(Q_n)$ and Krawtchouk poly.

- Let $V_j := \mathrm{Span}_{\mathbb{C}}\{\varepsilon_\alpha \mid \#\alpha = j\}$. Then $C(Q_n) = \bigoplus_{j=0}^n V_j$.
- $K_j : X \times X \to \mathbb{C}$: $K_j(\boldsymbol{x}, \boldsymbol{y}) := \sum_{\alpha \in 2^{[n]}, \#\alpha = j} \overline{\varepsilon_\alpha(\boldsymbol{x})} \varepsilon_\alpha(\boldsymbol{y})$

Remark 2

1. $\{V_j\}_{j=0}^n$ are the maximal common eigenspaces of the adjacency operators $\{A_i\}_{i=0}^n$, i.e., $\exists P_i(j) \in \mathbb{C}$ s.t. $A_i f = P_i(j) f$ for any $f \in V_j$.
2. K_j is the reproducing kernel of V_j, i.e.,
   - for $\boldsymbol{x} \in X$, $K_j(\boldsymbol{x}, \cdot) \in V_j$,
   - for $f \in V_j$, $(K_j(\boldsymbol{x}, \cdot), f) = f(\boldsymbol{x})$.

For any $\boldsymbol{x}, \boldsymbol{y} \in X$ with $\partial(\boldsymbol{x}, \boldsymbol{y}) = u$, the value $K_j(\boldsymbol{x}, \boldsymbol{y})$ depend only on u:

$$K_j(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=0}^{\min\{u,j\}} (-1)^k \binom{u}{k} \binom{n-u}{j-k}.$$

$K_j(u) := \sum_{k=0}^{j} (-1)^k \binom{u}{k} \binom{n-u}{j-k}$ is called the Krawtchouk polynomial.

Symmetric spaces and Antipodal sets

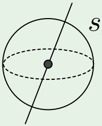
Definition 3

A Riemannian manifold M is called a (Riemanian) symmetric space if $\forall x \in M$, $\exists$point symmetry $s_x \colon M \to M$, where a point symmetry is an isometry satisfying

- *s_x is an involution,*
- *x is an isolated fixed point of s_x.*

Example 4

Sphere $S^d := \{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$ is a symmetric space. the point symmetry s_x is defined by $s_x(y) = -y + 2\langle x, y\rangle x$.



s_x $(180^\circ$ rotation)

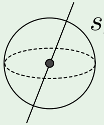
Antipodal sets

Definition 5

For a symmetric space M with point symmetries s, A subset S of M is called an antipodal set *if $s_x(y) = y$ for any $x, y \in S$.*

Example 6

$S = \{x\}$ (single point set) and $S = \{x, -x\}$ (a point and its antipodal point) are antipodal sets on S^d.



Some results for antipodal sets

Fact 1 (Chen–Nagano, Takeuchi, Sánchez, Tanaka–Tasaki)

For a compact symmetric space M and an antipodal set S,

1. $\#S < \infty$ *and* $\max\{\#S \mid S$ *: antipodal set* $\} < \infty$, *and this value is called the 2-number* $\#_2 M$ *of* M.
2. *there exist antipodal sets* S *with* $\#S = \#_2 M$. *This set* S *is called a great antipodal set (GAS).*
3. *If* M *is a symmetric* R*-space (it is a "good" symmetric space), a great antipodal set of* M *is unique up to congruences.*

GAS on $U(n)$

- $U(n) := \{A \in GL_n(\mathbb{C}) \mid A^*A = I_n\}$: the unitary group of degree n
- The point symmetry $s_x \colon U(n) \to U(n)$ of $x \in U(n)$ is defined by $s_x(y) = xy^{-1}x$. Then $U(n)$ is a compact symmetric space.

Fact 2 (Chen–Nagano)

- *$U(n)$ is a symmetric R-space.*
- *Each great antipodal set on $U(n)$ is congruent to*

$$S = \{\mathrm{diag}(x_1, x_2, \ldots, x_n) \in U(n) \mid x_1, x_2, \ldots, x_n \in \{\pm 1\}\},$$

  where $\mathrm{diag}(x_1, x_2, \ldots, x_n)$ is a diagonal matrix whose diagonal entries are x_i.
- $\#S = 2^n$.

Q_n and GAS

- S: GAS on $U(n)$
- $\mathrm{dist}\colon U(n) \times U(n) \to \mathbb{R}_{\geq 0}$: the distance function on $U(n)$
- $\mathrm{dist}_{\min}(S) := \min\{\mathrm{dist}(x,y) \mid x, y \in S,\ x \neq y\}$

Theorem 7 (K.-Okuda)

Let $E := \{\{x,y\} \in \binom{S}{2} \mid \mathrm{dist}(x,y) = \mathrm{dist}_{\min}(S)\}$. *Then* (S, E) *is a Hamming cube* Q_n.

cf: Other GAS's on symmetric R-spaces carry the structure of some distance-regular graphs

- GAS on $\mathrm{Gr}_k(\mathbb{F}^n)$ $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$ $\leftrightarrow$ Johnson graph $J(n,k)$
- GAS on $SO(2n)/U(n)$ $\leftrightarrow$ Halved Hamming cube $\frac{1}{2}Q_n$

etc. (K.-Okuda)

Design theory on $U(n)$

- $\widehat{U(n)}$: equivalence classes of irr. unitary rep. of $U(n)$
  $\cong (\mathbb{Z}^n)_+ := \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$
- $\mathcal{H}_\lambda$: subspace of $C(U(n))$ isomorphic to irr. unitary rep. indexed by λ
- $C(U(n)) \underset{\text{dense}}{\supset} \bigoplus_{\lambda \in (\mathbb{Z}^n)_+} \mathcal{H}_\lambda$ (Perter-Weyl's theorem)

Definition 8

Fix $\lambda \in (\mathbb{Z}^n)_+$. Let X be a subset of $U(n)$. X is called a λ-design if

$$\sum_{x,y \in X} K_\lambda(x,y) = 0 \quad \text{where } K_\lambda \text{ is the reproducing kernel of } \mathcal{H}_\lambda.$$

Remark 9

$K_\lambda(x,y) = s_\lambda($ eigenvalues of $y^{-1}x)$, where s_λ is the Schur polynomial.

Result 1

- $(1^j) := (\underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{n-j}) \in (\mathbb{Z}^n)_+$
- $K_{(1^j)}$: the reproducing kernel of $\mathcal{H}_{(1^j)}$
- Fact: If $\operatorname{dist}(x_1, y_1) = \operatorname{dist}(x_2, y_2)$, then $K_{(1^j)}(x_1, y_1) = K_{(1^j)}(x_2, y_2)$

Theorem 10 (K.)

$K_{(1^j)}|_{S \times S} = K_j$, *i.e., for* $x, y \in S$ *with* $\operatorname{dist}(x, y) = n - u$,
$K_{(1^j)}(x, y) = K_j(x, y) = K_j(u)$ *(Krawtchouk poly.)*

Corollary 11

Let $\mathcal{H}_{(1^j)}|_S := \{f|_S \mid f \in \mathcal{H}_{(1^j)}\}$. *Then* $\mathcal{H}_{(1^j)}|_S = V_j$.

Result 2

For $\lambda \in (\mathbb{Z}^n)_+$ and $k \in \mathbb{Z}$,
let $\lambda + 2k := (\lambda_1 + 2k, \lambda_2 + 2k, \ldots, \lambda_n + 2k) \in (\mathbb{Z}^n)_+$.

Lemma 12 (K.)

If S is a λ-design, then for each $k \in \mathbb{Z}$, S is a $\lambda + 2k$-design.

We consider the following equivalence relation on $(\mathbb{Z}^n)_+$:

$$\lambda \sim \lambda' \ \Leftrightarrow\ \exists k \in Z \text{ s.t. } \lambda' = \lambda + 2k$$

Let $[\lambda]$ be the equivalence class with λ. By Lemma 12, we can define a $[\lambda]$-design for S. On the other hand, the parity of $[(\lambda_1, \lambda_2, \ldots, \lambda_n)]$ is defined by the parity of $\sum_i \lambda_i$. It is well-defined.

Result 3

Theorem 13 (K.)

For a GAS S on $U(n)$,

1. *If $[\lambda]$ is odd, then S is a $[\lambda]$-design.*
2. *There are only finitely many even $[\lambda]$ such that S is a $[\lambda]$-design.*

Example 14

For small n, we get the condition that $[\lambda]$ carries that S is a $[\lambda]$-design.

1. GAS S on $U(2)$ is a $[\lambda]$-design $\Leftrightarrow$ $[\lambda]$ is odd or $[\lambda] = [(1,1)]$.
2. GAS S on $U(3)$ is a $[\lambda]$-design $\Leftrightarrow$ $[\lambda]$ is odd or $[\lambda] = [(1,1,0)], [(2,1,1)]$.

Example 14 (continued)

- S on $U(2)$ is an even $[\lambda]$-design $\Leftrightarrow$ $[\lambda] = [(1,1)]$. (1 class)
- S on $U(3)$ is an even $[\lambda]$-design $\Leftrightarrow$ $[\lambda] = [(1,1,0)], [(2,1,1)]$. (2 classes)
- S on $U(4)$ is an even $[\lambda]$-design $\Leftrightarrow$ $[\lambda] = [(1,1,0,0)], [(2,1,1,0)], [(1,1,1,1)], [(3,1,1,1)], [(2,2,1,1)], [(3,3,3,1)]$ (6 classes)
- S on $U(5)$ is an even $[\lambda]$-design $\Leftrightarrow$ 12 classes $[\lambda]$
- S on $U(6)$ is an even $[\lambda]$-design $\Leftrightarrow$ 26 classes $[\lambda]$
- S on $U(7)$ is an even $[\lambda]$-design $\Leftrightarrow$ 48 classes $[\lambda]$
- S on $U(8)$ is an even $[\lambda]$-design $\Leftrightarrow$ 91 classes $[\lambda]$
- S on $U(9)$ is an even $[\lambda]$-design $\Leftrightarrow$ 158 classes $[\lambda]$

Question 15

What is the sequence $1, 2, 6, 12, 26, 48, 91, 158, \ldots$?
(cf. OEIS A246584, number of overcubic partitions of n;
$1, 2, 6, 12, 26, 48, 92, 160, \ldots$)