Adjacency properties of graphs and related results

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Adjacency properties

(Erdős-Rényi 1963, Blass-Harary 1979)

G satisfies $\boldsymbol{\mathcal{P}}(\boldsymbol{l},\boldsymbol{m}) \underset{\text{def}}{\Leftrightarrow}$ $\forall A, B \subset V(G)\ (|A| = l, |B| = m\ , A \cap B = \emptyset)$,
$\exists z_{A,B} \notin A \cup B$ satisfying the following property $(*)_{A,B}$:

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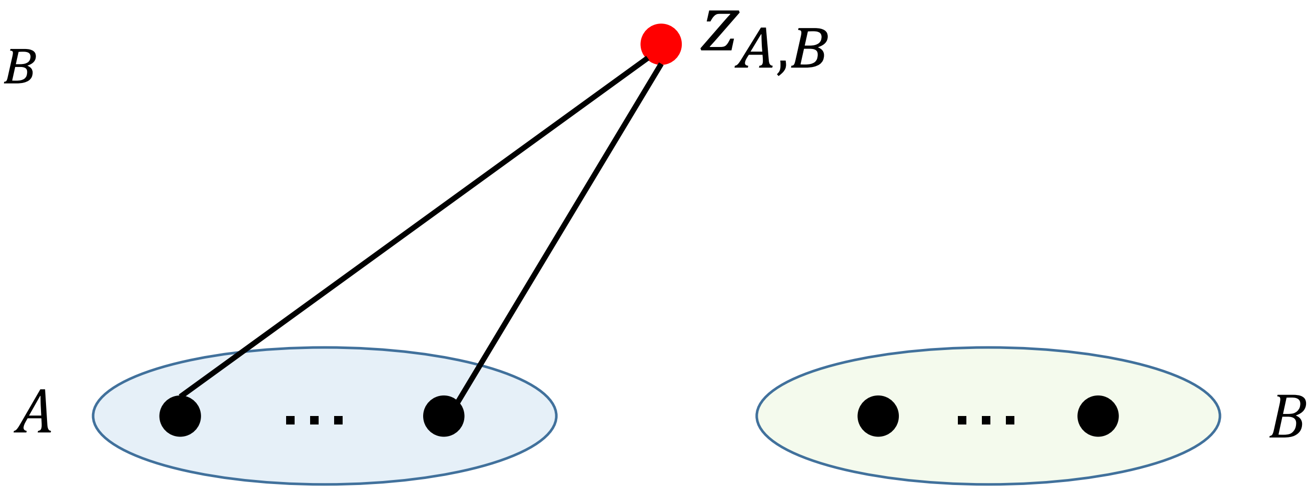
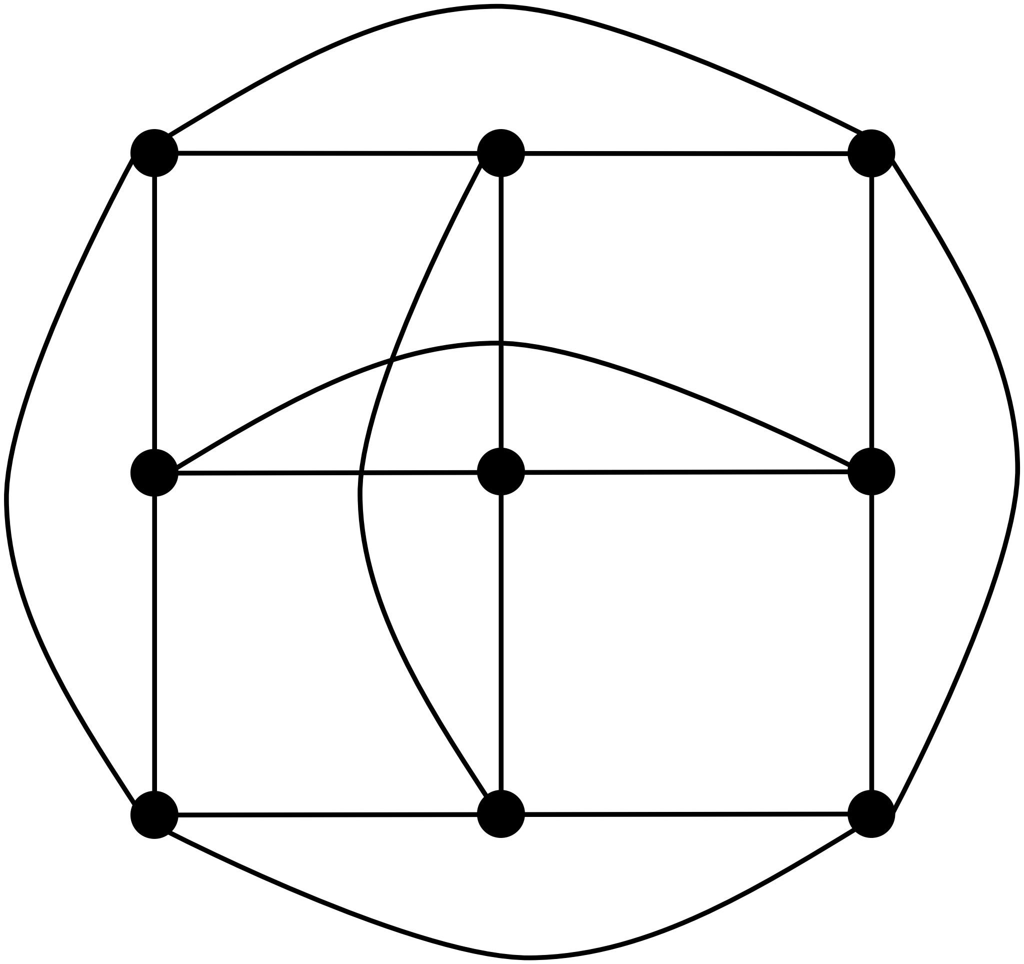


If G satisfies $\mathcal{P}(l,m)$ for all (l,m) s.t. $l + m = n$, G is called ***n*-e.c. (*n*-existentially closed).**

Adjacency properties



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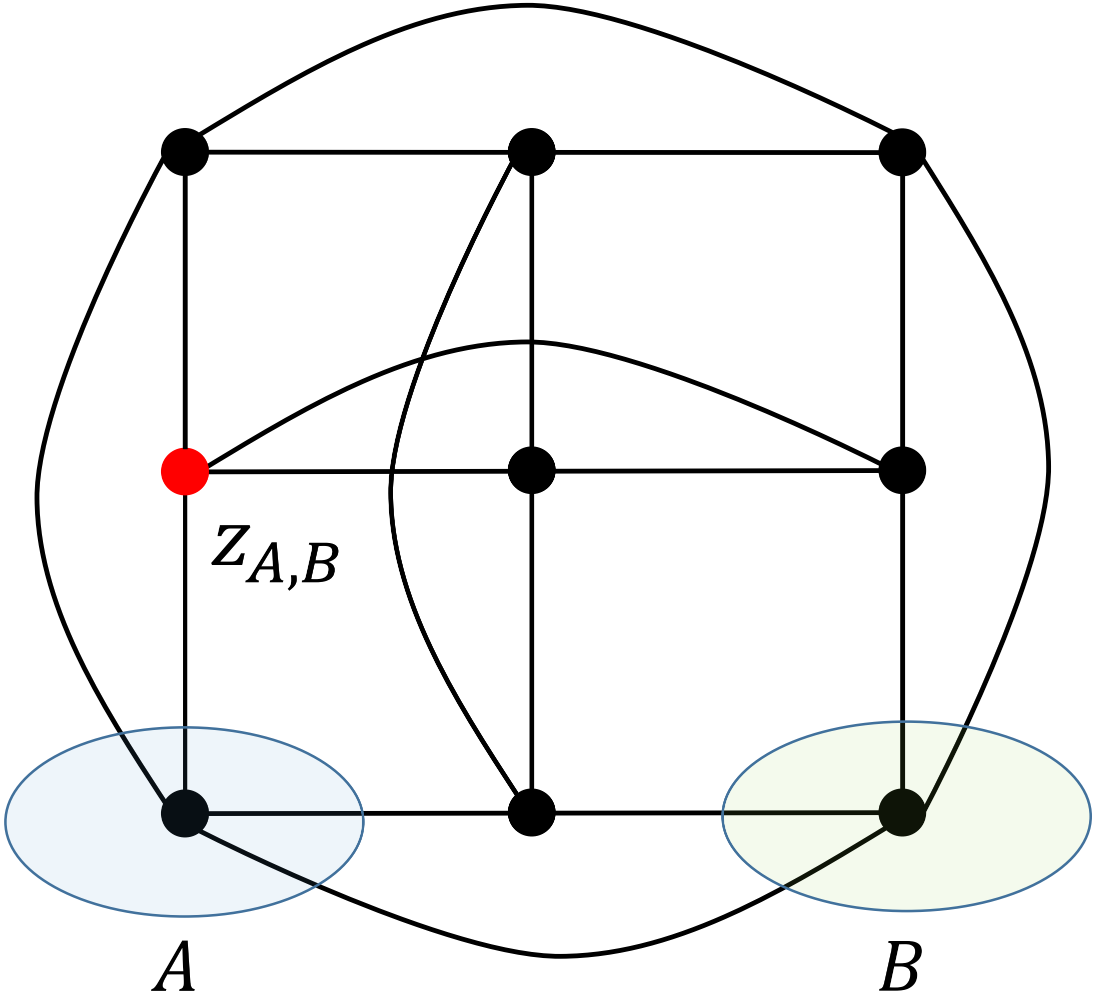

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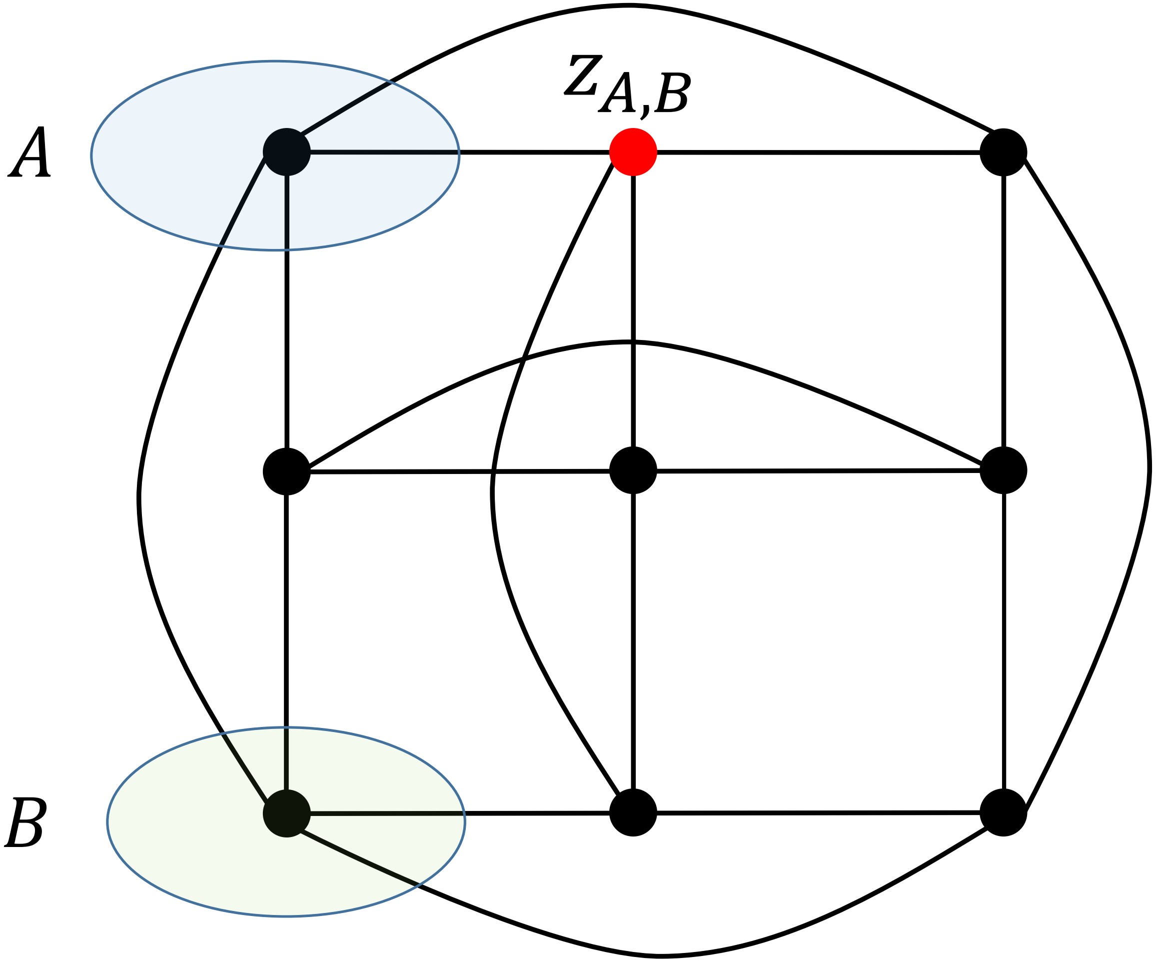

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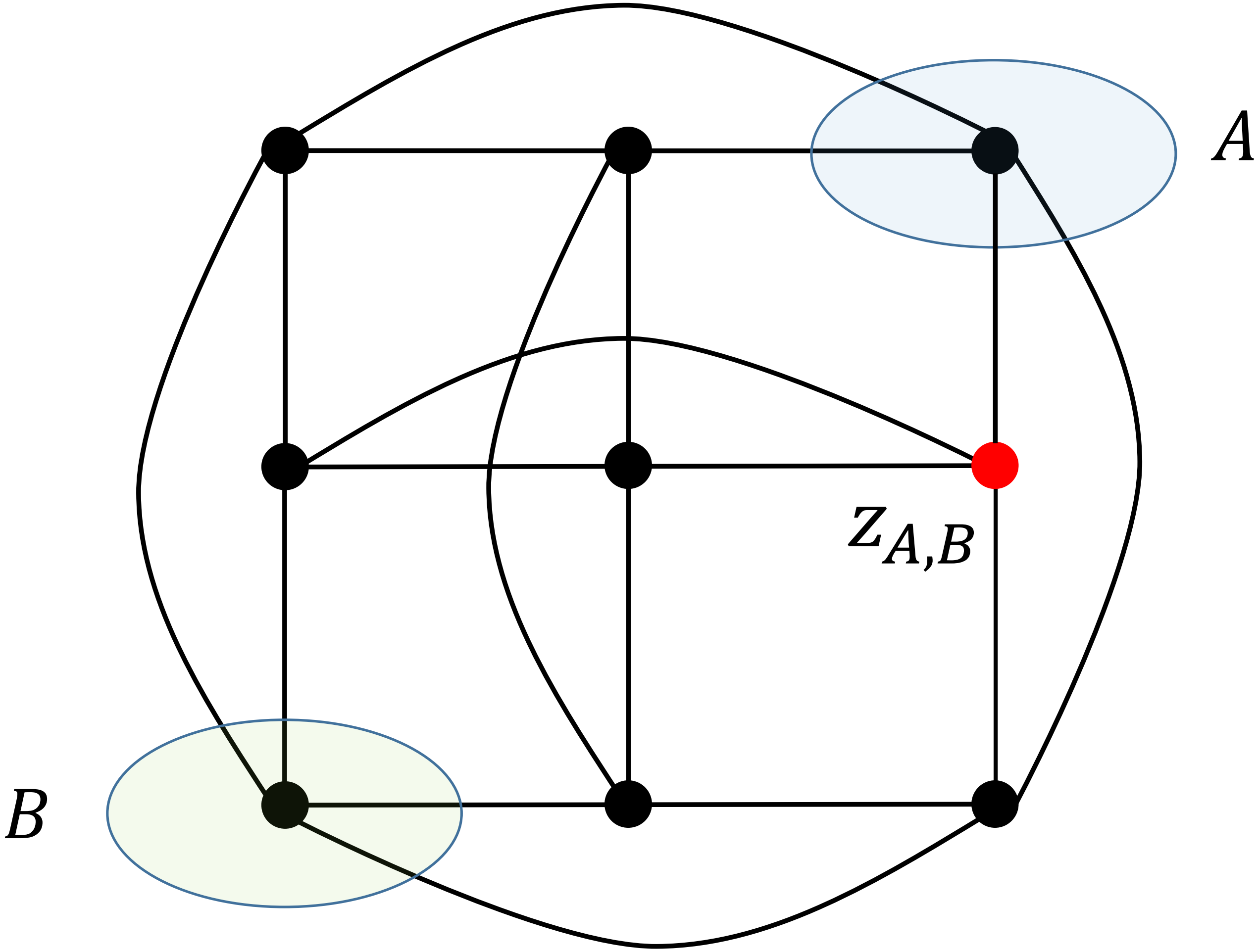

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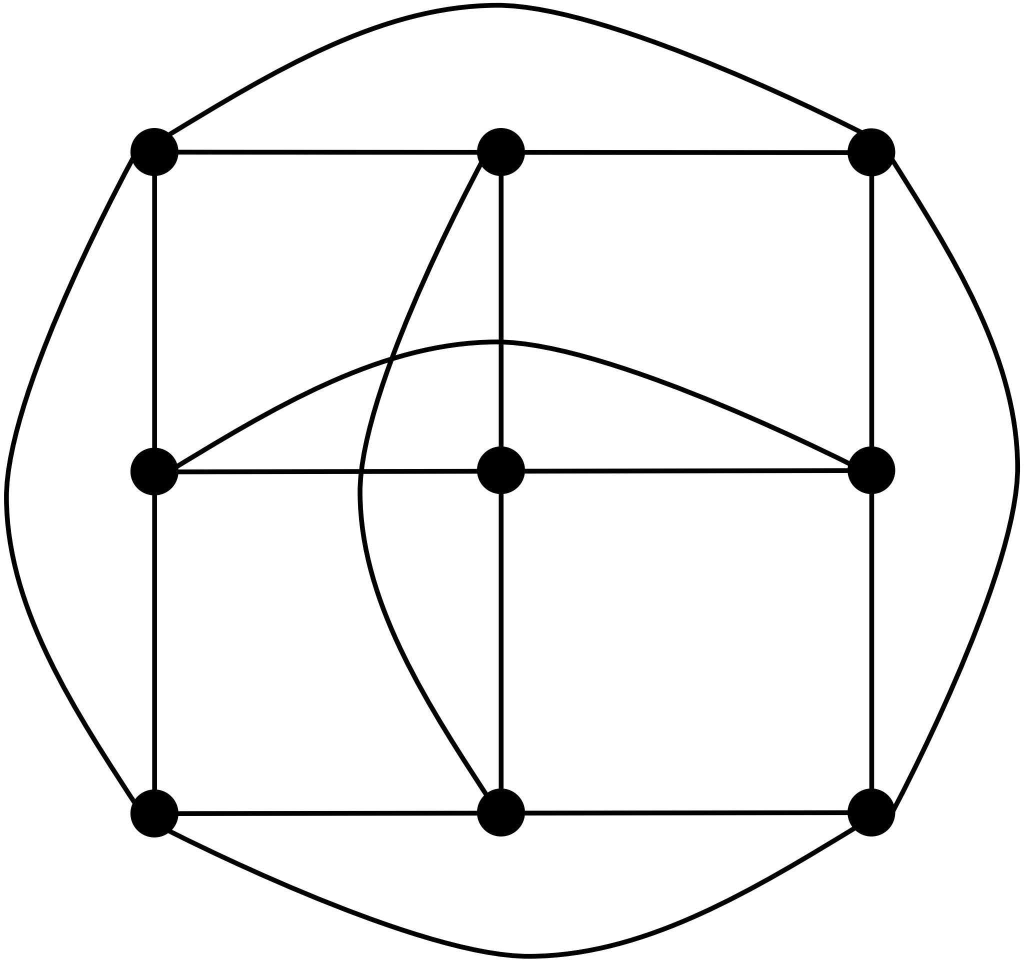

$\mathcal{P}(1,1)$ (moreover, 2-e. c.)

Main result

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Let e be odd and $p \equiv 1 \bmod 4$.

We define the graph G_{p^e} with vertex set $\mathbb{Z}_{p^e}$ as follows:

$$(x, y) \in E\left(G_{p^e}\right) \Leftrightarrow x - y = \mathrm{QR} \bmod p^e, (x - y, p) = 1.$$

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Theorem (S. 2018)

- For each l, m s.t. $l + m \leq n$, G_{p^e} satisfies $\mathcal{P}(l, m)$ if

$$p^e - \{(n-2)2^{n-1} + 1\}p^{e-\frac{1}{2}} - (n^2 + n)p^{e-1} - \frac{n(n+1)}{2} > 0.$$

Moreover, G_{p^e} is also n-e.c.

Random graphs a.a.s. satisfy n-e.c.

Let $G(m,p)$ be Erdős−Rényi random graph with edge probability p (constant).

Then, for each n, $G(m,p)$ is asymptotically almost surely (a.a.s.) n-e.c., that is,

$$\mathrm{Prob}[G(m,p) \text{ is } n\text{-e.c.}] \to 1 \ \ (m \to \infty).$$

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Proof

Let $p = \frac{1}{2}$ for simplicity. Then,

$$\mathrm{Prob}\left[G\left(m,\frac{1}{2}\right) \text{ is not } n\text{- e. c.}\right] \le \underbrace{\binom{m}{n}2^n}_{\#\{(A,B)\ \mid\ A\cap B=\emptyset,\ |A\cup B|=n\}} \underbrace{\left(1-\left(\frac{1}{2}\right)^n\right)^{m-n}}_{\mathrm{Prob}\left[\forall z\notin A\cup B,\ z \text{ doesn't satisfy } (*)_{A,B}\right]} \to 0$$

as $m \to \infty$.

□

Pseudo-randomness of random graphs

- Let $0 < p = p(m) < 1 \le \alpha$. G_m with m vertices is called **(p, α)-jumbled** if

$$\forall U \subset V(G), \left| |E(\underline{G_m[U]})| - p\binom{|U|}{2} \right| \le \alpha|U|. \quad \text{(Thomason, 1987)}$$

The induced subgraph of G_m induced by U

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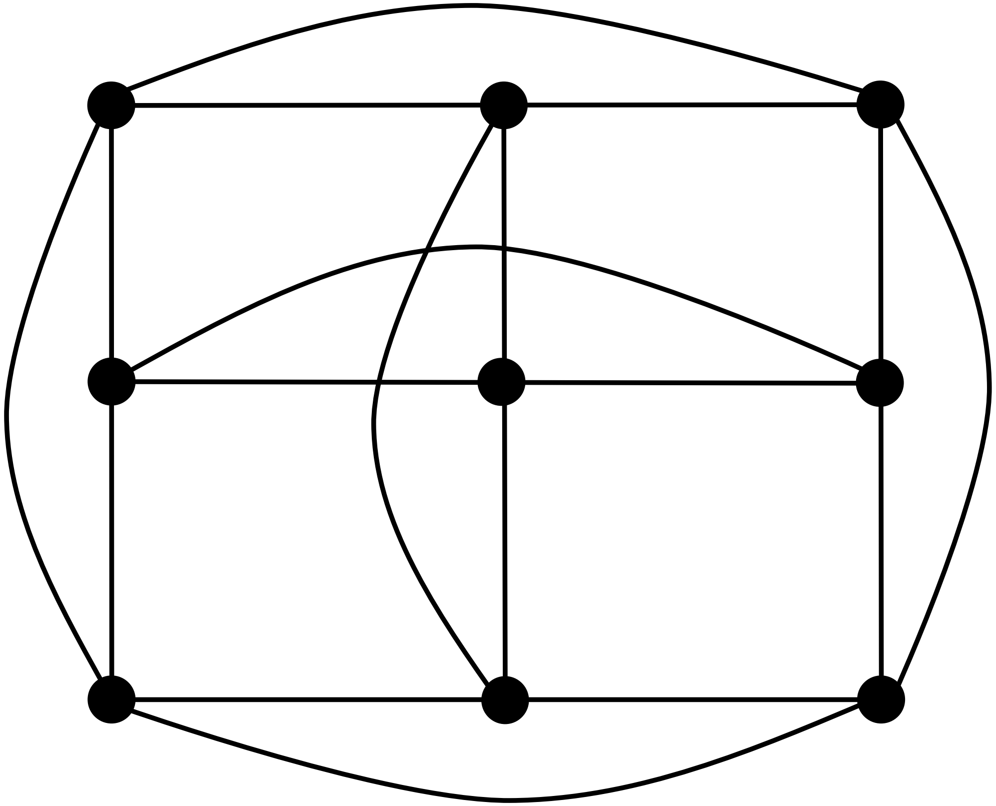

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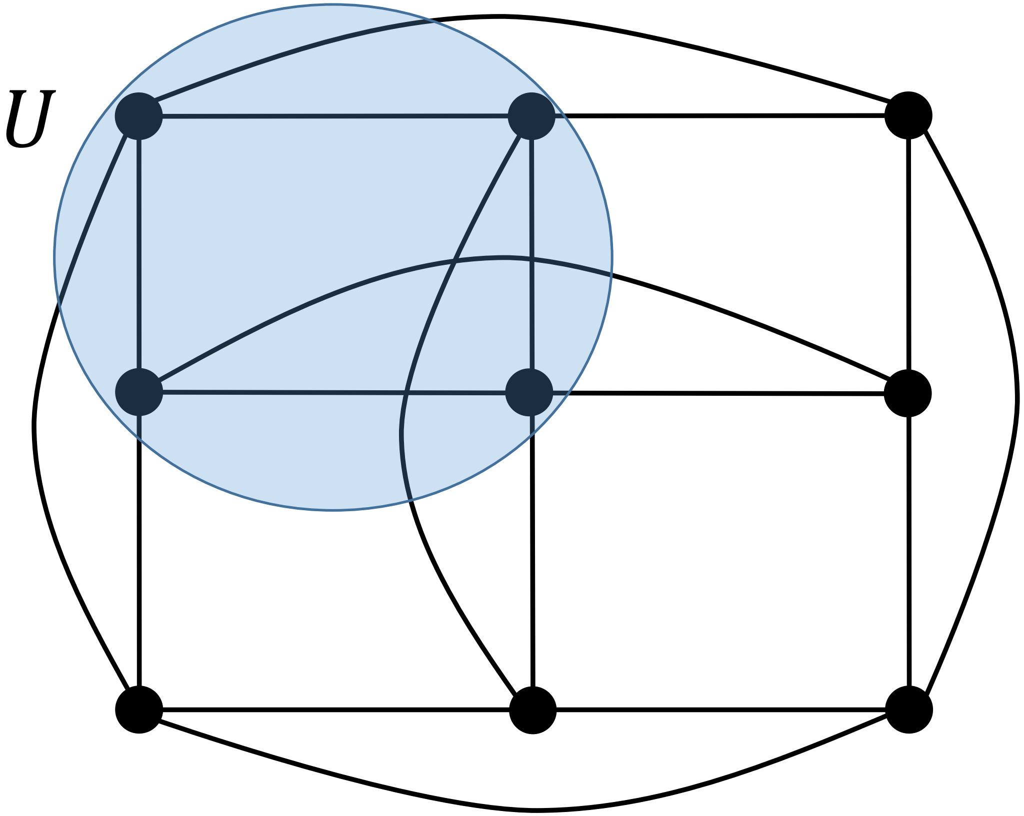

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$$\left|4 - \frac{4}{9}\binom{4}{2}\right| = 1.33 \ldots \leq 8 = 2 \cdot 4.$$

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Then, if $mp \to \infty$ and $m(1-p) \to \infty$, $\boldsymbol{G(m,p)}$ **is a.a.s.** $\left(\boldsymbol{p}, \boldsymbol{O}(\sqrt{\boldsymbol{mp}})\right)$**-jumbled**.

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- If G_m is non-bipartite and $k(m)$-regular,

$$\boldsymbol{G_m} \textbf{ is pseudo-random} \Leftrightarrow \boldsymbol{\lambda(G_m) = O\left(\sqrt{k(m)}\right)}$$

where

$$\lambda(G_m) = \max\{|\theta| \mid \theta\text{: eigenvalue of adjacency matrix of } G_m \text{ s.t. } |\theta| \neq k(m)\}.$$

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Theorem (S. 2018, Rewrite)

G_{p^e} is n-e.c. if $p^e - \{(n-2)2^{n-1} + 1\}p^{e-\frac{1}{2}} - (n^2 + n)p^{e-1} - \frac{n(n+1)}{2} > 0$.

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Corollary (S. 2018)

n-e.c. $\not\subset$ pseudo-random ($\forall n \geq 1$).

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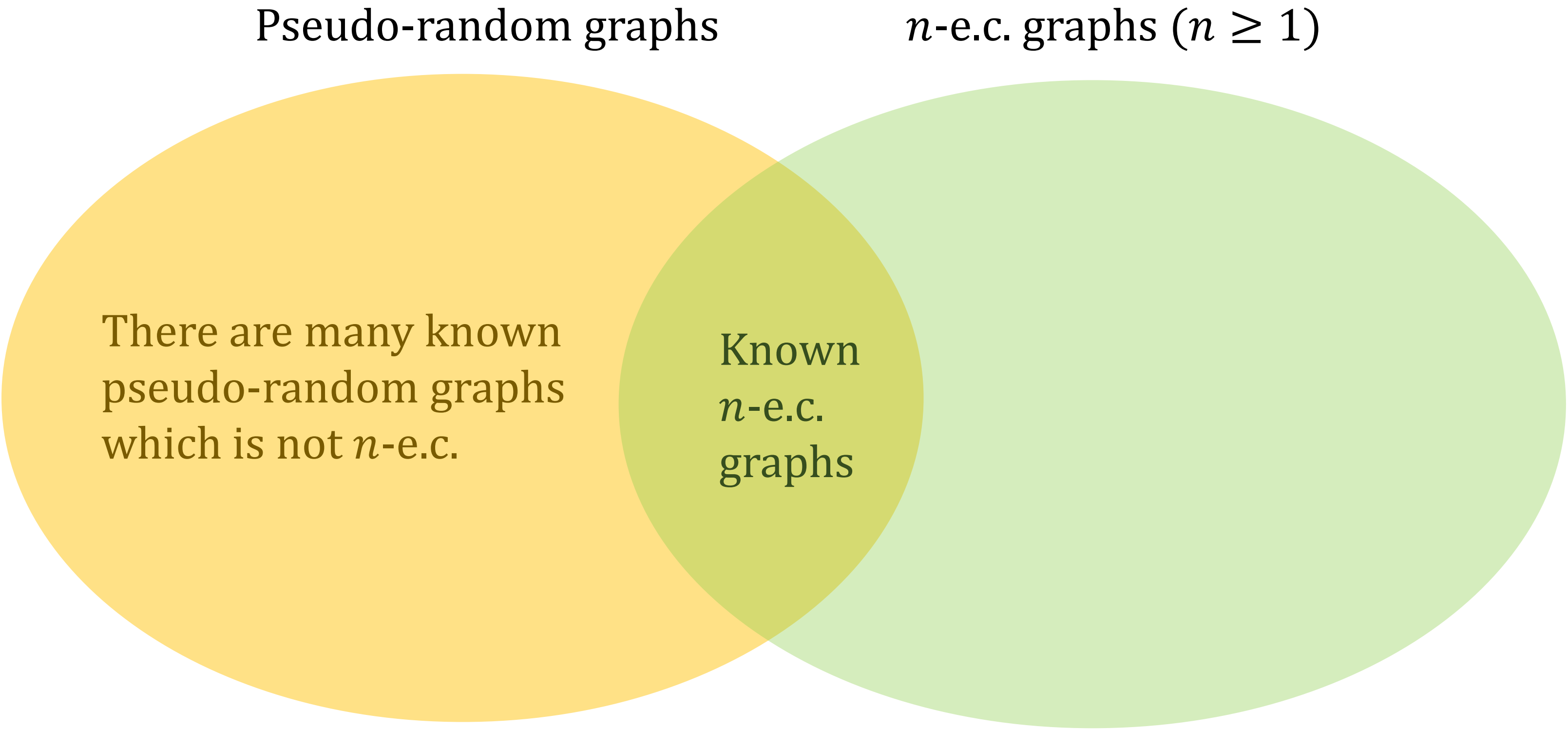

Pseudo-random graphs

n-e.c. graphs ($n \geq 1$)

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Known n-e.c. graphs

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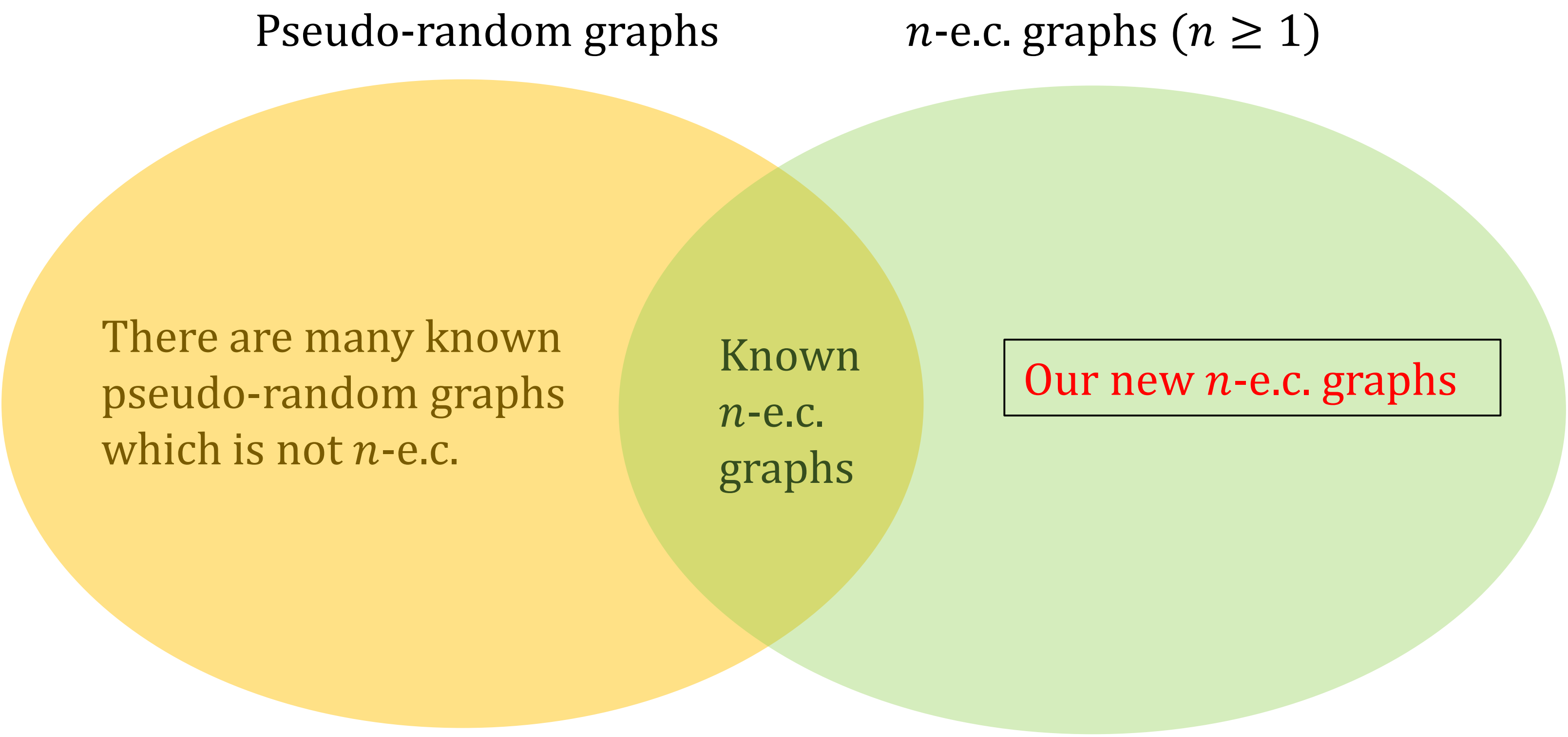

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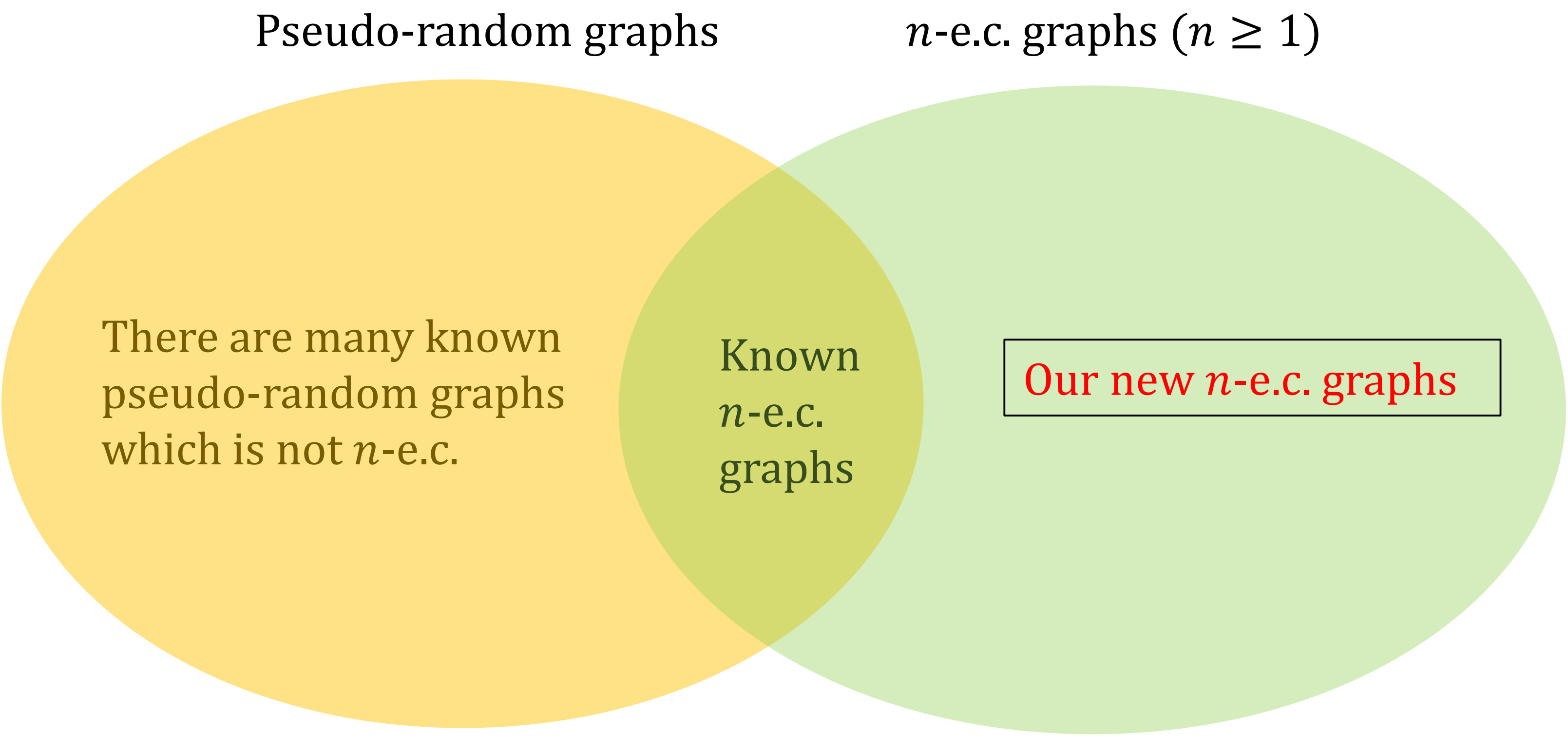

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Thank you for your attentions!!