Disjoint Cycles and Equitable Colorings in Graphs

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Outline

Corrádi-Hajnal Theorem

Corrádi-Hajnal, 1963

If G is a graph on n vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

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Examples:

- $k = 1$

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- Sharpness:

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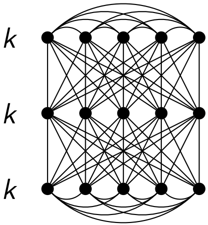
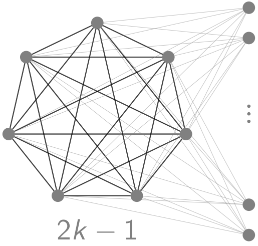
k is odd





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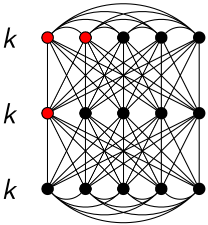

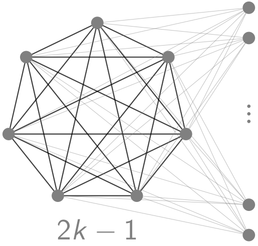

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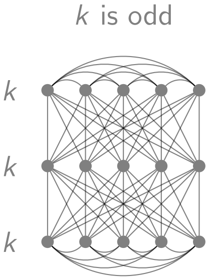

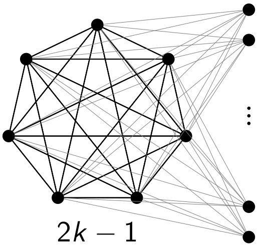

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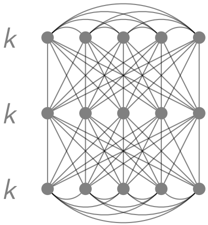

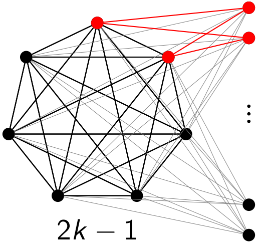

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Corrádi-Hajnal Theorem

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If G is a graph where all but one vertex has degree at least 2, then G contains a cycle.

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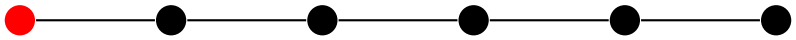

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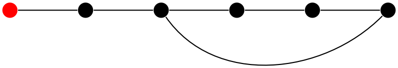
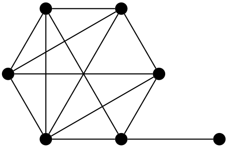


Dirac-Erdős, 1963



Let $V_{\geq c}$ be the number of vertices with degree at least c, etc.

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If $V_{\geq 2k} - V_{\leq 2k-2} \geq k^2 + 2k - 4$, $k \geq 3$, then G contains k disjoint cycles.

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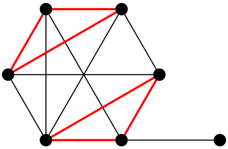

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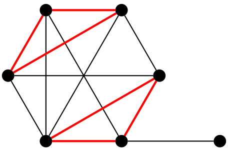

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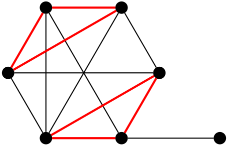

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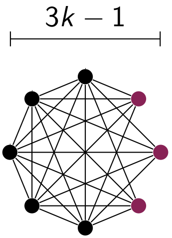

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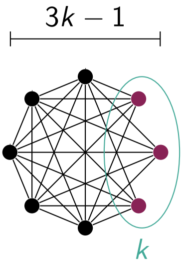

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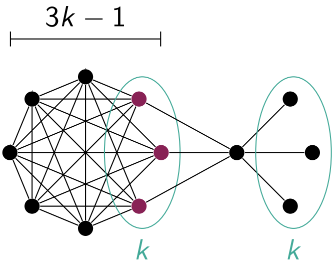

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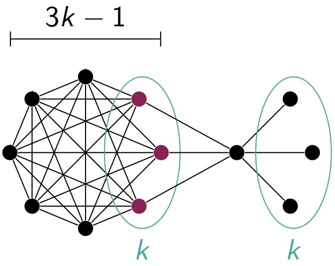

high degree: $3k$

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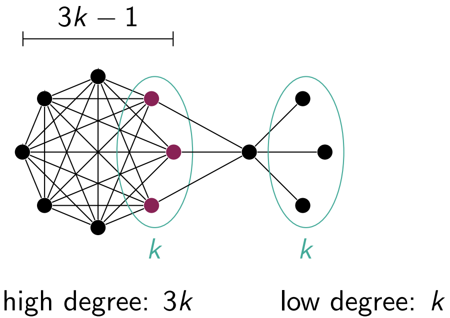

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Kierstead-Kostochka-McConvey, 2018 (link)

Let $k \geq 2$ be an integer and G be a graph with $|G| \geq 19k$ and $V_{\geq 2k} - V_{\leq 2k-2} \geq 2k$. Then G contains k disjoint cycles.

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Open

Characterize graphs G with $V_{\geq 2k} - V_{\leq 2k-2} \geq 2k$ and no k disjoint cycles.

Outline

1. Disjoint Cycles
   - Corrádi-Hajnal
   - Tolerance for some low-degree vertices
   - Ore condition (minimum degree-sum of nonadjacent vertices)
   - Generalized Degree-Sum Conditions
   - Connectivity
   - Neighborhood Union
2. Chorded Cycles
   - Degree conditions
   - Neighborhood Union
   - Multiply Chorded Cycles
3. Equitable Coloring
   - Definition
   - Connection to Cycles

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With a little work, implies Dirac-Erdős

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Proof (Enomoto)

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  - Remaining graph at least 3 vertices
- Minimize number of vertices in cycles
- Maximize longest path in remainder

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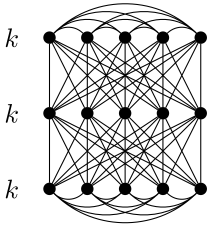

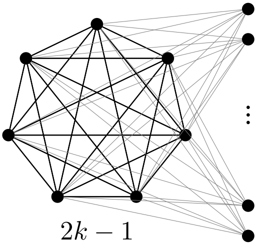

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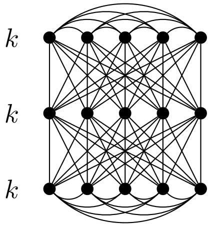

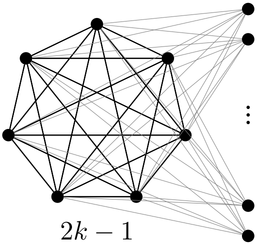

$2k-1$

$3k$ vertices

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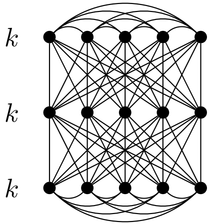

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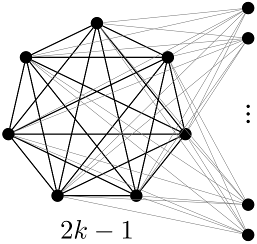


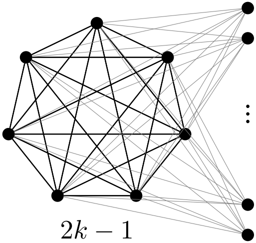
$\alpha(G)$ large

Kierstead-Kostochka-Yeager, 2017 (link)

Independence Number:

Observation:

$\alpha(G) \geq n - 2k + 1 \Rightarrow$ no k cycles



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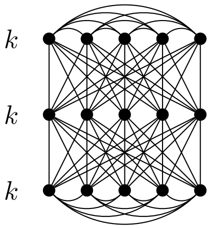
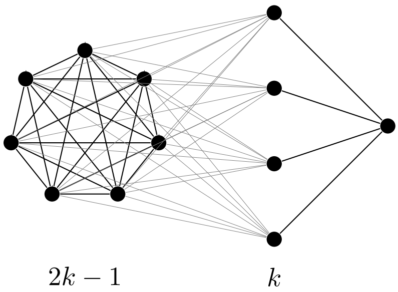
$$n \geq 3k+1$$





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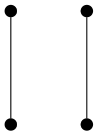

Kierstead-Kostochka-Yeager, 2017

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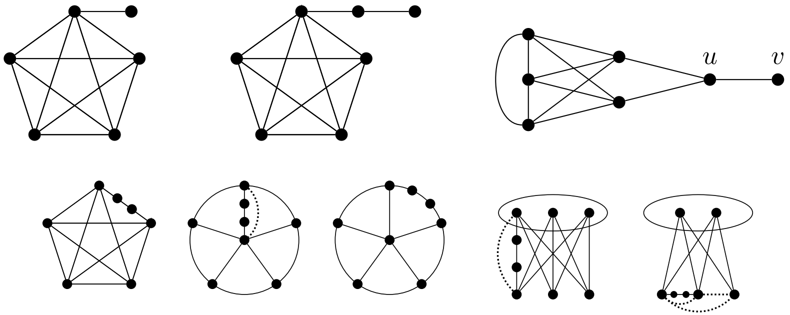

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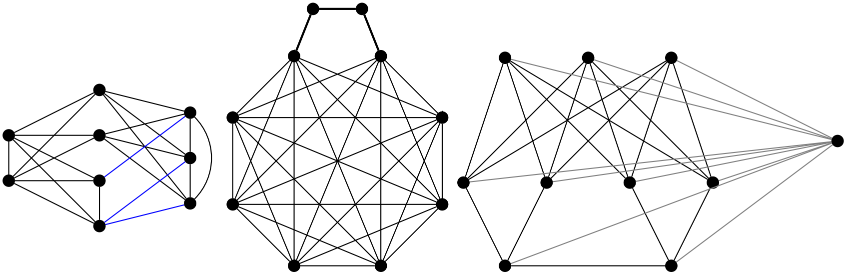

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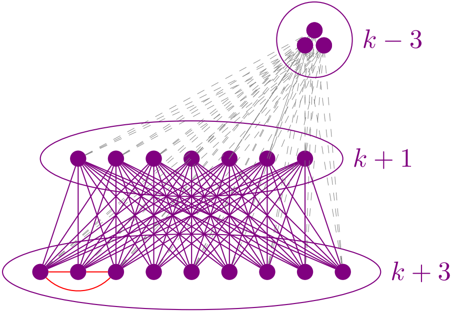
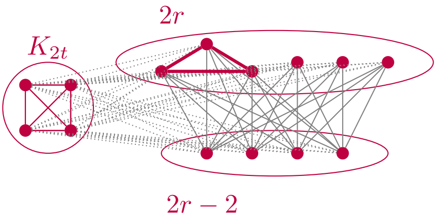
$$\sigma_2 = 4k - 4:$$





Outline

1. Disjoint Cycles
   - Corrádi-Hajnal
   - Tolerance for some low-degree vertices
   - Ore condition (minimum degree-sum of nonadjacent vertices)
   - Generalized Degree-Sum Conditions
   - Connectivity
   - Neighborhood Union
2. Chorded Cycles
   - Degree conditions
   - Neighborhood Union
   - Multiply Chorded Cycles
3. Equitable Coloring
   - Definition
   - Connection to Cycles

Extending Enomoto-Wang

Enomoto 1998, Wang 1999

If G is a graph on n vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k disjoint cycles.

$\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$

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Conjecture: Gould, Hirohata, Keller 2018 (link)

Let G be a graph of sufficiently large order. If $\sigma_t(G) \geq 2kt - t + 1$ for any two integers $k \geq 2$ and $t \geq 1$, then G contains k disjoint cycles.

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$t = 3$: Fujita, Matsumura, Tsugaki, Yamashita 2006 (link)
$t = 4$: proved in paper as evidence for conjecture

Ma, Yan

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Let G be a graph with $|G| \geq (2t+1)k$. If $\sigma_t(G) \geq 2kt - t + 1$ for any two integers $k \geq 2$ and $t \geq 5$, then G contains k disjoint cycles.

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Proof

In an edge-maximal counterexample, choose $k - 1$ disjoint cycles such that

- number of vertices in cycles is minimal, and
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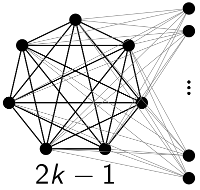

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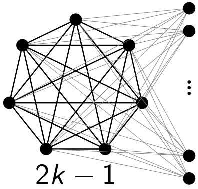

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Open

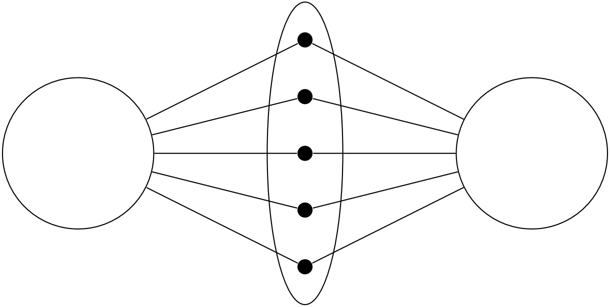
What is the best possible bound on $|G|$ in the Ma-Yan Theorem?
Can we characterize graphs G with $\sigma_t(G) \geq 2kt - t + 1$ but no k disjoint cycles?

Outline

Dirac: $(2k-1)$-connected without k disjoint cycles

Dirac, 1963 (link)

What $(2k-1)$-connected graphs do not have k disjoint cycles?



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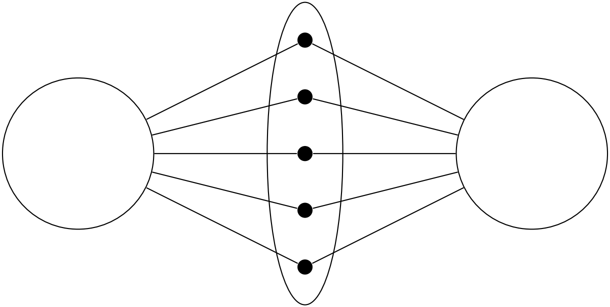

Observation:

G is $(2k-1)$ connected

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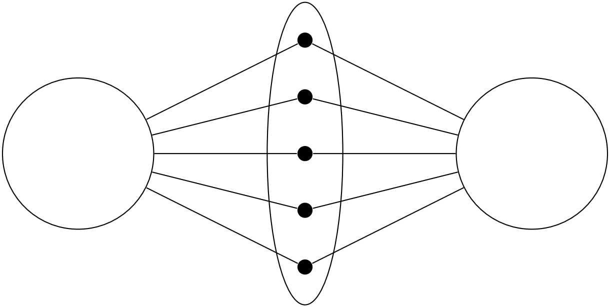

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G is $(2k-1)$ connected $\Rightarrow \delta(G) \geq 2k-1$

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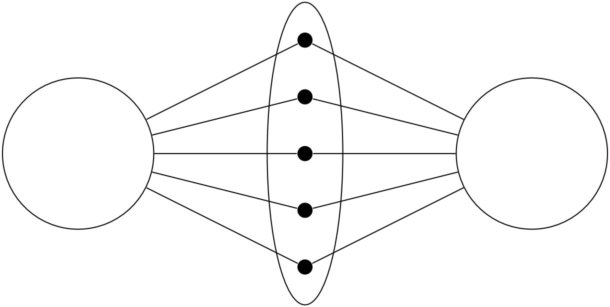

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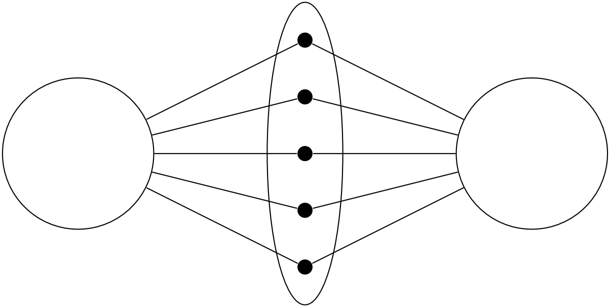

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KKY: Holds for $\sigma_2(G) \geq 4k-3$

Dirac: $(2k-1)$-connected without k disjoint cycles

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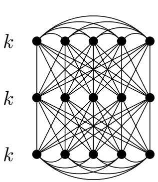
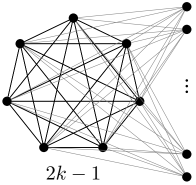
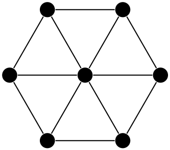
Answer to Dirac's Question for Simple Graphs (KKY 2017)

Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if

- if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$, and
- $\alpha(G) \leq |G| - 2k$, and
- if $k = 2$ then G is not a wheel.







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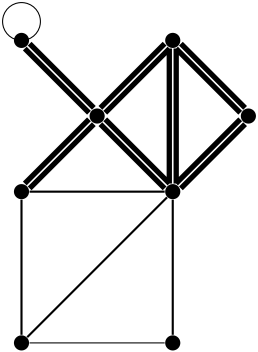
Further:

characterization for multigraphs

Simple Graphs → Multigraphs

Idea:



- Take all 1-vertex cycles

Simple Graphs → Multigraphs

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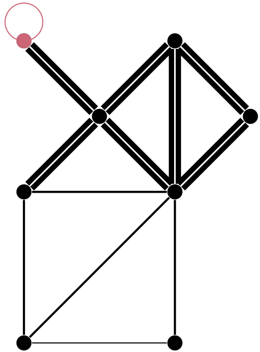

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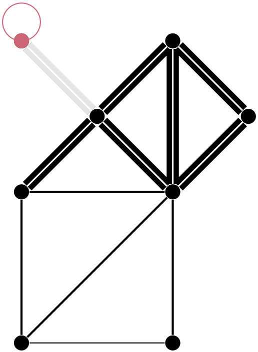
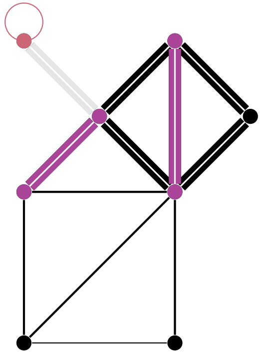


- Take all 1-vertex cycles
- Take as many 2-vertex cycles as possible (maximum matching)

Simple Graphs → Multigraphs

Idea:



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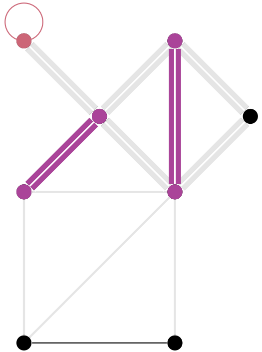


- Take all 1-vertex cycles
- Take as many 2-vertex cycles as possible (maximum matching)
- What's left is a simple graph

$(2k-1)$-connected multigraphs with no k disjoint cycles

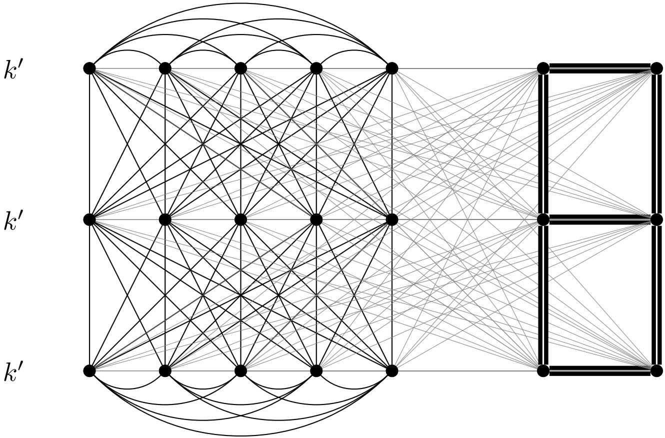
Answer to Dirac's Question for multigraphs: Kierstead-Kostochka-Yeager 2015 (link)

Let $k \geq 2$ and $n \geq k$. Let G be an n-vertex graph with simple degree at least $2k-1$ and no loops. Let F be the simple graph induced by the strong edgs of G, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Then G does not contain k disjoint cycles if and only if one of the following holds:

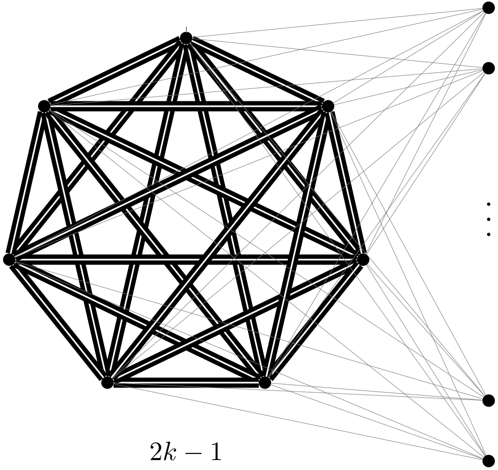
- $n + \alpha' < 3k$;
- $|F| = 2\alpha'$ (i.e., F has a perfect matching) and either (i) k' is odd and $G - F = Y_{k',k'}$, or (ii) $k' = 2 < k$ and $G - F$ is a wheel with 5 spokes;
- G is extremal and either (i) some big set is not incident to any strong edge, or (ii) for some two distinct big sets I_j and $I_{j'}$, all strong edges intersecting $I_j \cup I_{j'}$ have a common vertex outside of $I_j \cup I_{j'}$;
- $n = 2\alpha' + 3k'$, k' is odd, and F has a superstar $S = \{v_0, \ldots, v_s\}$ with center v_0 such that either (i) $G - (F - S + v_0) = Y_{k'+1,k'}$, or (ii) $s = 2$, $v_1 v_2 \in E(G)$, $G - F = Y_{k'-1,k'}$ and G has no edges between $\{v_1, v_2\}$ and the set X_0 in $G - F$;
- $k = 2$ and G is a wheel, where some spokes could be strong edges;
- $k' = 2$, $|F| = 2\alpha' + 1 = n - 5$, and $G - F = C_5$.

k' odd, F has a perfect matching

Example: $k = 8$, $\alpha' = 3$, $k' = 5$.

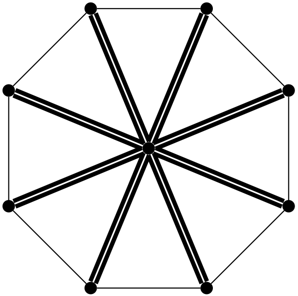


Big independent set, incident to no multiple edges



Wheel, with possibly some spokes multiple

Example: $k = 2$



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Characterization of multigraphs without k disjoint cycles that have minimum simple degree at least $2k-1$. That is, the underlying simple graph $\underline{G}$ has $\delta(\underline{G}) \geq 2k-1$.

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Open

Do the other results in this talk generalize nicely to multigraphs?

Outline

Neighborhood Union

Faudree-Gould, 2005 (link)

If G has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices x, y, then G contains k disjoint cycles.

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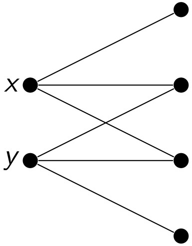

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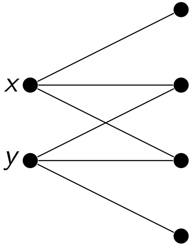

$d(x) + d(y) = 6$

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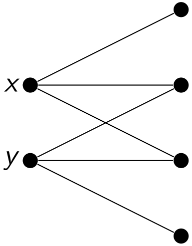

$$d(x) + d(y) = 6$$
$$|N(x) \cup N(y)| = 4$$

Neighborhood Union

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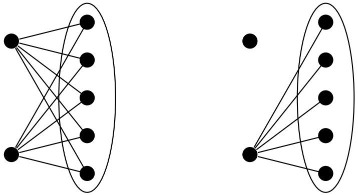
If G has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices x, y, then G contains k disjoint cycles.

Neither stronger nor weaker than Corrádi-Hajnal.

- If $\delta(G) = 2k$, then $\min_{xy \notin E(G)} \{|N(x) \cup N(y)|\} \geq 2k$.
- If $|N(x) \cup N(y)| \geq 3k$, then $\delta(G) \geq 0$.



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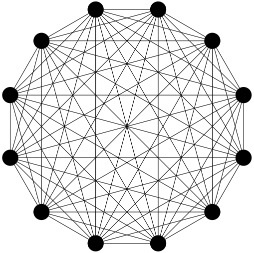
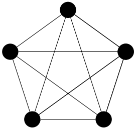
Sharpness:



K_{3k-4}



K_5

Gould-Hirohata-Horn, 2013

Faudree-Gould, 2005 (link)

If G has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices x, y, then G contains k disjoint cycles.

Gould-Hirohata-Horn, 2013 (link) (conjecture from FG'05)

Let G be a graph on $n > 30k$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 2k + 1$. Then G contains k disjoint cycles.

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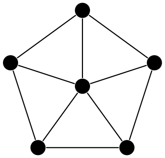
Sharpness of $|N(x) \cup N(y)| \geq 2k + 1$:



$k = 2$

Gould-Hirohata-Horn, 2013

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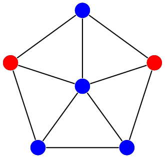

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$|N(x) \cup N(y)| \geq 4 = 2k$

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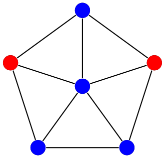

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No two disjoint cycles

Gould-Hirohata-Horn, 2013

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Open:

Perhaps $n > 30k$ is not best possible–can be reduced to $4k$?

Outline

Finkel, 2008

Posed by Pósa, 1961

Finkel, 2008 (link)

If G is a graph on $n \geq 4k$ vertices with $\delta(G) \geq 3k$, then G contains k disjoint chorded cycles.

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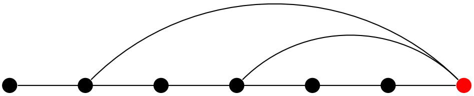

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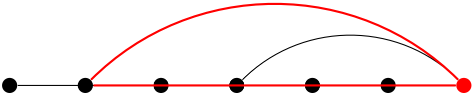

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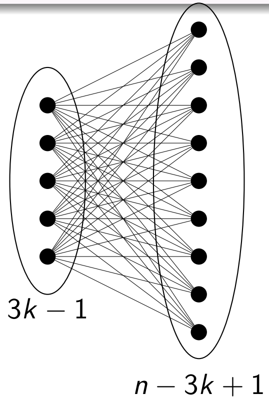

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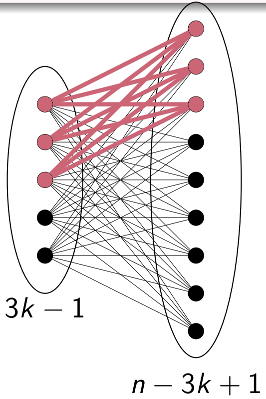

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If G is a graph on $n \geq 4k$ vertices with $\delta(G) \geq 3k$, then G contains k disjoint chorded cycles.

Proof (2 pages!)

In an edge-maximal counterexample, choose $k-1$ disjoint cycles such that

- number of vertices in cycles is minimal, and
- longest path in the remaining graph is maximal

Chorded + Unchorded Cycles

Conjecture: Bialostocki-Finkel-Gyárfás, 2008 (link)

If G is a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then G contains $r + s$ cycles, s of them chorded.

$s = 0$: Corrádi-Hajnal

$r = 0$: Finkel

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Chiba-Fujita-Gao-Li, 2010 (link)

Let r and s be integers with $r + s \geq 1$, and let G be a graph on $n \geq 3r + 4s$ vertices. If $\boxed{\sigma_2(G) \geq 4r + 6s - 1}$, then G contains $r + s$ disjoint cycles, s of them chorded cycles.

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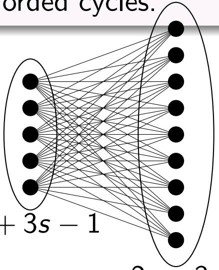

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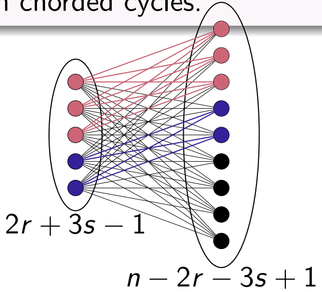

Chorded + Unchorded Cycles: How Sharp Is It?

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Let G be a graph on $n \geq 4s$ vertices. If $\sigma_2(G) \geq 6s - 1$, then G contains s disjoint chorded cycles.

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Corollary

Let G be a graph on $n \geq 4s$ vertices. If $\sigma_2(G) \geq 6s - 1$, then G contains s disjoint chorded cycles.

Molla-Santana-Yeager, 2017 (link)

For $s \geq 2$, let G be a graph $n \geq 4s$ vertices. If $\sigma_2(G) \geq 6s - 2$, then G does not contain s disjoint chorded cycles if and only if $G \in \{K_{3s-1,n-3s+1}, K_{3s-2,3s-2,1}\}$.

Chorded + Unchorded Cycles: How Sharp Is It?

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Let r and s be integers with $r + s \geq 1$, and let G be a graph on $n \geq 3r + 4s$ vertices. If $\sigma_2(G) \geq 4r + 6s - 1$, then G contains $r + s$ disjoint cycles, s of them chorded cycles.

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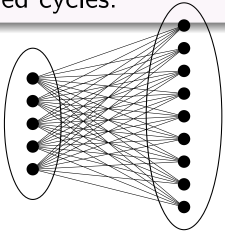

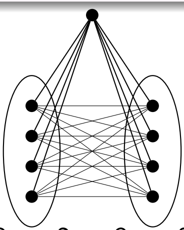

Chorded + Unchorded Cycles: How Sharp Is It?

Chiba-Fujita-Gao-Li, 2010 (link)

Corollary: If G is a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then G contains $r + s$ cycles, s of them chorded.

Chorded + Unchorded Cycles: How Sharp Is It?

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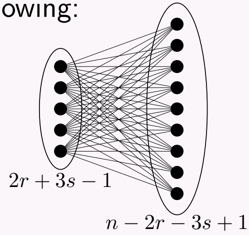
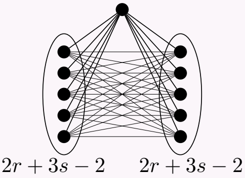
Corollary: If G is a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then G contains $r + s$ cycles, s of them chorded.

Molla-Santana-Yeager, 2018+

Let r and s be integers with $r + s \geq 1$, and let G be a graph on $n \geq 3r + 4s$ vertices. If $\delta(G) \geq 2r + 3s - 1$, then G fails to contain a collection of $r + s$ disjoint cycles, s of them chorded, if and only if G is one of the following:





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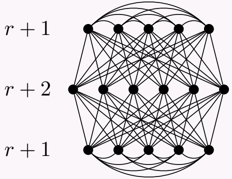
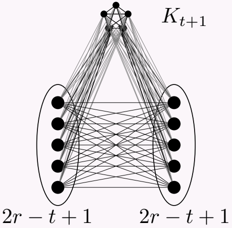
$s = 1$:





Chorded + Unchorded Cycles: Open

Chiba-Fujita-Gao-Li, 2010 (link)

Let r and s be integers with $r + s \geq 1$, and let G be a graph on $n \geq 3r + 4s$ vertices. If $\sigma_2(G) \geq 4r + 6s - 1$, then G contains $r + s$ disjoint cycles, s of them chorded cycles.

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For $s \geq 2$, let G be a graph $n \geq 4s$ vertices. If $\sigma_2(G) \geq 6s - 2$, then G does not contain s disjoint chorded cycles if and only if $G \in \{K_{3k-1,n-3k+1}, K_{3k-2,3k-2,1}\}$.

Open

We know what happens if $\sigma_2(G) \geq 6s - 2$; what if $\sigma_2(G) \geq 6s - 3$?

Degree-sum condition: chorded?

Ma, Yan 2018+ (link)

Let G be a graph with $|G| \geq (2t+1)k$. If $\sigma_t(G) \geq 2kt - t + 1$ for any two integers $k \geq 2$ and $t \geq 5$, then G contains k disjoint cycles.

Open

Is there a chorded-cycles analogue to the Ma-Yan Theorem?

Outline

Neighborhood-Union Conditions

Qiao, 2012 (link)

Let r, s be nonnegative integers, and let G be a graph on at least $3r + 4s$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 3r + 4s + 1$. Then G contains $r + s$ disjoint cycles, s of them chorded.

Neighborhood-Union Conditions

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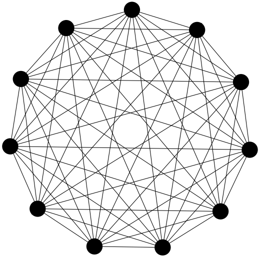
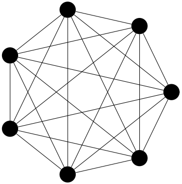
Sharpness ($r = 0$):



K_{2s+3}



K_{2s-1}

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Let G be a graph on at least $4s$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 4s + 1$. Then G contains s disjoint chorded cycles.

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Let G be a graph on at least $4s$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 4s + 1$. Then G contains s disjoint chorded cycles.

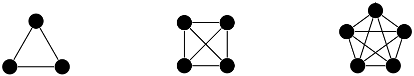
Open:

Can this be improved for large n, like for (not-necessarily-chorded) cycles?

Outline

Multiply Chorded Cycles

We define $f(c)$ to be the number of chords in K_{c+1}, viewed as a cycle. That is, $f(c) = \frac{(c+1)(c-2)}{2}$.



$f(2) = 0$ $f(3) = 2$ $f(4) = 5$

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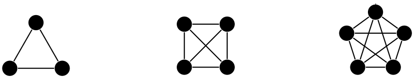

$f(2) = 0$ $\quad$ $f(3) = 2$ $\quad$ $f(4) = 5$

Conjecture: Gould-Horn-Magnant, 2014

If $|G| \geq k(c+1)$ and $\delta(G) \geq ck$, then G contains k disjoint cycles, each with at least $f(c)$ chords.

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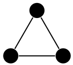

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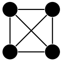
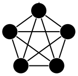


$f(3) = 2$



$f(4) = 5$

Conjecture: Gould-Horn-Magnant, 2014

If $|G| \geq k(c+1)$ and $\delta(G) \geq ck$, then G contains k disjoint cycles, each with at least $f(c)$ chords.

If $c = 2$, then $f(c) = 0$, so the conjecture states:
If $|G| \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles

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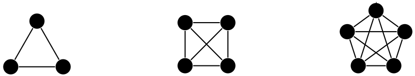

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Corrádi-Hajnal

Multiply Chorded Cycles

Conjecture: (GHM 2014)

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Multiply Chorded Cycles

Conjecture: (GHM 2014)

If $|G| \geq k(c+1)$ and $\delta(G) \geq ck$, then G contains k disjoint cycles, each with at least $f(c)$ chords.

If $c = 3$, then $f(c) = 2$, so the conjecture states:

If $|G| \geq 4k$ and $\delta(G) \geq 3k$, then G contains k disjoint cycles, each with at least 2 chords.

Multiply Chorded Cycles

Conjecture: (GHM 2014)

If $|G| \geq k(c+1)$ and $\delta(G) \geq ck$, then G contains k disjoint cycles, each with at least $f(c)$ chords.

If $c = 3$, then $f(c) = 2$, so the conjecture states:
If $|G| \geq 4k$ and $\delta(G) \geq 3k$, then G contains k disjoint cycles, each with at least 2 chords.

Qiao-Zhang, 2010 (link)

Let G be a graph on $n \geq 4k$ vertices with $\delta(G) \geq \lfloor 7k/2 \rfloor$. Then G contains k disjoint, doubly chorded cycles.

Gould-Hirohata-Horn, 2015 (link)

If G is a graph on $n \geq 6k$ vertices with $\delta(G) \geq 3k$, then G contains k vertex-disjoint doubly chorded cycles.

Multiply Chorded Cycles

Conjecture: (GHM 2014)

If $|G| \geq k(c+1)$ and $\delta(G) \geq kc$, then G contains k disjoint cycles, each with at least $f(c)$ chords.

Chiba-Lichiardopol, 2017 (link)

Let k and c be integers, $c \geq 2$, $k \geq 1$.
If G is a graph with $\delta(G) \geq k(c+1) - 1$, then G contains k disjoint cycles, each with at least $f(c)$ chords.

Multiply Chorded Cycles

Conjecture: (GHM 2014)

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Open

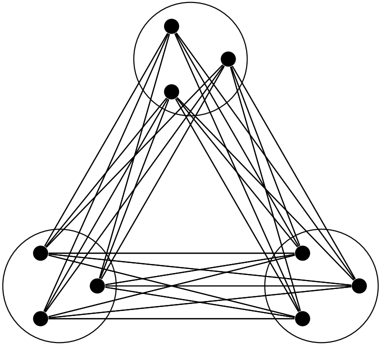
Is $\delta(G) \geq k(c+1)-1$ the most fitting bound?

Outline

Equitable Coloring

Definition

An *equitable k-coloring* of a graph G is a proper coloring of $V(G)$ such that any two color classes differ in size by at most one.



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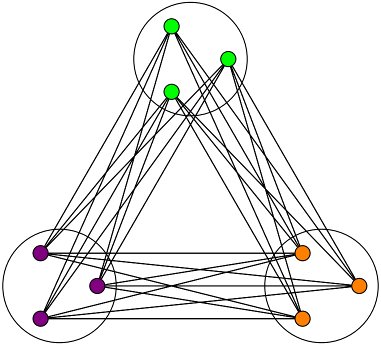

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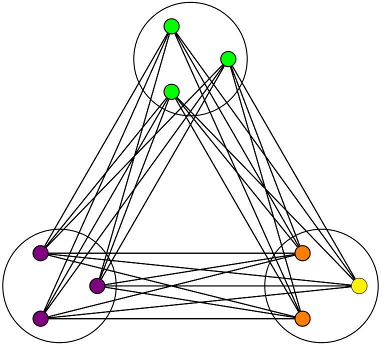

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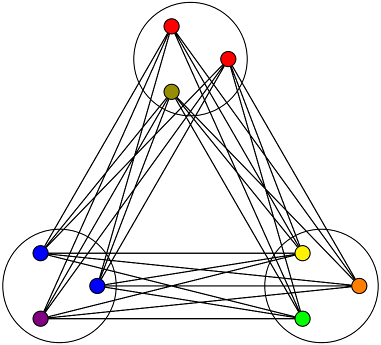

Outline

Equitable Coloring and Cycles

$n = 3k$

If G has $n = 3k$ vertices, then G has an equitable k-coloring iff $\overline{G}$ has k disjoint cycles (all triangles).

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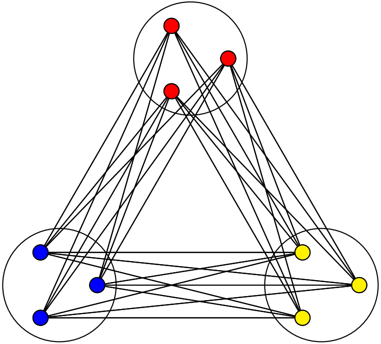

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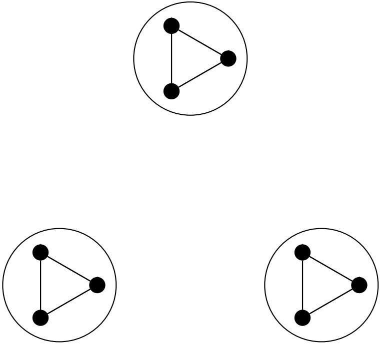

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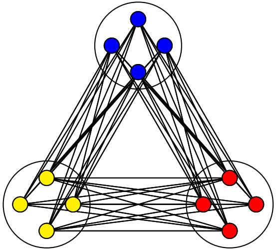

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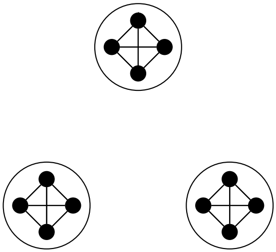

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What's Really Going On

- If G has $3k$ vertices and k cycles, those cycles are cliques
- If G has $4k$ vertices and k doubly chorded cycles, those cycles are cliques
- The complement of a clique is an independent set (color class)

Equitable Coloring and Cycles

Enomoto 1998, Wang 1999

If G is a graph on n vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k disjoint cycles.

(minimum degree sum of nonadjacent vertices)

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$n = 3k$

Equivalent when $n = 3k$: 2(3k-1)-(2k-1)=4k-1

Chen-Lih-Wu

Hajnal-Szemerédi, 1970

If $k \geq \Delta(G) + 1$, then G is equitably k-colorable.

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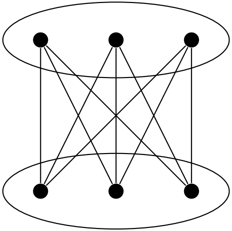

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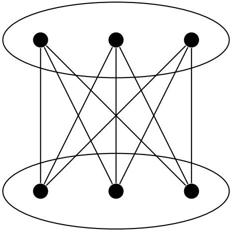

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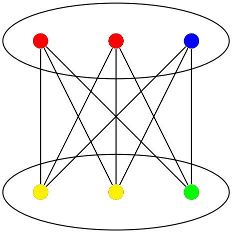

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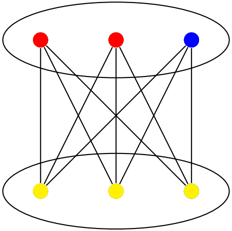

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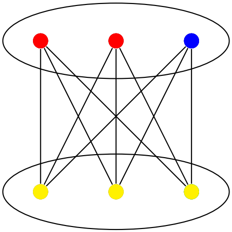

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Chen-Lih-Wu Conjecture, 1994 (link)

A connected graph G is equitably $\Delta(G)$ colorable if G is different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$ for every $m \geq 1$.

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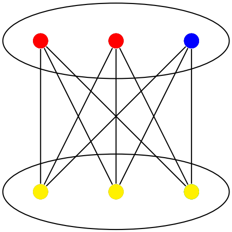

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Many special cases proved; still open in general

Ore Conditions

Chen-Lih-Wu Conjecture Re-stated

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then G is equitably k-colorable.

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If G is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles, or is one of several exceptions.

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KKY, 2017

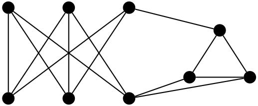
For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

Exceptions

$|G| = 3k$, $\chi(\overline{G}) \leq k$, $\sigma_2(G) \geq 4k - 3$, no k disjoint cycles.

- $k = 3$

Equitable coloring:



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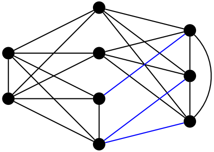

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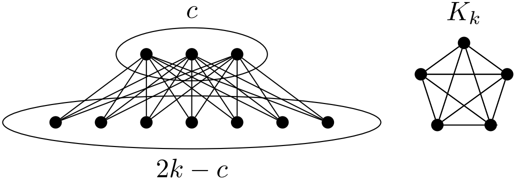

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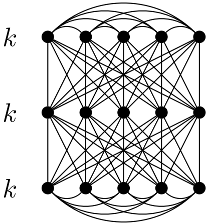

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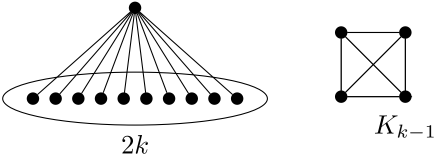

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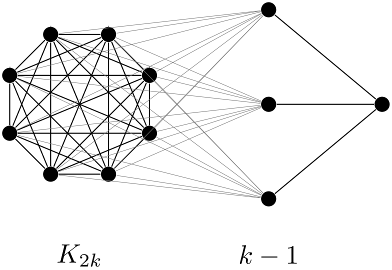
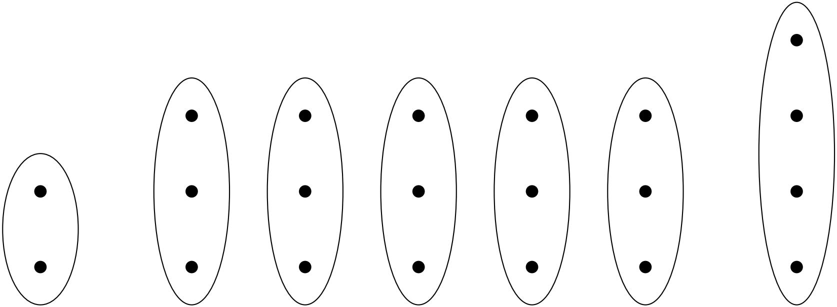


Proof of KKMY 2016



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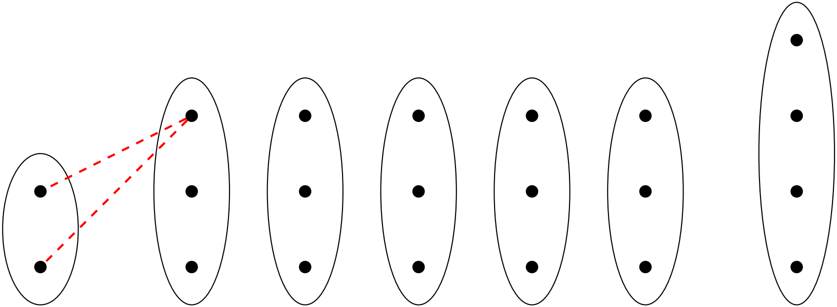

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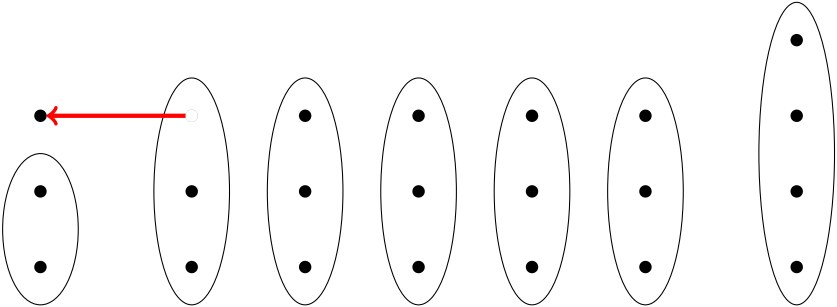

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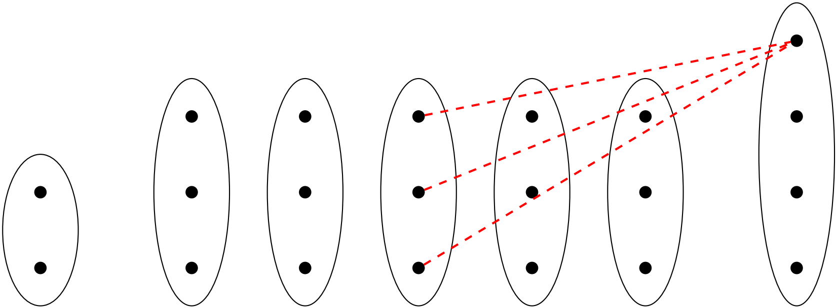

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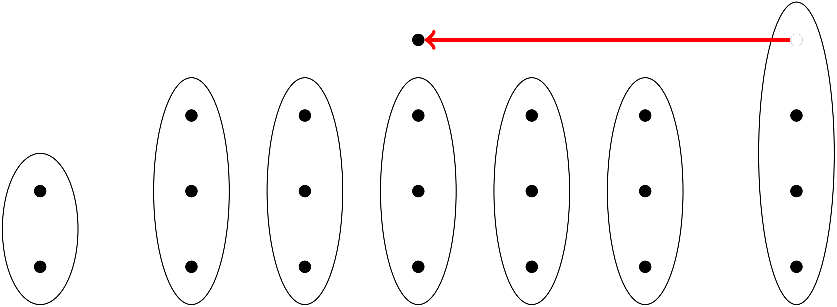

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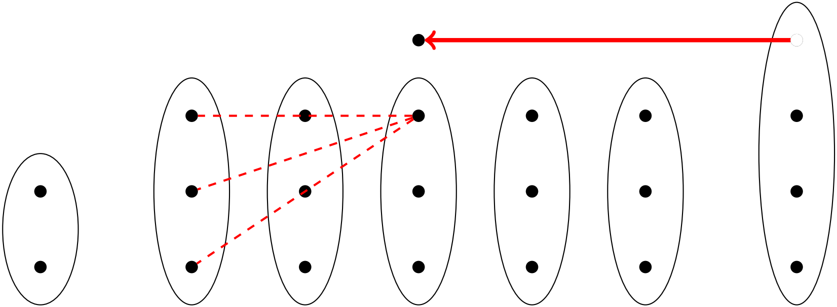

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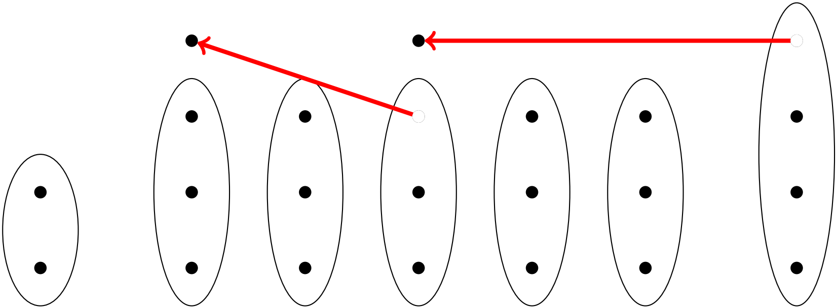

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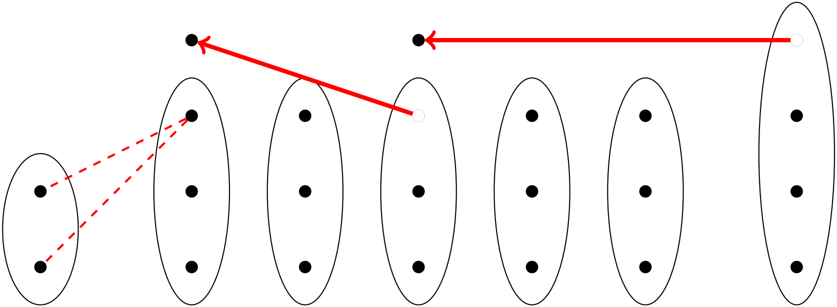

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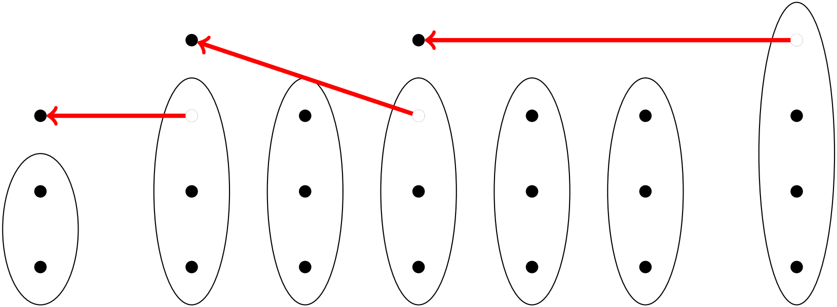

Slides available at:
`http://www.math.ubc.ca/~elyse/Talk_Sendai18.pdf`

Thanks!