Workshop in Mathematical Programming

Model building in Mathematical Programming
Oct. 10 – Nov. 14, 2006 Akiko Yoshise
Materials are available at
http://infoshako.sk.tsukuba.ac.jp/~yoshise/Course/MC/

Schedule：

I. Oct. 10

- What is Mathematical Programming
- How to get XPRESS-MP
- Case study I

II. Oct. 17

- Some Special Types of Mathematical Programming
- Case study I I
- **Assignment #1**
  - Due date Oct. 30

Schedule：

III. Oct. 24:

- Building Integer Programming Model
- Case study III
- **Assignment #2**
  - Due date Nov. 20

IV. Nov. 1:

- Solving Linear Programming Model
- Solving Integer Programming Model

V. Nov. 8:

- Discussions

VI. Nov. 15:

- **Presentation of Assignment#2**

Model 1:

$$\text{Maximize} \sum_{i=1}^{3}\sum_{j=1}^{5} b_{ij}x_{ij} - \sum_{l=1}^{3}\sum_{k>i}\sum_{j=1}^{3}\sum_{i=1}^{5} c_{jl}q_{ik}x_{ij}x_{kl}$$

$$\text{subject to} \sum_{j=1}^{3} x_{ij} = 1 \quad (i = 1,2,3,4,5)$$

$$x_{ij} \in \{0,1\} \quad (i = 1,2,3,4,5,\ j = 1,2,3)$$

Model 1:

Quadratic function

$$\text{Maximize} \sum_{i=1}^{3}\sum_{j=1}^{5} b_{ij}x_{ij} - \sum_{l=1}^{3}\sum_{k>i}^{}\sum_{j=1}^{3}\sum_{i=1}^{5} c_{jl}q_{ik}x_{ij}x_{kl}$$

$$\text{subject to} \sum_{j=1}^{3} x_{ij} = 1 \quad (i = 1,2,3,4,5)$$

$$x_{ij} \in \{0,1\} \quad (i = 1,2,3,4,5,\ j = 1,2,3)$$

New representation (variable):

$$y_{ijkl} = \begin{cases} 1 & \text{if } \; x_{ij} = 1 \text{ and } x_{kl} = 1 \\ 0 & \text{otherwize} \end{cases}$$

$$\text{Note:} \;\; y_{ijkl} \text{ is only defined}$$

$$\text{for } i < k \text{ and } q_{ik} \neq 0$$

Constraints for the new variables

$$y_{ijkl} = \begin{cases} 1 & \text{if } \; x_{ij} = 1 \text{ and } x_{kl} = 1 \\ 0 & \text{otherwize} \end{cases}$$

$$\Updownarrow$$

$$y_{ijkl} \in \{0,1\}$$

$$y_{ijkl} = 1 \rightarrow x_{ij} = 1 \text{ and } x_{kl} = 1$$

$$x_{ij} = 1 \text{ and } x_{kl} = 1 \rightarrow y_{ijkl} = 1$$

Constraints for the new variables

For $y_{ijkl} \in \{0,1\}$ and $x_{ij} \in \{0,1\}$ $(i < k)$

$$y_{ijkl} = 1 \rightarrow x_{ij} = 1 \text{ and } x_{kl} = 1$$

$$\Updownarrow$$

$$\begin{cases} y_{ijkl} - x_{ij} \le 0 \\ y_{ijkl} - x_{kl} \le 0 \end{cases}$$

Constraints for the new variables

For $y_{ijkl} \in \{0,1\}$ and $x_{ij} \in \{0,1\}$ $(i < k)$

$$x_{ij} = 1 \text{ and } x_{kl} = 1 \rightarrow y_{ijkl} = 1$$

$$\Updownarrow$$

$$x_{ij} + x_{kl} - y_{ijkl} \leq 1$$

Generalization:
Logical Condition and 0-1 Variables

H. Paul Williams, *Model Building in Mathematical Programming*

- 0-1 variables are often introduced into an LP (or sometimes an IP) model as decision variables or indicator variables.
- Having introduced such variables it is then possible to represent logical connections between different decisions or states by linear constraints involving 0-1 variables.
- It is at first slight rather surprising that so many different types of logical condition can be imposed in this way.

Propositional Logic

p, q, r : proposition

t : true, f : false

$\vee$: or, $\wedge$: and, $\neg$: not

$p \equiv q$: p is equivallent to q,

$p \rightarrow q$: p implies q

Basic Properties:

1. $p \vee p \equiv p, \quad p \wedge p \equiv p$
2. $(p \vee q) \vee r \equiv p \vee (q \vee r), \quad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
3. $p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$
4. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r),$

   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5. $p \vee t \equiv t, \quad p \wedge t \equiv p, \quad p \vee f \equiv p, \quad p \wedge f \equiv f$
6. $p \vee \neg p \equiv t, \quad p \wedge \neg p \equiv f, \quad \neg t \equiv f, \quad \neg f \equiv t$
7. $\neg\neg p \equiv p$

Basic Properties:

8. $\neg(p \vee q) \equiv \neg p \wedge \neg q, \qquad \neg(p \wedge q) \equiv \neg p \vee \neg q$

9. $p \rightarrow q \equiv \neg p \vee q$

   (hence $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$)

10. $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$

    $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$

    $(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$

    $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

11. $\neg(p \vee q) \equiv \neg p \wedge \neg q, \quad \neg(p \wedge q) \equiv \neg p \vee \neg q$

Propositions and Indicator Variables

P_i : proposition

δ_i : indicator variable, $\delta_i \in \{0,1\}$

$$P_i \equiv [\delta_i = 1]$$

1. $P_1 \vee P_2 \equiv [\delta_1 + \delta_2 \geq 1]$
2. $P_1 \wedge P_2 \equiv [\delta_1 = 1, \delta_2 = 1]$
3. $\neg P_1 \equiv [\delta_1 = 0] \equiv [1 - \delta_1 = 1]$
4. $P_1 \rightarrow P_2 \equiv \neg P_1 \vee P_2 \equiv [(1 - \delta_1) + \delta_2 \geq 1] \equiv [\delta_2 - \delta_1 \geq 0]$

   $P_1 \leftrightarrow P_2 \equiv [\delta_2 - \delta_1 = 0]$

Indicator Variables and Real Variables

$x \geq 0:$ real variable

$M > 0:$ (realistic) upper bound of $x > 0$

$m > 0:$ (realistic) lower bound of $x > 0$

$$x > 0 \rightarrow m \leq x \leq M$$

1. $[\, x > 0 \rightarrow \delta = 1] \equiv [x - M\delta \leq 0]$
2. $[\, \delta = 1 \rightarrow x > 0] \equiv [x - m\delta \geq 0]$

Exercise:

- Represent the following constraint as inequalities using indicator variables

If Product A or Product B is produced then at least one of three products, Product C, Product D or Product E should be produced.

Intoroduce five indicator variables

$\delta_A : [\delta_A = 1] \equiv X_A$ where X_A denotes "Product A is produced"

$\delta_B : [\delta_B = 1] \equiv X_B$ where X_B denotes "Product B is produced"

$\delta_C : [\delta_C = 1] \equiv X_C$ where X_C denotes "Product C is produced"

$\delta_D : [\delta_D = 1] \equiv X_D$ where X_D denotes "Product D is produced"

$\delta_E : [\delta_E = 1] \equiv X_E$ where X_E denotes "Product E is produced"

The constraint can be represented by

$$X_A \vee X_B \to X_C \vee X_D \vee X_E$$

$$X_A \vee X_B \equiv [\delta_A + \delta_B \geq 1]$$
$$X_C \vee X_D \vee X_E \equiv [\delta_C + \delta_D + \delta_E \geq 1]$$

Let δ be another indicator variable which denotes

$$X_A \vee X_B \rightarrow [\delta = 1]$$

To ensure the above, we should impose that

$$[\delta_A + \delta_B \geq 1] \rightarrow [\delta = 1]$$

Since

$$\delta_A + \delta_B > 0 \leftrightarrow \delta_A + \delta_B \geq 1$$

$$\delta_A + \delta_B \leq M = 2,$$

$$[\delta_A + \delta_B \geq 1] \rightarrow [\delta = 1]$$

can be represented by

$$\delta_A + \delta_B - M\delta \leq 0$$

i.e.,

$$\delta_A + \delta_B - 2\delta \leq 0$$

Next let us represent

$$[\delta = 1] \rightarrow X_C \vee X_D \vee X_E$$

i.e.,

$$[\delta = 1] \rightarrow [\delta_C + \delta_D + \delta_E \geq 1]$$

Since

$$\delta_C + \delta_D + \delta_E \geq m = 1 \leftrightarrow \delta_C + \delta_D + \delta_E > 0,$$

$$[\delta = 1] \rightarrow [\delta_C + \delta_D + \delta_E \geq 1]$$

can be represented by

$$(\delta_C + \delta_D + \delta_E) - m\delta \geq 0$$

i.e.,

$$(\delta_C + \delta_D + \delta_E) - \delta \geq 0$$

Summarizing, we have

$$X_A \vee X_B \rightarrow [\delta = 1]$$

is represented by

$$(\delta_A + \delta_B) - 2\delta \geq 0$$

and

$$[\delta = 1] \rightarrow X_C \vee X_D \vee X_E$$

is represented by

$$[\delta = 1] \rightarrow [\delta_C + \delta_D + \delta_E \geq 1]$$

$$(\delta_C + \delta_D + \delta_E) - m\delta \geq 0$$

i.e.,

$$(\delta_C + \delta_D + \delta_E) - \delta \geq 0$$

If Product A or Product B is produced then at least one of three products, Product C, Product D or Product E should be produced.

Thus the above constraint is given by

$$(\delta_A + \delta_B) - 2\delta \leq 0, \quad (\delta_C + \delta_D + \delta_E) - \delta \geq 0$$

where

$$\delta_A \in \{0,1\}, \delta_B \in \{0,1\}, \delta_C \in \{0,1\}, \delta_D \in \{0,1\}, \delta_E \in \{0,1\}$$

Linearization of Quadratic (Bilinear) Form I

Represent the quadratic term $\delta_1 \cdot \delta_2$ by δ

i.e.,

$$\delta = \begin{cases} 1 & \delta_1 = 1 \text{ and } \delta_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

To do this, we should impose that

$$[\delta = 1] \rightarrow [\delta_1 = 1 \text{ and } \delta_2 = 1]$$

$$[\delta_1 = 1 \text{ and } \delta_2 = 1] \rightarrow [\delta = 1]$$

Linearization of Quadratic (Bilinear) Form I

$$\left[\delta_1 = 1 \text{ and } \delta_2 = 1\right] \rightarrow \left[\delta = 1\right]$$

$$\Updownarrow$$

$$\left[\delta_1 = 1 \rightarrow \delta = 1\right] \vee \left[\delta_2 = 1 \rightarrow \delta = 1\right]$$

$$\Updownarrow$$

$$\left[\delta - \delta_1 \geq 0\right] \vee \left[\delta - \delta_2 \geq 0\right]$$

$$\Updownarrow$$

$$\left[\delta - \delta_1 + 1 \geq 1\right] \vee \left[\delta - \delta_2 + 1 \geq 1\right]$$

$$\Updownarrow$$

$$\left(\delta - \delta_1 + 1\right) + \left(\delta - \delta_2 + 1\right) \geq 1$$

$$\text{(since } \left(\delta - \delta_1 + 1\right) \geq 0,\ \left(\delta - \delta_2 + 1\right) \geq 0\text{)}$$

Linearization of Quadratic (Bilinear) Form I

$$[\delta = 1] \rightarrow [\delta_1 = 1 \,\text{and}\, \delta_2 = 1]$$

$$\Updownarrow$$

$$[\delta = 1 \rightarrow \delta_1 = 1] \wedge [\delta = 1 \rightarrow \delta_2 = 1]$$

$$\Updownarrow$$

$$\delta - \delta_1 \geq 0, \quad \delta - \delta_2 \geq 0$$

Linearization of Quadratic (Bilinear) Form I

Thus, $\delta_1 \cdot \delta_2$ is represented by δ adding the inequalities

$$\begin{cases} \delta_1 - \delta \geq 0, \quad \delta_2 - \delta \geq 0 \\ (\delta - \delta_1 + 1) + (\delta - \delta_2 + 1) \geq 1 \end{cases}$$

i.e.,

$$\begin{cases} \delta_1 - \delta \geq 0, \quad \delta_2 - \delta \geq 0 \\ \delta_1 + \delta_2 - 2\delta \leq 1 \end{cases}$$

Considering $\delta, \delta_1, \delta_2 \in \{0,1\}$, the above system is equivatent to

$$\begin{cases} \delta_1 - \delta \geq 0, \quad \delta_2 - \delta \geq 0 \\ \delta_1 + \delta_2 - \delta \leq 1 \end{cases}$$

The above inequalities has been used in Assignment #2

Linearization of Quadratic (Bilinear) Form II

Represent the quadratic term $x \cdot \delta$ by y

where x and y are real variables

i.e.,

$$\begin{cases} \delta = 0 \rightarrow y = 0 \\ \delta = 1 \rightarrow y = x \end{cases}$$

By a similar discussion, we can see that the above is equivalent to

$$\begin{cases} y - M\delta \leq 0 \\ -x + y \leq 0 \\ x - y + M\delta \leq M \end{cases}$$