

## LAGRANGIAN RELAXATION AND PEGGING TEST FOR LINEAR ORDERING PROBLEMS

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*Abstract* We develop an algorithm for the linear ordering problem, which has a large number of applications such as triangulation of input-output matrices, minimizing total weighted completion time in one-machine scheduling, and aggregation of individual preferences. The algorithm is based on the Lagrangian relaxation of a binary integer linear programming formulation of the problem. Since the number of the constraints is proportional to the third power of the number of items and grows rapidly, we propose a modified subgradient method that temporarily ignores a large part of the constraints and gradually adds constraints whose Lagrangian multipliers are likely to be positive at an optimal multiplier vector. We also propose an improvement on the ordinary pegging test by using the problem structure.

**Keywords:** Combinatorial optimization, linear ordering problem, Lagrangian relaxation, Lagrangian dual, subgradient method, pegging test

### 1. Introduction

The problem we consider in this paper is to find a linear ordering of  $n$  items when their pairwise comparison data are given. The data are given by an  $n \times n$  matrix  $C := [c_{ij}]_{i,j=1,\dots,n}$  such that its  $(i, j)$ th element carries the amount of profit made when item  $i$  is ranked prior to item  $j$ . Choosing an appropriate matrix  $C$  lets the problem embrace the ranking aggregation problem, which is known as Kemeny's problem, the minimum violations ranking problem, and Slater's problem. See the survey paper by Charon and Hudry [8] and the book by Reinelt [25]. The problem is formulated as a linear integer programming problem. The polytope being the convex hull of binary vectors each corresponding to a linear ordering was named the *linear ordering polytope* and investigated by Grötschel et al. [17]. They introduced some facet-defining valid inequalities of the polytope, and proposed a linear-programming-relaxation-based algorithm for the problem in [16]. For subsequent researches on the linear ordering polytope, see [4, 10, 20, 23]. Their approach was further extended by Mitchell and Borchers [21, 22], who proposed a cutting plane algorithm based on a primal-dual interior point method, and solved problems with as many as 250 items. Since the problem is *NP*-hard, see e.g., Section 2 of [8], there have been proposed several heuristic methods, e.g., Lagrangian heuristic method in [3], scatter search method in [5], linear ordering construction heuristics in [9], Goddard's method in [14], variable neighborhood local search method in [15]. Charon and Hudry [7] made an experiment of a branch-and-bound method with Lagrangian relaxation and some heuristics.

The binary integer programming formulation of the linear ordering problem has an  $O(n^3)$  of inequality constraints. This feature makes the problem hard to solve. We propose a Lagrangian relaxation algorithm that considers a small fraction of the inequality constraints, and a pegging test that takes advantage of the problem structure. The algorithm is a

combination of well-known and widely-used techniques of optimization, however, it would be well worth reporting how they function together and how they make the algorithm efficient in an integrated manner.

This paper is organized as follows. We define the linear ordering problem in Section 2 and introduce its formulations in Section 3. In Section 4, we propose relaxation techniques, that is, the constraints relaxation and the Lagrangian relaxation. Some basic results concerning these relaxations are given in Section 5. In Section 6, we propose an improvement on the ordinary pegging test. In Section 7, we explain the subgradient method to solve the Lagrangian dual problem. In Sections 8 and 9, we describe some heuristics for good incumbents, and a technique to check the feasibility of the relaxed solutions. Explaining our algorithm in Section 10, we report the computational results and concluding comments in Section 11.

Throughout this paper we will use the following symbols:

$$\begin{aligned} N &:= \{1, 2, \dots, n\}, & N^2 &:= \{(i, j) \mid i, j \in N\}, \\ N_{\neq}^2 &:= \{(i, j) \mid i, j \in N, i \neq j\}, & N_{\neq}^3 &:= \{(i, j, k) \mid i, j, k \in N, i \neq j, j \neq k, k \neq i\}, \\ N_{<}^2 &:= \{(i, j) \mid i, j \in N, i < j\}, & N_{<}^3 &:= \{(i, j, k) \mid i, j, k \in N, i < j < k\}. \end{aligned}$$

## 2. Linear Ordering Problem

### 2.1. Ranking aggregation

Suppose we have several different rankings of  $n$  items, and want to aggregate them to a single ranking or linear ordering. If each ranking comes from the ratings of items, summing up the ratings that item  $i$  receives to its overall rating and sorting them for a final linear ordering is a possible and widely used method. Our starting point is not the ratings of items but their rankings. One of the well-known method for aggregation of rankings is the Borda method which was first proposed in the 18th century. As Kemeny proposed in [18], a natural solution would be a linear ordering that is “close” to all given rankings. Let  $\sigma_1, \dots, \sigma_\kappa, \dots, \sigma_K$  be given rankings of items  $N$ . Let for  $\alpha \in [0, 1]$

$$c_{ij}^{(1)} := \alpha |\{ \kappa \mid \sigma_\kappa(i) < \sigma_\kappa(j) \}| - (1 - \alpha) |\{ \kappa \mid \sigma_\kappa(i) > \sigma_\kappa(j) \}|, \quad (2.1)$$

$$c_{ij}^{(2)} := \sum_{\kappa=1}^K \alpha [\sigma_\kappa(j) - \sigma_\kappa(i)]^+ - (1 - \alpha) [\sigma_\kappa(i) - \sigma_\kappa(j)]^+, \quad (2.2)$$

where  $|\cdot|$  denotes the cardinality of the corresponding set, and  $[t]^+ = \max\{t, 0\}$ . The coefficient  $c_{ij}^{(1)}$  is a weighted difference of the number of rankings that put  $i$  above  $j$  and those that put  $i$  below  $j$ , and  $c_{ij}^{(2)}$  is the weighted sum of differences between the rankings of  $i$  and  $j$ . The parameter  $\alpha$  should be determined according to which of “aye” and “nay” is considered more important. When  $\alpha = 1/2$ ,  $c_{ij}^{(2)} = (1/2) \sum_{\kappa=1}^K (\sigma_\kappa(j) - \sigma_\kappa(i))$ .

Let  $\pi$  denote an aggregated linear ordering. The values  $c_{ij}^{(\nu)}$  for  $\nu = 1, 2$  shows how the linear ordering  $\pi$  and given rankings agree about the order of  $i$  and  $j$  when  $\pi(i) < \pi(j)$ . Hence the overall degree of agreement is given by

$$\sum_{(i,j):\pi(i)<\pi(j)} c_{ij}^{(\nu)}.$$

A linear ordering that maximizes this function should be accepted as the “closest” aggregated linear ordering, hence our problem is formulated as

$$(LOP) \quad \left\{ \begin{array}{l} \text{maximize} \quad \sum_{(i,j):\pi(i)<\pi(j)} c_{ij}^{(\nu)} \\ \text{subject to} \quad \pi \text{ is a linear ordering,} \end{array} \right.$$

which we will refer to as the *Linear Ordering Problem* (*LOP* for short).

## 2.2. Minimum violations ranking

Ali et al. [1] and Pedings et al. [24] proposed the *minimum violations ranking*. Suppose we are given a matrix  $D := [d_{ij}]_{(i,j) \in N^2}$  such that  $d_{ij}$  is the points by which team  $i$  beats team  $j$  in their matchup, where we take the convention that  $d_{ii} = 0$ . They call this matrix a *point differential matrix* and define a *hillside form*.

**Definition 2.1.** *A matrix  $D$  is in hillside form if*

$$\begin{aligned} d_{ki} &\leq d_{kj} && (\text{ascending order across rows}) \\ d_{ik} &\geq d_{jk} && (\text{descending order down columns}) \end{aligned}$$

for all  $i, j, k \in N_{\neq}^3$  such that  $i < j$ .

They proposed to find such a hidden hillside form by a simultaneous reordering of rows and columns of the given point differential matrix  $D$ , and showed that the problem is formulated\* as (*LOP*) in the previous section with the following objective function coefficient for  $\alpha = 1/2$

$$c_{ij}^{(3)} := \alpha |\{k \in N \setminus \{i, j\} \mid d_{ki} \leq d_{kj}\}| + (1 - \alpha) |\{k \in N \setminus \{i, j\} \mid d_{ik} \geq d_{jk}\}|. \quad (2.3)$$

Another choice of the objective function coefficient would be

$$c_{ij}^{(4)} := \sum_{k \in N \setminus \{i, j\}} \alpha (d_{kj} - d_{ki}) + (1 - \alpha) (d_{ik} - d_{jk}). \quad (2.4)$$

## 3. Formulation of Linear Ordering Problem

### 3.1. Quadratic assignment formulation

Let  $y_{ki}$  be the binary variable such that

$$y_{ki} = \begin{cases} 1 & \text{if } k\text{th ranking is given to item } i \\ 0 & \text{otherwise.} \end{cases}$$

Then these variables satisfy

$$\sum_{k \in N} y_{ki} = 1 \quad \text{for all } i \in N, \quad (3.1)$$

$$\sum_{i \in N} y_{ki} = 1 \quad \text{for all } k \in N. \quad (3.2)$$

The cost concerning the ordered pair  $(i, j)$  is given by

$$c_{ij} \sum_{k=1}^{n-1} y_{ki} \left( \sum_{l=k+1}^n y_{lj} \right),$$

\*To be very precise, Pedings et al. [24] formulated the problem as a minimization of violations of the hillside structure.

and the total agreement is

$$\sum_{(i,j) \in N_{\neq}^2} c_{ij} \sum_{k=1}^{n-1} y_{ki} \left( \sum_{l=k+1}^n y_{lj} \right).$$

The quadratic assignment formulation is to maximize the total agreement under the assignment constraints (3.1) and (3.2) together with the binary variable constraints. This is a well-known  $NP$ -hard problem and already a challenging problem when  $n = 25$ . See Çela [6].

### 3.2. Integer linear programming formulation

For a given linear ordering  $\pi$  let binary variables  $x_{ij}$  for  $(i, j) \in N_{\neq}^2$  be defined as

$$x_{ij} = \begin{cases} 1 & \text{if } \pi(i) < \pi(j) \\ 0 & \text{otherwise,} \end{cases}$$

then the linear ordering problem is formulated as

$$(LOP) \quad \left\{ \begin{array}{ll} \text{maximize} & \sum_{(i,j) \in N_{\neq}^2} c_{ij} x_{ij} \\ \text{subject to} & x_{ij} \in \{0, 1\} \quad \text{for all } (i, j) \in N_{\neq}^2 \quad (\text{binary}) \\ & x_{ij} + x_{ji} = 1 \quad \text{for all } (i, j) \in N_{\neq}^2 \quad (\text{antisymmetry}) \\ & x_{ij} + x_{jk} + x_{ki} \leq 2 \quad \text{for all } (i, j, k) \in N_{\neq}^3 \quad (\text{transitivity}). \end{array} \right.$$

The point is that the problem has  $n(n-1)$  binary variables,  $n(n-1)/2$  equality constraints and  $n(n-1)(n-2)/3$  inequality constraints, all of which grow very rapidly as  $n$  grows.

### 3.3. Variable reduction

Substituting  $1 - x_{ij}$  for  $x_{ji}$  for all  $i, j \in N$  with  $i < j$  halves the decision variables and yields the following equivalent problem ( $P$ ):

$$(P) \quad \left\{ \begin{array}{ll} \text{maximize} & \sum_{(i,j) \in N_{<}^2} \bar{c}_{ij} x_{ij} + \sum_{(i,j) \in N_{<}^2} c_{ji} \\ \text{subject to} & x_{ij} \in \{0, 1\} \quad \text{for all } (i, j) \in N_{<}^2 \\ & x_{ij} + x_{jk} - x_{ik} \leq 1 \quad \text{for all } (i, j, k) \in N_{<}^3 \quad (\text{type 1}) \\ & -x_{ij} - x_{jk} + x_{ik} \leq 0 \quad \text{for all } (i, j, k) \in N_{<}^3 \quad (\text{type 2}), \end{array} \right.$$

where

$$\bar{c}_{ij} := c_{ij} - c_{ji}.$$

We will call the inequality constraint of the first half the *transitivity constraint of type 1*, and one of the latter half *type 2*, and we will denote the optimal objective function value of ( $P$ ) by  $\omega(P)$ .

## 4. Relaxation

### 4.1. Relaxation of inequality constraints

A possible relaxation is to temporarily discard some of the inequality constraints of ( $P$ ). Namely let  $U$  and  $V$  be subsets of  $N_{<}^3$  and solve

$$(P(U, V)) \left\{ \begin{array}{l} \text{maximize} \quad \sum_{(i,j) \in N_{<}^2} \bar{c}_{ij} x_{ij} \\ \text{subject to} \quad x_{ij} \in \{0, 1\} \quad \text{for all } (i, j) \in N_{<}^2 \\ \quad \quad \quad x_{ij} + x_{jk} - x_{ik} \leq 1 \quad \text{for all } (i, j, k) \in U \\ \quad \quad \quad -x_{ij} - x_{jk} + x_{ik} \leq 0 \quad \text{for all } (i, j, k) \in V. \end{array} \right.$$

Clearly if the optimal solution of  $(P(U, V))$  satisfies all the transitivity constraints, it is an optimal solution of problem  $(P)$ .

### 4.2. Lagrangian relaxation

Problem  $(P(U, V))$  is still a difficult problem to solve unless no favorable structure can be assumed on  $U$  and  $V$ . One of the common tricks to deal with the problem would be the Lagrangian relaxation. Namely, introducing a nonnegative multiplier  $u_{ijk}$  for each constraint of type 1 and also a nonnegative multiplier  $v_{ijk}$  for each constraint of type 2, we consider the following integer linear programming with only a simple binary variable constraint:

$$(LR(\mathbf{u}, \mathbf{v})) \left\{ \begin{array}{l} \text{maximize} \quad \sum_{(i,j) \in N_{<}^2} \bar{c}_{ij} x_{ij} + \sum_{(i,j,k) \in U} u_{ijk}(1 - x_{ij} - x_{jk} + x_{ik}) \\ \quad \quad \quad + \sum_{(i,j,k) \in V} v_{ijk}(0 + x_{ij} + x_{jk} - x_{ik}) \\ \text{subject to} \quad x_{ij} \in \{0, 1\} \quad \text{for all } (i, j) \in N_{<}^2. \end{array} \right.$$

Omitting  $U$  and  $V$ , we denote this problem simply by  $(LR(\mathbf{u}, \mathbf{v}))$ , where  $\mathbf{u}$  and  $\mathbf{v}$  denote multiplier vectors  $(u_{ijk})_{(i,j,k) \in U}$  and  $(v_{ijk})_{(i,j,k) \in V}$ , respectively. Let  $r(\mathbf{u}, \mathbf{v})_{ij}$  denote the coefficient of variable  $x_{ij}$  in the objective function. It is written as

$$\begin{aligned} r(\mathbf{u}, \mathbf{v})_{ij} = & \bar{c}_{ij} - \sum_{k:(i,j,k) \in U} u_{ijk} - \sum_{k:(k,i,j) \in U} u_{kij} + \sum_{k:(i,k,j) \in U} u_{ikj} \\ & + \sum_{k:(i,j,k) \in V} v_{ijk} + \sum_{k:(k,i,j) \in V} v_{kij} - \sum_{k:(i,k,j) \in V} v_{ikj}. \end{aligned} \tag{4.1}$$

Due to the simple constraint, an optimal solution  $\mathbf{x}(\mathbf{u}, \mathbf{v}) = (x(\mathbf{u}, \mathbf{v})_{ij})_{(i,j) \in N_{<}^2}$  of problem  $(LR(\mathbf{u}, \mathbf{v}))$  can be obtained by

$$x(\mathbf{u}, \mathbf{v})_{ij} = \begin{cases} 1 & \text{if } r(\mathbf{u}, \mathbf{v})_{ij} > 0 \\ 0 & \text{if } r(\mathbf{u}, \mathbf{v})_{ij} \leq 0. \end{cases} \tag{4.2}$$

Furthermore, the optimal objective function value, which we will denote by  $\omega(LR(\mathbf{u}, \mathbf{v}))$ , provides an upper bound of the optimal objective function value  $\omega(P)$  of problem  $(P)$ .

### 5. Optimality and Duality Gap

The following theorem is well known, see e.g., Geoffrion [13].

**Theorem 5.1.** *Let  $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) := ((\bar{u}_{ijk})_{(i,j,k) \in N_{<}^3}, (\bar{v}_{ijk})_{(i,j,k) \in N_{<}^3})$  be a Lagrangian multiplier vector corresponding to all the transitivity constraints, and let  $\mathbf{x}$  be an optimal solution of the Lagrangian relaxation problem of  $(P)$  with  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ . If  $\mathbf{x}$  is feasible to problem  $(P)$  and satisfies the complementarity condition*

$$\begin{aligned} \bar{u}_{ijk}(1 - x_{ij} - x_{jk} + x_{ik}) &= 0 \quad \text{for all } (i, j, k) \in N_{<}^3, \\ \bar{v}_{ijk}(0 + x_{ij} + x_{jk} - x_{ik}) &= 0 \quad \text{for all } (i, j, k) \in N_{<}^3, \end{aligned}$$

*then it is an optimal solution of  $(P)$ .*

**Definition 5.1.** We say that  $\mathbf{x}$  satisfies the restricted complementarity condition with  $(\mathbf{u}, \mathbf{v})$  when

$$\begin{aligned} u_{ijk}(1 - x_{ij} - x_{jk} + x_{ik}) &= 0 \quad \text{for all } (i, j, k) \in U, \\ v_{ijk}(0 + x_{ij} + x_{jk} - x_{ik}) &= 0 \quad \text{for all } (i, j, k) \in V. \end{aligned}$$

We readily see the following corollary.

**Corollary 5.1.** If an optimal solution  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  of  $(LR(\mathbf{u}, \mathbf{v}))$  is feasible to problem  $(P)$  and satisfies the restricted complementarity condition with  $(\mathbf{u}, \mathbf{v})$ , then it is an optimal solution of  $(P)$ .

*Proof.* We readily see that the Lagrangian relaxation problem  $(LR(\mathbf{u}, \mathbf{v}))$  is an ordinary Lagrangian relaxation problem of  $(P)$  with multipliers  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$  such that

$$\bar{u}_{ijk} = \begin{cases} u_{ijk} & \text{for } (i, j, k) \in U \\ 0 & \text{for } (i, j, k) \in N_{<}^3 \setminus U, \end{cases} \quad \bar{v}_{ijk} = \begin{cases} v_{ijk} & \text{for } (i, j, k) \in V \\ 0 & \text{for } (i, j, k) \in N_{<}^3 \setminus V. \end{cases}$$

When  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  meets the restricted complementarity condition with  $(\mathbf{u}, \mathbf{v})$  in Definition 5.1, it also satisfies the complementarity condition for all constraints with  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ . This together with the feasibility of  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  yields the desired result.  $\square$

A feasible solution of problem  $(P)$  that has the largest objective function value among the feasible solutions found thus far is called an *incumbent solution*, and its objective function value is called an *incumbent value*. The difference of  $\omega(LR(\mathbf{u}, \mathbf{v}))$  and the incumbent value is called the *duality gap*.

## 6. Pegging Test

### 6.1. Ordinary pegging test

By the information obtained from the optimal solution  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  of the Lagrangian relaxation problem  $(LR(\mathbf{u}, \mathbf{v}))$  we can see which variable takes one and which takes zero at the optimal solution of problem  $(P)$ . Let us choose  $(s, t) \in N_{<}^2$  and suppose that problem  $(P)$  has an optimal solution with  $x_{st} = \xi$  for some  $\xi \in \{0, 1\}$ . Then problem  $(P)$  with an additional constraint  $x_{st} = \xi$  is equivalent to problem  $(P)$  in the sense that optimal values of the two problems coincide. Suppose further we have an incumbent value  $\omega_{\text{low}}$ . Then clearly

$$\omega(P|x_{st} = \xi) = \omega(P) \geq \omega_{\text{low}}.$$

Since  $(P(U, V))$  is a relaxation of problem  $(P)$ , and it is further relaxed to  $(LR(\mathbf{u}, \mathbf{v}))$ , we obtain

$$\omega(LR(\mathbf{u}, \mathbf{v})|x_{st} = \xi) \geq \omega(P(U, V)|x_{st} = \xi) \geq \omega(P|x_{st} = \xi),$$

hence

$$\omega(LR(\mathbf{u}, \mathbf{v})|x_{st} = \xi) \geq \omega_{\text{low}}.$$

**Lemma 6.1.** Let  $\xi$  be either zero or one. If  $\omega(LR(\mathbf{u}, \mathbf{v})|x_{st} = \xi) < \omega_{\text{low}}$ , then  $x_{st} = 1 - \xi$  for any optimal solution of problem  $(P)$ .

*Proof.* Straightforward from the above discussion.  $\square$







where

$$\begin{aligned}
A' &= (((S_1 \times T_1) \cap N_{\leq}^2 \cup P_1) \cap \{(i, j) \mid x(\mathbf{u}, \mathbf{v})_{ij} = 0\}) \\
&\quad \cup (((T_1 \times S_1) \cap N_{<}^2 \cup P_0) \cap \{(i, j) \mid x(\mathbf{u}, \mathbf{v})_{ij} = 1\}) \\
&\quad \text{when } x(\mathbf{u}, \mathbf{v})_{st} = 0, \text{ and} \\
A' &= (((T_0 \times S_0) \cap N_{\leq}^2 \cup P_1) \cap \{(i, j) \mid x(\mathbf{u}, \mathbf{v})_{ij} = 0\}) \\
&\quad \cup (((S_0 \times T_0) \cap N_{<}^2 \cup P_0) \cap \{(i, j) \mid x(\mathbf{u}, \mathbf{v})_{ij} = 1\}) \\
&\quad \text{when } x(\mathbf{u}, \mathbf{v})_{st} = 1.
\end{aligned} \tag{6.6}$$

The first subset  $A'$  corresponds to the variables that should be one but takes zero at  $\mathbf{x}(\mathbf{u}, \mathbf{v})$ , and the second subset  $A'$  to those that should be zero but takes one at  $\mathbf{x}(\mathbf{u}, \mathbf{v})$ .

### 6.3. Transitive closure

As was seen in the previous subsection, it would be useful and save computation time to peg as many variables as possible. This can be done by computing the transitive closure of the directed graph consisting of node set  $N$  and arc set  $A(P_0, P_1)$  of (6.3). The *transitive closure* of  $(N, A(P_0, P_1))$  is a directed graph  $(N, \bar{A})$  such that  $(i, j) \in \bar{A}$  if and only if there is a directed path from  $i$  to  $j$  in  $A(P_0, P_1)$ . Once we have made the transitive closure, the four sets in Definition 6.2 are readily obtained by

$$\begin{aligned}
S_1 &:= \{s\} \cup \{i \in N \mid (i, s) \in \bar{A}\}, & T_1 &:= \{t\} \cup \{j \in N \mid (t, j) \in \bar{A}\}, \\
S_0 &:= \{s\} \cup \{i \in N \mid (s, i) \in \bar{A}\}, & T_0 &:= \{t\} \cup \{j \in N \mid (j, t) \in \bar{A}\}.
\end{aligned}$$

We apply the well-known algorithm for computing the transitive closure proposed by Warshall in 1962, see e.g., Section 19.3 of Sedgewick [26].

## 7. Subgradient Method for Lagrangian Dual Problem

For the sake of simplicity we abbreviate  $\omega(LR(\mathbf{u}, \mathbf{v}, P_0, P_1))$  to  $\omega(\mathbf{u}, \mathbf{v})$  in this section. The *Lagrangian dual problem*, denoted by  $(LD)$ , is a problem for finding the smallest upper bound of  $\omega(P)$ . Namely, it searches for a nonnegative multiplier vector  $(\mathbf{u}, \mathbf{v})$  that minimizes  $\omega(\mathbf{u}, \mathbf{v})$ :

$$(LD) \quad \left\{ \begin{array}{l} \text{minimize } \omega(\mathbf{u}, \mathbf{v}) \\ \text{subject to } \mathbf{u}, \mathbf{v} \geq \mathbf{0}. \end{array} \right.$$

The function  $\omega(\mathbf{u}, \mathbf{v})$  is piecewise linear convex and not differentiable on the intersection of pieces. One of the most widely used methods for this problem is the *subgradient method*. See for example Fisher [11].

**Definition 7.1.**  $(\mathbf{g}^u, \mathbf{g}^v)$  is said to be a subgradient of  $\omega$  at  $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \geq \mathbf{0}$  when

$$\omega(\bar{\mathbf{u}}, \bar{\mathbf{v}}) + \langle \mathbf{g}^u, \mathbf{u} - \bar{\mathbf{u}} \rangle + \langle \mathbf{g}^v, \mathbf{v} - \bar{\mathbf{v}} \rangle \leq \omega(\mathbf{u}, \mathbf{v})$$

holds for any  $(\mathbf{u}, \mathbf{v}) \geq \mathbf{0}$ , where  $\langle \cdot, \cdot \rangle$  means the inner product.

The following lemma is well known.

**Lemma 7.1.** Let  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  denote an optimal solution of the Lagrangian relaxation problem  $(LR(\mathbf{u}, \mathbf{v}, P_0, P_1))$ . Then  $(\mathbf{g}^u, \mathbf{g}^v)$  such that

$$\begin{aligned}
g_{ijk}^u &:= 1 - x(\mathbf{u}, \mathbf{v})_{ij} - x(\mathbf{u}, \mathbf{v})_{jk} + x(\mathbf{u}, \mathbf{v})_{ik} \quad \text{for } (i, j, k) \in U, \\
g_{ijk}^v &:= 0 + x(\mathbf{u}, \mathbf{v})_{ij} + x(\mathbf{u}, \mathbf{v})_{jk} - x(\mathbf{u}, \mathbf{v})_{ik} \quad \text{for } (i, j, k) \in V
\end{aligned}$$

is a subgradient of  $\omega$  at  $(\mathbf{u}, \mathbf{v})$ .

We use the following rule to update the multiplier vector  $(\mathbf{u}, \mathbf{v})$  to the next iterate  $(\mathbf{u}^+, \mathbf{v}^+)$ .

$$u_{ijk}^+ := \max \left\{ 0, u_{ijk} - \mu \frac{\omega(\mathbf{u}, \mathbf{v}) - \omega_{\text{low}}}{\|(\mathbf{g}^u, \mathbf{g}^v)\|^2} g_{ijk}^u \right\} \quad \text{for } (i, j, k) \in U, \quad (7.1)$$

$$v_{ijk}^+ := \max \left\{ 0, v_{ijk} - \mu \frac{\omega(\mathbf{u}, \mathbf{v}) - \omega_{\text{low}}}{\|(\mathbf{g}^u, \mathbf{g}^v)\|^2} g_{ijk}^v \right\} \quad \text{for } (i, j, k) \in V, \quad (7.2)$$

where  $\mu$  is the step size control parameter initially set to 2 and  $\|\cdot\|$  is the Euclidean norm. It is known that if  $\omega_{\text{low}}$  in the update formulas is replaced by the optimal value  $\omega(P)$ , the sequence generated will converge to an optimal solution of the Lagrangian dual problem (LD), see e.g., Geoffrion [13] and Larsson et al. [19]. However the value  $\omega(\mathbf{u}, \mathbf{v})$  does not necessarily decrease when the multiplier vector is updated. We count the number of consecutive failures to decrease the value, and when it amounts to 5, we halve the step size control parameter  $\mu$ . When  $\mu$  falls below 0.005, we increment the constraint index sets  $U$  and  $V$  and reset  $\mu$  to its initial value 2. See the steps 6 and 8 of the algorithm in Section 10 for the details.

## 8. Heuristics for Good Incumbents

For a given  $n \times n$  binary matrix  $X := [x_{ij}]_{(i,j) \in N^2}$  let

$$w_i := \sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji},$$

row-column difference, for each  $i \in N$ . Ali et al. [1] showed the following lemma and used it in their linear ordering problem formulation.

**Lemma 8.1.** *Let  $X := [x_{ij}]_{(i,j) \in N^2}$  be an  $n \times n$  binary matrix with zero diagonal elements. Then  $\{w_1, w_2, \dots, w_n\}$  ranges over  $\{n-1, n-3, \dots, -(n-3), -(n-1)\}$  if and only if  $X$  represents a linear ordering.*

When the matrix  $X$  satisfies the antisymmetry, i.e.,  $x_{ji} = 1 - x_{ij}$ , the row-column difference  $w_i$  could be replaced by the row sum  $r_i := \sum_{j \in N} x_{ij}$ , which is sometimes called the *Copeland score*.

**Corollary 8.1.** *Suppose that  $n \times n$  binary matrix  $X := [x_{ij}]_{(i,j) \in N^2}$  satisfies the antisymmetry and has zero diagonal elements. Then the row sum  $r_i := \sum_{j \in N} x_{ij}$  ranges over  $\{n-1, n-2, \dots, 1, 0\}$  if and only if  $X$  represents a linear ordering.*

*Proof.* Since  $x_{ji} = 1 - x_{ij}$ , we see that  $w_i = 2r_i - n + 1$  holds. Substituting this for  $w_i$  in Lemma 8.1 completes the proof.  $\square$

For a solution  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  of  $(LR(\mathbf{u}, \mathbf{v}, P_0, P_1))$ , let  $\bar{X} := [\bar{x}_{ij}]_{(i,j) \in N^2}$  be the matrix such that

$$\bar{x}_{ij} := \begin{cases} x(\mathbf{u}, \mathbf{v})_{ij} & \text{for } i < j \\ 0 & \text{for } i = j \\ 1 - x(\mathbf{u}, \mathbf{v})_{ji} & \text{for } i > j. \end{cases} \quad (8.1)$$

The row sum  $r_i := \sum_{j \in N} \bar{x}_{ij}$  of  $\bar{X}$  is given by

$$r_i = \sum_{j:j>i} x(\mathbf{u}, \mathbf{v})_{ij} - \sum_{j:j<i} x(\mathbf{u}, \mathbf{v})_{ji} + i - 1. \quad (8.2)$$

It is reasonable to think that the item with a larger value of  $r_i$  should be ranked higher. However,  $r_i$  may not accurately reflect the information about which variables have been pegged, and it could happen that  $r_i < r_j$  even when  $x_{ij}$  has been pegged at one or  $x_{ji}$  at zero. The descending ordering of the values of  $r_i$  may violate the order that we have known is met by every optimal solution. On the other hand, the transitive closure  $\bar{B}$  of the arc set

$$B := \{ (i, j) \mid (j, i) \in P_0 \text{ or } (i, j) \in P_1 \}.$$

precisely reflects the information of the pegged variables. Namely, the row-column difference  $\bar{w}_i$  of the adjacency matrix of  $\bar{B}$  satisfies

$$\bar{w}_i > \bar{w}_j \text{ if } (j, i) \in P_0 \text{ or } (i, j) \in P_1.$$

However, lots of row-column differences may fall into a tie before the pegged variables build up.

For a pegged variable  $x_{ij}$ , let  $\delta_{ij}$  be the difference of the positions of item  $i$  and  $j$  in the optimal linear ordering  $\hat{\pi}$ , i.e.,  $\delta_{ij} := |\hat{\pi}^{-1}(i) - \hat{\pi}^{-1}(j)|$ . Clearly for  $k = 1, 2, \dots, n - 1$ , there are  $(n - k)$  pairs such that  $\delta_{ij} = k$ . We consider the variables pegged by the first application of the pegging test. Figure 1 is a scatter plot of

$$\frac{|\{ (i, j) \in N_{<}^2 \mid \delta_{ij} = k, x_{ij} \text{ is pegged by the first application of the pegging test } \}|}{n - k}$$

versus  $k$  for  $k = 1, 2, \dots, n - 1$  for the data `DsumC` of  $n = 347$  items. We observed that all pairs with  $\delta_{ij} \geq 120$  and 99% of pairs with  $\delta_{ij} \geq 42$  were pegged by only the first application of the pegging test. This confirms that  $\bar{w}_i$  is credible as the sorting key.

Then we propose to sort the items according to the two keys:  $\bar{w}_i$  as the primary key and  $r_i$  as the secondary key, which will serve as a tie breaker. The sorting can be done by sorting firstly according to the secondary key, and then according to the primary key by a *stable* sorting algorithm. See [26] for stable sorting algorithms.

As heuristics for a good incumbent we first arrange the items as above, and then apply a local search for a further improvement. We observed from some preliminary experiment that 2-opt or 3-opt heuristics is not worth their computational cost, which agrees with the observation reported in Belloni and Lucena [3]. Then we use the following simple heuristic method. Given a linear ordering  $\pi$ , we take an item, say  $i = \pi^{-1}(k)$ , at the  $k$ th position, and search for a position in the range of  $[\max\{1, k - \beta\}, \min\{n, k + \beta\}]$  such that moving

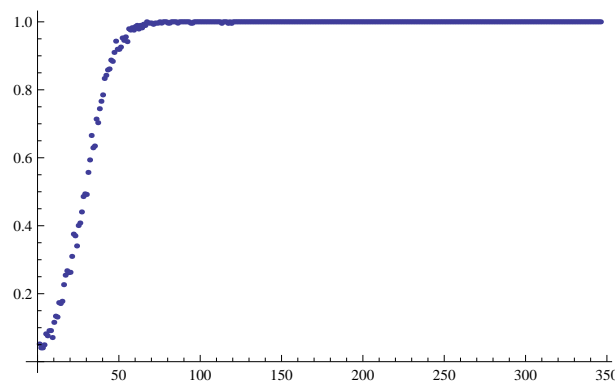


Figure 1: Percentage of pegged variables with  $\delta_{ij} = k$  vs  $k$

item  $i$  to the position improves the objective function value, where  $\beta$  is a fixed positive number. Then we accept it as a temporary incumbent and take the next item  $\pi^{-1}(k + 1)$  for a possible further improvement.

### 9. Feasibility Check

When  $\mu$  becomes less than 0.005, we decide that there is no chance of improving the upper bound unless we expand  $U$  or  $V$ . We add the transitivity constraints violated by the latest optimal solution  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  of  $(LR(\mathbf{u}, \mathbf{v}, P_0, P_1))$ . Namely, we use the following rule to update  $(U, V)$  to the next iterate  $(U^+, V^+)$ :

$$U^+ := U \cup \{ (i, j, k) \in N_{<}^3 \mid x(\mathbf{u}, \mathbf{v})_{ij} + x(\mathbf{u}, \mathbf{v})_{jk} - x(\mathbf{u}, \mathbf{v})_{ik} > 1 \},$$

$$V^+ := V \cup \{ (i, j, k) \in N_{<}^3 \mid -x(\mathbf{u}, \mathbf{v})_{ij} - x(\mathbf{u}, \mathbf{v})_{jk} + x(\mathbf{u}, \mathbf{v})_{ik} > 0 \}.$$

To avoid checking an enormous number of transitivity constraints one by one, we first make the arc set

$$A(\mathbf{x}(\mathbf{u}, \mathbf{v})) := \{ (i, j) \in N_{\neq}^2 \mid x(\mathbf{u}, \mathbf{v})_{ij} = 1 \text{ or } x(\mathbf{u}, \mathbf{v})_{ji} = 0 \}$$

$$= \{ (i, j) \in N_{\neq}^2 \mid \bar{x}_{ij} = 1 \},$$

where  $\bar{x}_{ij}$  is defined by (8.1). Then we compute row sum  $r_i$  of (8.2), sort the items according to it, and then look for a pair of items such that

$$r_j < r_i \text{ and } (j, i) \in A(\mathbf{x}(\mathbf{u}, \mathbf{v})),$$

which we call an *upward arc*. Tracing the arcs of  $A(\mathbf{x}(\mathbf{u}, \mathbf{v}))$  starting from an upward arc, we look for another item, say  $k$ , such that the three arcs  $(j, i)$ ,  $(i, k)$  and  $(k, j)$  form a directed cycle. Clearly this triple violates the transitivity constraint. Furthermore, we see the following lemma.

**Lemma 9.1.** *The arc set  $A(\mathbf{x}(\mathbf{u}, \mathbf{v}))$  contains no upward arcs if and only if  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  is a linear ordering.*

*Proof.* Suppose that  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  is not a linear ordering. Then it violates one of the transitivity constraints. When  $x_{ij} + x_{jk} - x_{ik} \leq 1$  is violated,  $A(\mathbf{x}(\mathbf{u}, \mathbf{v}))$  contains a directed cycle  $\{(i, j), (j, k), (k, i)\}$  of length three, and at least one of its arcs form an upward arc. We also see that there is a directed cycle  $\{(i, k), (k, j), (j, i)\}$  when  $-x_{ij} - x_{jk} + x_{ik} \leq 0$  is violated.

When  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  is a linear ordering, its row sum  $r_i$  of  $\bar{X}$  ranges over  $\{n - 1, n - 2, \dots, 1, 0\}$  as in Corollary 8.1. Rearrange the columns and rows simultaneously in the descending order of  $r_i$ . Note that the diagonal elements are zero. Clearly the first row consists of a single zero followed by  $n - 1$  ones, i.e.,  $(\underline{0}, \underbrace{1, 1, \dots, 1}_{n-1})$ . As the induction hypothesis we assume that

the  $h$ th row is  $h$  zeros followed by  $n - h$  ones for  $h = 1, 2, \dots, k$ . The case of  $k = 3$  is shown below, where diagonal elements are underlined.

$$\bar{X} = \begin{bmatrix} \underline{0} & 1 & 1 & 1 & \dots & 1 \\ 0 & \underline{0} & 1 & 1 & \dots & 1 \\ 0 & 0 & \underline{0} & 1 & \dots & 1 \\ 0 & 0 & 0 & \underline{0} & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & & & \end{bmatrix}$$

The  $k + 1$ st row must have  $k + 1$  zeros and  $n - k - 1$  ones, and the first  $k$  elements are zero by the antisymmetry and the  $k + 1$ st element, which is a diagonal element, is also zero. Therefore it is  $k + 1$  zeros followed by  $n - k - 1$  ones, i.e.,  $(\underbrace{0, 0, \dots, 0}_{k+1}, \underbrace{1, 1, \dots, 1}_{n-k-1})$ . We see

that the matrix  $\bar{X}$  is upper triangular, meaning that  $A(\mathbf{x}(\mathbf{u}, \mathbf{v}))$  has no upward arcs.  $\square$

## 10. Algorithm

The algorithm is composed of the inner and outer cycles. The inner cycle consisting of Steps 2 to 7 generate a sequence of Lagrangian multiplier vectors  $(\mathbf{u}, \mathbf{v})$ , and a sequence of incumbent solutions and values  $\omega_{\text{low}}$ . Some variables are pegged there. The outer cycle expands the constraint index sets  $U$  and  $V$ .

Step 1 (Initialization)

- (a) Arrange the items according to the row-column difference  $\sum_{j \in N} c_{ij} - \sum_{j \in N} c_{ji}$  of cost coefficients and let the linear ordering obtained be the first incumbent solution and let  $\omega_{\text{low}}$  be its objective function value.
- (b) For each consecutive triple  $(i, j, k)$  in the incumbent linear ordering, add the transitivity constraints of type 1 and 2 to  $U$  and  $V$ , respectively.
- (c)  $l \leftarrow 0$ ,  $\mu \leftarrow 2.0$ ,  $(\mathbf{u}, \mathbf{v}) \leftarrow (\mathbf{0}, \mathbf{0})$ .
- (d)  $P_0, P_1 \leftarrow \emptyset$ .
- (e)  $\omega_{\text{up}} \leftarrow +\infty$ .

Step 2 (Solving  $(LR(\mathbf{u}, \mathbf{v}, P_0, P_1))$ )

- (a) Compute  $r(\mathbf{u}, \mathbf{v})_{ij}$  by (4.1).
- (b) Set  $x(\mathbf{u}, \mathbf{v})_{ij}$  according to (4.2).
- (c)  $\omega_{\text{up}} \leftarrow \min\{\omega_{\text{up}}, \omega(LR(\mathbf{u}, \mathbf{v}, P_0, P_1))\}$ .
- (d) If  $\omega_{\text{up}}$  is not improved,  $l \leftarrow l + 1$ . Otherwise,  $l \leftarrow 0$ .

Step 3 (Termination)

- (a) If  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  satisfies the optimality condition in Corollary 5.1 with  $(\mathbf{u}, \mathbf{v})$ , then terminate.
- (b) If  $\omega_{\text{up}} - \omega_{\text{low}} < \varepsilon$ , then terminate, where  $\varepsilon$  is a predetermined tolerance to the duality gap.

Step 4 (Heuristics)

- (a) Apply the heuristic method in Section 8 to  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  for a better solution  $\tilde{\mathbf{x}}$ .
- (b)  $\omega_{\text{low}} \leftarrow \max\{\omega_{\text{low}}, \text{objective function value of } \tilde{\mathbf{x}}\}$ .

Step 5 (Pegging Test)

- (a) When  $\omega_{\text{up}} - \omega_{\text{low}} < \eta$ , then apply the improved pegging test (or the pegging test when  $P_0 = P_1 = \emptyset$ ).
- (b) Let  $P_0$  and  $P_1$  be the index sets of variables pegged at zero and one, respectively.

Step 6 (Update of  $\mu$ )

- (a) If  $\mu \leq 0.005$ , then  $\mu \leftarrow 2.0$  and go to Step 8.
- (b) If  $l$  reaches 5,  $\mu \leftarrow \mu/2$ .

Step 7 (Update of  $(\mathbf{u}, \mathbf{v})$ )

- (a) Update  $(\mathbf{u}, \mathbf{v})$  according to (7.1) and (7.2).
- (b) Go to Step 2.

Step 8 (Update of  $U, V$ )

- (a) Find the transitivity constraints violated by  $\mathbf{x}(\mathbf{u}, \mathbf{v})$  and add them to  $U$  and  $V$ .
- (b)  $u_{ijk}, v_{ijk} \leftarrow 0$  for newly added indices  $(i, j, k)$ .
- (c) Go to Step 2.

## 11. Computational Results

We coded the algorithm in Java, and run it on a PC with an Intel Core i3, 3.33 GHz processor and 2 GB of memory. All results are based on the formulation ( $P$ ) in Subsection 3.3. The problem `DsumC` is a minimum violations ranking problem provided by K. Pedings, College of Charleston. The cost matrix  $C$  is based on the point differential matrix of 347 teams in NCAA college basketball for the 2008–2009 season. The problem has 60,031 binary variables and 13,807,130 transitivity constraints. Note that since the cost matrix is an integer matrix, objective function takes an integer value. See Pedings et al. [24] for the details.

Tables 1 to 3 show the results in which

- **iteration** shows the number of updates of the Lagrangian multiplier vector  $(\mathbf{u}, \mathbf{v})$ ,
- **lower bound** shows the incumbent value  $\omega_{\text{low}}$ ,
- **upper bound** shows the upper bound  $\omega_{\text{up}}$  or  $[\omega_{\text{up}}]$  when  $C$  is an integer matrix,
- **duality gap** is the difference  $\omega_{\text{up}} - \omega_{\text{low}}$ ,
- $|U| + |V|$  shows the number of transitivity constraints in  $U \cup V$ ,
- $\%^1$  shows the percentage of  $|U| + |V|$  to the total number of transitivity constraints,
- $|P_0| + |P_1|$  shows the number of pegged variables,
- $\%^2$  shows the percentage of the pegged variables,
- **time(sec)** shows the computation time in second.

These statistics are given for every 500th iteration.

Table 1 gives the result of the algorithm without pegging tests. After 2,212 iterations, the duality gap reduced to zero, and the incumbent at hand turned out to be an optimal solution. Note that the transitivity constraints being considered account for 0.08%, just a fraction of a percent of the total.

Since the pegging test places a burden on the computation, we did it every 500th iteration. Table 2 gives the result of the algorithm with the ordinary pegging test. It terminated after 2,141 iterations in 11.40 seconds, slightly shorter than the computation time when no pegging test was done. Note that about 92% of the variables were eventually pegged.

Table 3 shows the result of the algorithm with the improved pegging test. We applied the ordinary pegging test at the 500th iteration and then the improved pegging test from the 1,000th iteration at intervals of 500 iterations. The algorithm found an optimal solution at the 316th iteration, and proved its optimality at the 2,193rd iteration when the duality gap fell to zero. It took the longest computation time due to the burden of the improved pegging test; however, about 95% of the variables were eventually pegged. If we failed to prove the optimality of the incumbent solution by a possible abortion of computation, this would still provide much information about an optimal solution. We observed that  $U$  and  $V$  were updated for the first time in the 51st iteration, which led to a sharp decline of the upper bound. Figure 2 shows how the upper and lower bounds converge, and Figure 3 shows how the pegged variables grow as the computation goes.

Table 4 shows the result for the problem `DavgC` provided by Pedings based on the same data as `DsumC`. The difference is that the cost matrix consists of fractional cost coefficients. We stopped the computation after 5,772 iterations when the duality gap reduced to less

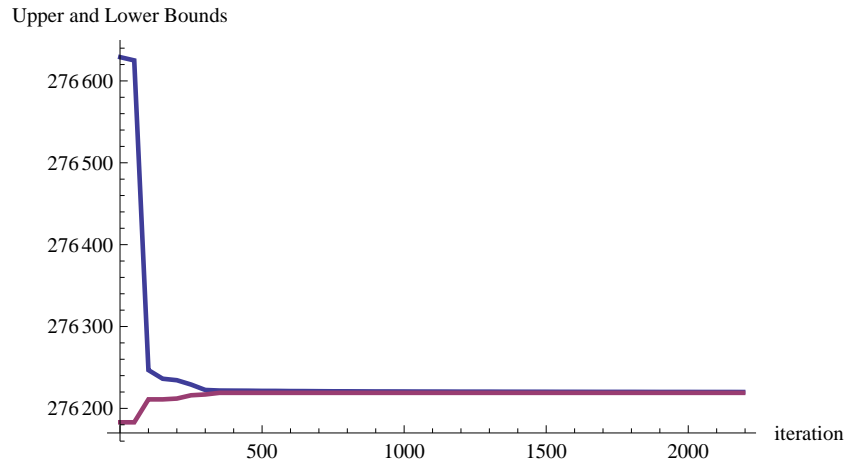


Figure 2: Upper and lower bounds vs iteration

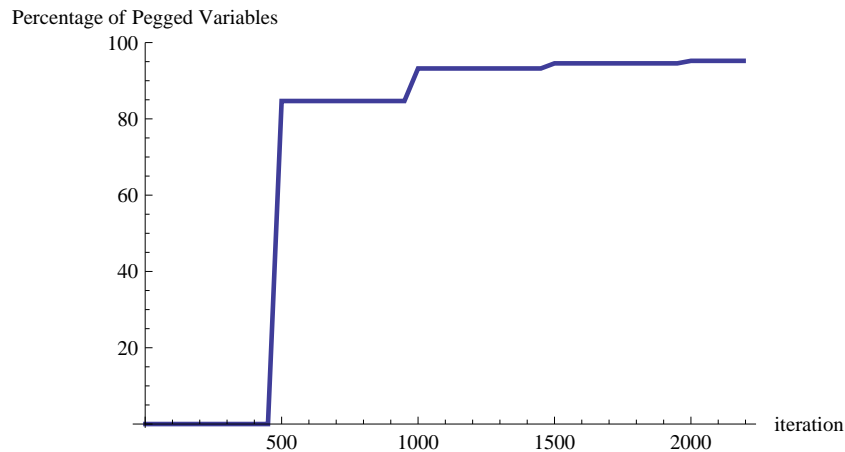


Figure 3: Percentage of pegged variables vs iteration

than one. The final incumbent may not be optimal, however, more than 95% of variables were pegged.

We solved the problem `Judges100` in Charon and Hudry [7]. It assumes that 50 judges rank 100 candidates. Each judge prefers  $i$  to  $j$  with a probability  $0.5 + 0.35(j - i)/(n - 1)$ ; if he does not prefer  $i$  to  $j$ , he prefers  $j$  to  $i$ . The cost coefficient  $c_{ij}$  is the difference between the number of judges preferring  $i$  to  $j$  and the number of judges preferring  $j$  to  $i$ . See  $c_{ij}^{(1)}$  of (2.1). We solved the linear programming relaxation of the problem. The optimal value, denote by  $\omega_{up}^{LP}$ , of the linear programming relaxation is the smallest possible value of the upper bound that the Lagrangian relaxation could provide. We set the iteration limit to 10,000 in the proposed algorithm and solved ten instances. In Table 5 the column %<sup>0</sup> gives the ratio  $(\omega_{up} - \omega_{up}^{LP})/\omega_{up}^{LP}$  in percent to show how close  $\omega_{up}$  is to  $\omega_{up}^{LP}$ . The column “relative duality gap” shows that a narrow duality gap still remains, and it prevents the proof of optimality of the incumbent solution. We compared the proposed algorithm and Xpress Optimizer 21.01.06 run on a PC with an Intel Core i7, 2.80GHz processor and 6 GB of memory. The column %<sup>3</sup> shows the ratio of the best incumbent value obtained by the proposed algorithm and the optimal value. Although our algorithm failed to complete the computation with an optimality proof in most cases, it provided a good lower bound at an

Table 1: Result for DsumC : no pegging test

iteration	lower bound	upper bound	duality gap	$ U  +  V $	% <sup>1</sup>	$ P_0  +  P_1 $	% <sup>2</sup>	time (sec)
1	276183.00	276629.00	446.00	345	0.00	0	0.00	0.01
500	276219.00	276221.00	2.00	10410	0.08	0	0.00	2.35
1000	276219.00	276220.00	1.00	10693	0.08	0	0.00	4.91
1500	276219.00	276220.00	1.00	10865	0.08	0	0.00	7.60
2000	276219.00	276220.00	1.00	11016	0.08	0	0.00	10.32
2212	276219.00	276219.00	0.00	11060	0.08	0	0.00	11.48

Table 2: Result for DsumC : ordinary pegging test

iteration	lower bound	upper bound	duality gap	$ U  +  V $	% <sup>1</sup>	$ P_0  +  P_1 $	% <sup>2</sup>	time (sec)
1	276183.00	276629.00	446.00	345	0.00	0	0.00	0.01
500	276219.00	276221.00	2.00	10410	0.08	50836	84.68	2.39
1000	276219.00	276220.00	1.00	10693	0.08	54421	90.65	5.08
1500	276219.00	276220.00	1.00	10916	0.08	55076	91.75	7.87
2000	276219.00	276220.00	1.00	11015	0.08	55386	92.26	10.66
2141	276219.00	276219.00	0.00	11073	0.08	55386	92.26	11.40

Table 3: Result for DsumC : improved pegging test

iteration	lower bound	upper bound	duality gap	$ U  +  V $	% <sup>1</sup>	$ P_0  +  P_1 $	% <sup>2</sup>	time (sec)
1	276183.00	276629.00	446.00	345	0.00	0	0.00	0.02
500	276219.00	276221.00	2.00	10410	0.08	50836	84.68	2.48
1000	276219.00	276220.00	1.00	10693	0.08	55953	93.21	8.66
1500	276219.00	276220.00	1.00	10882	0.08	56760	94.55	13.16
2000	276219.00	276220.00	1.00	11046	0.08	57156	95.21	17.36
2193	276219.00	276219.00	0.00	11091	0.08	57156	95.21	18.45

Table 4: Result for DavgC : improved pegging test

iteration	lower bound	upper bound	duality gap	$ U  +  V $	% <sup>1</sup>	$ P_0  +  P_1 $	% <sup>2</sup>	time (sec)
1	276269.00	276727.00	458.00	345	0.00	0	0.00	0.02
500	276303.00	276306.05	3.05	11248	0.08	47290	78.78	2.28
1000	276303.00	276305.31	2.31	11548	0.08	53840	89.69	9.27
1500	276303.00	276304.95	1.95	11780	0.09	55361	92.22	14.31
2000	276303.00	276304.77	1.77	11948	0.09	55919	93.15	18.83
2500	276303.00	276304.62	1.62	12025	0.09	55930	93.17	23.22
3000	276303.00	276304.45	1.45	12062	0.09	56384	93.92	27.64
3500	276303.00	276304.35	1.35	12089	0.09	56570	94.23	31.88
4000	276303.00	276304.24	1.24	12138	0.09	56747	94.53	35.94
4500	276303.00	276304.15	1.15	12154	0.09	56896	94.78	39.96
5000	276303.00	276304.09	1.09	12171	0.09	56996	94.94	44.06
5500	276303.00	276304.02	1.02	12187	0.09	57122	95.15	47.88
5772	276303.00	276303.99	0.99	12191	0.09	57122	95.15	49.24



early stage and also the pegging test worked well.

Table 5: Result for Judges100

instance	LRPT					Xpress		
	duality gap	relative duality gap	% <sup>0</sup>	% <sup>2</sup>	time (sec)	optimal value	time (sec)	% <sup>3</sup>
Judges100a	25.00	4.1e-4	3.8e-3	58.71	12.98	60834.00	118.00	99.99
Judges100b	21.00	3.4e-4	4.9e-3	65.45	12.26	61752.00	102.10	99.99
Judges100c	30.00	4.9e-4	6.4e-3	52.26	13.43	60598.00	111.10	99.97
Judges100d	0.00	0.0	0.0	91.45	9.81	61592.00	85.70	100.00
Judges100e	43.00	7.0e-4	9.4e-3	35.84	12.93	61120.00	113.70	99.95
Judges100f	20.00	3.3e-4	4.6e-3	65.11	12.36	60220.00	143.70	99.97
Judges100g	10.00	1.6e-4	4.1e-3	75.23	12.57	60888.00	107.60	99.99
Judges100h	33.00	5.5e-4	1.1e-2	48.99	13.98	60490.00	249.70	99.99
Judges100i	10.00	1.7e-4	1.1e-3	73.09	13.03	60400.00	130.40	99.99
Judges100j	60.00	1.0e-3	1.9e-2	11.70	14.76	59946.00	853.80	99.97

Another problem **Median39** in [7] is a 39-item problem made by drawing the orientation of the arc between  $i$  and  $j$  with a probability 0.5 for each orientation, and choosing its weight randomly from a uniform distribution between 1 and 10. We observe from the column %<sup>0</sup> of Table 6 that the Lagrangian relaxation worked well, and the duality gap shrank at an early stage.

Table 6: Result for Median39

instance	LRPT					Xpress		
	duality gap	relative duality gap	% <sup>0</sup>	% <sup>2</sup>	time (sec)	optimal value	time (sec)	% <sup>3</sup>
Median39a	70.00	2.4e-2	1.7e-1	0.00	7.52	2898.00	257.40	99.34
Median39b	68.00	2.4e-2	1.7e-1	0.00	7.38	2889.00	397.60	99.72
Median39c	44.00	1.5e-2	8.0e-2	0.00	7.46	2966.00	169.80	99.87
Median39d	54.00	1.8e-2	1.1e-1	0.00	6.54	2958.00	245.00	100.00
Median39e	51.00	1.8e-2	8.9e-2	0.00	7.86	2868.00	243.00	100.00
Median39f	83.00	2.9e-2	4.2e-2	0.00	8.60	2905.00	404.00	99.72
Median39g	80.00	2.8e-2	8.2e-2	0.00	7.50	2900.00	1282.70	99.93
Median39h	94.00	3.3e-2	1.1e-1	0.00	8.27	2878.00	2784.70	99.83
Median39i	71.00	2.5e-2	9.0e-2	0.00	7.41	2902.00	489.20	99.90
Median39j	63.00	2.2e-2	1.1e-1	0.00	7.05	2860.00	397.90	99.90

The first two problems **DsumC** and **DavgC** stem from the real-world data, and **Judges100** imitates them. The proposed algorithm works successfully for those problems. We observe from the column %<sup>0</sup> of Tables 5 and 6 that the Lagrangian relaxation gives an upper bound which is very close to the one provided by the linear programming relaxation. This shows the potential that the Lagrangian relaxation could replace the linear programming relaxation in the branch-and-bound algorithm. The problems we solved are so limited that more well-organized experiments should be carried out before any conclusion is made.

## Acknowledgment

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