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Abstract

In this paper we investigate a mathematical programming approach for tightening the bounds of the price of European-type contingent claims in incomplete markets. Arbitrage is generalized by using coherent risk measures. Although this direction of generalization has been proposed in the literature, by concentrating on the case of discrete model described via a scenario tree, we can fully enjoy the duality theory for semi-infinite linear programming. As a result, bounding prices is reduced to solving a pair of convex optimization problems. In particular, when conditional value-at-risk (CVaR) is employed as risk measure, the resulting problems are linear programs. Besides, due to the dual representation of coherent risk measures, the hedging portfolio problem can be considered as a robust optimization where the valuation will be robust against the uncertainty in probability estimation, which indicates that robust hedging portfolio links to asset pricing in incomplete markets. Numerical examples illustrate that the gap of the bounds shrinks to a unique price, which can be considered as a fair price in the sense that seller and buyer face the same risk.

1. Introduction

The pricing theory of financial contingent claims has an illustrious history which starts in 1973 when Black and Scholes presented the option pricing model, and then Merton extended their model in several important ways. Since their seminal works, numerous researchers have committed to establish the mathematical foundation for the pricing of contingent claims, and most of them model the uncertainty of asset market in either of two ways: via continuous models and via discrete models. The former assumes that investors can trade a set of assets in a market at any moment when they would like to do, and that there are infinitely many possible states while the latter assumes that trade can be made only at finite times within a time horizon and the number of possible states is usually set to be finite.

As in the Black-Scholes model, the continuous model can provide an explicit formula by assuming that the underlying asset price follows a specific stochastic process such as geometric Brownian motion. One of the drawbacks of the continuous models is, however, that the assumption can be too restrictive to adopt the model directly to real market data. In addition, most of them can treat only a few underlying assets simultaneously.

In order to enable us to fit the models to real situation, several directions have been considered for relaxing the restrictive assumptions on the underlying distribution. One direction is to introduce a general class of stochastic processes such as Lévi process (e.g., [26]) and stochastic volatility model (e.g., [12]). Another direction is to abandon the use of parametric distributions, for example, by supposing that only partial information of the distribution is available. For example, several researches [20, 4, 9, 19] provide convex optimization approaches to bounding the

price of European type contingent claims when only information on moments of the underlying asset price distribution is available.

On the other hand, discrete model can deal with general distributions by way of some scenario tree approximation. Cox, Ross and Rubinstein model [5] and several interest rate tree models such as Hull [15] are well-known useful examples. Another advantage of the discrete model is that it can easily treat multiple underlying assets (see, e.g., [21, 17]). Therefore it provides a basis for numerically computing the price of contingent claims which have complex structures such as exotic options on multiple underlying assets for which no analytical formula can be achieved.

Interestingly, linear programming has played a role in developing discrete models. For example, Ritchken [22] formulates linear programming problems for computing a pair of upper and lower bounds on the price of a European style contingent claim in a single period model, and Ritchken and Kuo [23] and Basso and Pianca [2] extend the approach so as to reduce the gap of the pricing bounds by introducing risk attitude of investors. King and his coauthor [17, 18] point out, however, that such a reduction cannot contribute to explain pricing unless the gap vanishes because the upper bound implies *ask price*, i.e., lower bound of seller's price for selling the asset, while the lower bound implies *bid price*, i.e., upper bound of buyer's price for buying the asset, and therefore the existence of the gap means that neither buyer nor seller is motivated to make a deal.

It is worth noting that how to price contingent claims in incomplete market is still a big outstanding problem for the finance theory since real world markets are obviously incomplete. One of the advantages of the discrete model is that it can treat the incomplete market situation in a similar manner to the complete one.

In this paper, we investigate the linear programming approach for computing a pair of upper and lower bounds of price of a contingent claim in incomplete markets by reconsidering how it is decided via hedging with the risk measure one uses. More specifically, first introducing a generalized version of arbitrage based on coherent risk measures, which was first introduced by Artzner et al.[1], we pose a semi-infinite programming problem for establishing the equivalence between nonexistence of the generalized arbitrage and existence of a martingale probability in a similar manner to King [17]. A smaller gap of the upper and lower bounds is then obtained by considering a pair of semi-infinite programming problems which represent super-replicating problems for seller and buyer, respectively. As a result, a fair price of a contingent claim in incomplete markets can be calculated based on optimization over a subset of martingale probability measures that is decided by the coherent risk measure one employed.

The contribution of this research is threefold. First, we generalize the fundamental theory of asset pricing in a discrete state model discussed in King [17] by replacing the traditional no-risk condition in the definition of arbitrage with an alternative condition associated with coherent risk measures. This direction for generalization has been already discussed in the literature, e.g., [16], but, by focusing on discrete model, the existing duality theorems for (semi-infinite) linear programming can be fully utilized. Second, we present an explanation for decreasing and vanishing the gap, which suggests that there is a way of pricing contingent claims in incomplete market in a similar manner to complete market as in [17]. In particular, when CVaR [25] is employed as the coherent risk measure, the resulting optimization problems turns out to be linear programs. Third, we illustrate that computing the price bounds based on coherent risk measures links to a robust optimization for pricing, where underlying probability plays a role of nominal vector for uncertainty set of probability measures. See, e.g., [3] for a survey of robust optimization. This indicates that the gap of the bounds is directly related to the size of uncertainty set for probability parameter.

Also, this paper provides a clear view of the pricing of contingent claims in incomplete market only on the basis of optimization theory and convex analysis so that optimization people can treat contingent claim pricing problem without being preoccupied with probability theory more than necessary.

The structure of this paper is as follows. In the next section, we briefly review the classical results of the well-known no-arbitrage option pricing in a discrete market model. In Section 3, we describe coherent risk measures and introduce a generalized notion of arbitrage. By exploiting a duality theorem for semi-infinite linear programming, a generalized version of the fundamental theorem of discrete market model is shown. Section 4 is devoted to evaluating a pair of upper and lower bounds of a European type contingent claim price on the basis of semi-infinite linear program. In Section 5, some numerical examples for illustrating how to shrink the gap. Section 6 concludes the paper with some remarks.

2. Pricing Bounds and Arbitrage Opportunity

In this section, we briefly review the methods in which mathematical programming techniques are utilized for computing the upper and lower bounds of the price of contingent claims.

2.1. **Notations and basic assumptions.** We first introduce a scenario tree which describes the uncertainty of contingent claims' values according to [17].

We denote the depth of the scenario tree by T and write $[T] := \{0, 1, \ldots, T\}$. Let $\mathcal{N}_t$ be the set of all nodes at depth t in the scenario tree for $t \in [T]$ and let $\mathcal{N}$ be the set of all nodes, i.e., $\mathcal{N} := \bigcup_{0 \le t \le T} \mathcal{N}_t$. We denote the number of nodes of $\mathcal{N}_t$ by n_t. The set $\mathcal{N}_0$ consists of the initial node 0 alone, i.e., $n_0 = 1$. We abbreviate $\mathcal{N} \setminus \mathcal{N}_t$ to $\mathcal{N}_{-t}$. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, 2, \ldots, T$ has a unique parent node in $\mathcal{N}_{t-1}$ denoted by $a(n)$, and every node $n \in \mathcal{N}_t$ for $t = 0, 1, \ldots, T-1$ has a nonempty set of child nodes in $\mathcal{N}_{t+1}$, denoted by $c(n)$. The original probability distribution $\boldsymbol{p}$ is modeled by attaching a positive weight p_n to each terminal node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. Each non-terminal node is given the probability $\tilde{p}_n$ defined inductively by $\tilde{p}_n := \sum_{m \in c(n)} \tilde{p}_m$, where $\tilde{p}_n := p_n$ for $n \in \mathcal{N}_T$.

We suppose that the market has $J+1$ tradable securities indexed by $j = 0, \ldots, J$ whose prices at node n are denoted by the vector $\boldsymbol{s}_n := (s_n^0, \ldots, s_n^J)^\top$. We assume that the security 0, which we refer to as *numeraire*, always has a positive value, i.e., $s_n^0 > 0$ for all $n \in \mathcal{N}$. Let us denote the discount rate by $\delta_n := 1/s_n^0$ and $\boldsymbol{z}_n := \delta_n \boldsymbol{s}_n$. Then the vector $\boldsymbol{z}_n$ is the relative price vector with respect to the numeraire. Obviously, $z_n^0 = 1$ for all $n \in \mathcal{N}$.

It is noteworthy that we can do without introducing the notion of random variable because any random variable in the tree can be represented by a finite dimensional vector attached with a node. For convenience, we reserves the upper case characters for denoting random variables. For $t \in [T]$ we denote the random variable vector on $\mathcal{N}_t$ with $\boldsymbol{Z}_t(n) = \boldsymbol{z}_n$ for each $n \in \mathcal{N}_t$ by $\boldsymbol{Z}_t$.

For $j = 0, \ldots, J$ and $n \in \mathcal{N}$ let θ_n^j denote the amount of security j held by the investor at node n and let $\boldsymbol{\theta}_n := (\theta_n^0, \ldots, \theta_n^J)^\top$. We call $\{\boldsymbol{\theta}_n : n \in \mathcal{N}\}$ a *trading strategy*. We sometimes denote it simply by $\boldsymbol{\theta}$.

Definition 2.1. *We say that a trading strategy $\boldsymbol{\theta}$ is self-financing if it satisfies*

$$\boldsymbol{z}_n^\top (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{a(n)}) = 0 \qquad (n \in \mathcal{N}_{-0}).$$

2.2. **Fundamental theorem of asset pricing.** Let us first introduce the definition of traditional arbitrage strategy.

Definition 2.2 (Traditional arbitrage strategy). *Arbitrage is a self-financing trading strategy $\boldsymbol{\theta}$ that begins with zero initial value at time 0, maintains a non-negative value at each terminal node $n \in \mathcal{N}_T$ and has a positive expected value at the maturity date T. Mathematically, arbitrage is a trading strategy $\boldsymbol{\theta}$ that is defined by*

$$(2.1) \qquad \boldsymbol{z}_0^\top \boldsymbol{\theta}_0 = 0$$

$$(2.2) \qquad \boldsymbol{z}_n^\top (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{a(n)}) = 0 \qquad (n \in \mathcal{N}_{-0})$$

$$(2.3) \qquad \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \ge 0 \qquad (n \in \mathcal{N}_T)$$

$$(2.4) \qquad \sum_{n \in \mathcal{N}_T} p_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n > 0.$$

The market is said to be *arbitrage free* if there is no chance of arbitrage in the market. Harrison and Kreps [11] proved that the absence of arbitrage is essentially equivalent to the existence of an *equivalent martingale probability measure* $\tilde{\boldsymbol{q}}$ such that

    1) $\tilde{\boldsymbol{q}}$ agrees with $\tilde{\boldsymbol{p}}$ on impossible events, i.e., $\tilde{q}_n = 0$ if and only if $\tilde{p}_n = 0$ for all $n \in \mathcal{N}$,

    2) $\{\, \boldsymbol{Z}_t \mid t \in [T] \,\}$ is a martingale process under the probability measure $\tilde{\boldsymbol{q}}$, i.e.,

$$(2.5) \qquad \tilde{q}_n \boldsymbol{z}_n = \sum_{m \in c(n)} \tilde{q}_m \boldsymbol{z}_m \quad (n \in \mathcal{N}_{-T}).$$

Further, the no arbitrage price of security Y is given by the expected value of Y at the maturity date T based on an arbitrarily chosen equivalent martingale probability measure.

Theorem 2.3 (Fundamental theorem of asset pricing)**.**

    1) There is no arbitrage if and only if there is a martingale probability measure $\tilde{\boldsymbol{q}}$ equivalent to $\tilde{\boldsymbol{p}}$.

    2) In the arbitrage free market, the lower and upper bounds of the price of a contingent claim Y are given, respectively, by

$$\frac{1}{\delta_0} \min_{\tilde{\boldsymbol{q}} \in \mathcal{M}} \sum_{n \in \mathcal{N}_T} \tilde{q}_n \delta_n y_n \quad and \quad \frac{1}{\delta_0} \max_{\tilde{\boldsymbol{q}} \in \mathcal{M}} \sum_{n \in \mathcal{N}_T} \tilde{q}_n \delta_n y_n,$$

    where $\mathcal{M}$ is the set of martingale probability measures, and y_n is the future value of contingent claim Y at node n.

Proof. See Theorem 1 of [17], for example. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

If the price of contingent claim Y is less than or equal to the lower bound in Theorem 2.3, there exists an arbitrage for buyer; on the other hand, a price which is greater than or equal to the upper bound induces an arbitrage for seller. Thus the interval between the two bounds in the theorem is called the *no arbitrage interval of Y*.

3. ACCEPTANCE SET, COHERENT RISK MEASURE AND ARBITRAGE

In this section, we introduce concepts of the acceptance set and coherent risk measure following Artzner et al. [1], and then describe a generalization of the arbitrage. The condition for the absence of arbitrage based on the coherent risk measure is also given.

3.1. **Generalization of arbitrage and relation to coherent measure.** We define the *acceptance set* of the investor by the set of net worths of a contingent claim that the investor is willing to take, and denote it by $\mathcal{A}$. Given an acceptance set $\mathcal{A}$, a trading strategy $\boldsymbol{\theta}$ is said to be *acceptable* if $(\boldsymbol{z}_n^{\top} \boldsymbol{\theta}_n)_{n \in \mathcal{N}_T} \in \mathcal{A}$.

Definition 3.1 (Generalized arbitrage)**.** *A self-financing trading strategy $\boldsymbol{\theta}$ is said to be a generalized arbitrage associated with $\mathcal{A}$ if it begins with zero initial value, maintains an acceptable net worth at each terminal node $n \in \mathcal{N}_T$, and has a positive expected value at the maturity date T. Mathematically, the generalized arbitrage associated with an acceptance set $\mathcal{A}$ is a trading strategy $\boldsymbol{\theta}$ such that*

$$(3.1) \qquad \boldsymbol{z}_0^{\top} \boldsymbol{\theta}_0 = 0$$

$$(3.2) \qquad \boldsymbol{z}_n^{\top} (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{a(n)}) = 0 \qquad (n \in \mathcal{N}_{-0})$$

$$(3.3) \qquad (\boldsymbol{z}_n^{\top} \boldsymbol{\theta}_n)_{n \in \mathcal{N}_T} \in \mathcal{A}$$

$$(3.4) \qquad \sum_{n \in \mathcal{N}_T} p_n \boldsymbol{z}_n^{\top} \boldsymbol{\theta}_n > 0.$$

Note that the arbitrage associated with

$$\mathbb{R}_+^{n_T} := \{\, \boldsymbol{x} \in \mathbb{R}^{n_T} \mid \boldsymbol{x} \geq \boldsymbol{0} \,\}$$

coincides with the traditional arbitrage in Definition 2.2. If $\mathcal{A}$ contains $\mathbb{R}^{n_T}_+$, the no risk condition for the arbitrage associated with $\mathcal{A}$ is weaker than that for the traditional arbitrage.

Definition 3.2 (Risk measure associated with an acceptance set [1])**.** *Risk measure ρ is a mapping from the set of all net worths $\boldsymbol{x} \in \mathbb{R}^{n_T}$ to $\mathbb{R}$. The risk measure $\rho_{\mathcal{A}}$ associated with an acceptance set $\mathcal{A}$ is defined by*

$$\rho_{\mathcal{A}}(\boldsymbol{x}) := \inf\{ c \mid c\boldsymbol{e} + \boldsymbol{x} \in \mathcal{A} \},$$

where $\boldsymbol{e}$ is the vector of ones in $\mathbb{R}^{n_T}$.

Definition 3.3 (Acceptance set associated with a risk measure)**.** *Given a risk measure ρ, the acceptance set associated with ρ, denoted by $\mathcal{A}_\rho$, is defined by*

$$\mathcal{A}_\rho := \{ \boldsymbol{x} \in \mathbb{R}^{n_T} \mid \rho(\boldsymbol{x}) \le 0 \}.$$

Definition 3.4 (Coherent risk measure)**.** *A risk measure ρ is said to be* coherent *if ρ has the following four properties.*

1) *Translation invariance: For all $\boldsymbol{x}$ and real number c, $\rho(\boldsymbol{x} + c\boldsymbol{e}) = \rho(\boldsymbol{x}) - c$.*
2) *Subadditivity: For all $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, $\rho(\boldsymbol{x}_1 + \boldsymbol{x}_2) \le \rho(\boldsymbol{x}_1) + \rho(\boldsymbol{x}_2)$.*
3) *Positive homogeneity: For all $\lambda \ge 0$ and net worth $\boldsymbol{x}$, $\rho(\lambda \boldsymbol{x}) = \lambda \rho(\boldsymbol{x})$.*
4) *Monotonicity: For all $\boldsymbol{x}$ and $\boldsymbol{y}$ with $\boldsymbol{x} \le \boldsymbol{y}$, it holds that $\rho(\boldsymbol{y}) \le \rho(\boldsymbol{x})$.*

Artzner et al. [1] proved that if ρ is a coherent risk measure, then the acceptance set $\mathcal{A}_\rho$ has the following properties and vice versa.

1) $\mathcal{A}_\rho$ is a closed convex cone.
2) $\mathbb{R}^{n_T}_+ \subseteq \mathcal{A}_\rho$ and $\mathcal{A}_\rho \cap \mathbb{R}^{n_T}_{--} = \phi$, where $\mathbb{R}^{n_T}_{--} := \{ \boldsymbol{x} \in \mathbb{R}^{n_T} \mid \boldsymbol{x} < \boldsymbol{0} \}$.

Let $\mathbb{E}_{\boldsymbol{\pi}}(\boldsymbol{x}) = \boldsymbol{\pi}^\top \boldsymbol{x}$. They also showed if $\mathcal{P}_\rho$ is defined as

(3.5)　　$\mathcal{P}_\rho := \{ \boldsymbol{\pi} \mid \boldsymbol{\pi}$ is a probability measure on $\mathcal{N}_T$ and $\mathbb{E}_{\boldsymbol{\pi}}(-\boldsymbol{x}) \le \rho(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^{n_T} \}$,

then $\rho(\cdot)$ satisfies

(3.6)　　　　　　　　　　$\rho(\boldsymbol{x}) = \sup\{ \mathbb{E}_{\boldsymbol{\pi}}(-\boldsymbol{x}) \mid \boldsymbol{\pi} \in \mathcal{P}_\rho \}.$

We also see that $\rho(-(\boldsymbol{z}_n^\top \boldsymbol{\theta}_n)_{n \in \mathcal{N}_T}) \le 0$ if and only if

(3.7)　　　　　　　　　$\sum_{n \in \mathcal{N}_T} \pi_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \ge 0 \qquad (\boldsymbol{\pi} \in \mathcal{P}_\rho).$

Lemma 3.5. *$\mathcal{P}_\rho$ is a compact convex subset of $\mathbb{R}^{n_T}$ that does not contain the origin.*

Proof. By definition,

$$\mathcal{P}_\rho = \bigcap_{\boldsymbol{x} \in \mathbb{R}^{n_T}} \{ \boldsymbol{\pi} \mid -\boldsymbol{\pi}^\top \boldsymbol{x} \le \rho(\boldsymbol{x}) \} \cap \{ \boldsymbol{\pi} \mid \boldsymbol{\pi} \ge \boldsymbol{0}, \boldsymbol{e}^\top \boldsymbol{\pi} = 1 \},$$

all of which are closed convex sets, and the right most set is a compact set that does not contain the origin. $\qquad\square$

3.2. **Absence of arbitrage.** We hereafter consider the acceptance set $\mathcal{A}_\rho$ defined by a coherent risk measure ρ. We define ρ-*arbitrage* as follows.

Definition 3.6 (ρ-arbitrage)**.** *For any coherent risk measure ρ, a ρ-arbitrage is a trading strategy $\boldsymbol{\theta} = (\boldsymbol{\theta}_n)_{n \in \mathcal{N}}$ that satisfies*

(3.8)　　　　　　　　　　$\boldsymbol{z}_0^\top \boldsymbol{\theta}_0 = 0$

(3.9)　　　　　　　　　　$\boldsymbol{z}_n^\top (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{a(n)}) = 0 \qquad (n \in \mathcal{N}_{-0})$

(3.10)　　　　　　　　　$\sum_{n \in \mathcal{N}_T} \pi_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \ge 0 \qquad (\boldsymbol{\pi} \in \mathcal{P}_\rho)$

(3.11)　　　　　　　　　$\sum_{n \in \mathcal{N}_T} p_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n > 0.$

Note that the system (3.8), (3.9) and (3.10) is always consistent since $\boldsymbol{\theta} = (\boldsymbol{0})_{n \in \mathcal{N}}$ satisfies it. We say that the system satisfies the *Slater constraint qualification* when there is a vector $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_n)_{n \in \mathcal{N}}$ satisfying

$$\boldsymbol{z}_0^\top \hat{\boldsymbol{\theta}}_0 = 0$$
$$\boldsymbol{z}_n^\top (\hat{\boldsymbol{\theta}}_n - \hat{\boldsymbol{\theta}}_{a(n)}) = 0 \qquad (n \in \mathcal{N}_{-0})$$
$$\sum_{n \in \mathcal{N}_T} \pi_n \boldsymbol{z}_n^\top \hat{\boldsymbol{\theta}}_n > 0 \qquad (\boldsymbol{\pi} \in \mathcal{P}_\rho).$$

Based on the above generalized notion of arbitrage, we show that the existence of a martingale measure is a necessary condition for the no ρ-arbitrage. For a probability measure $\tilde{\boldsymbol{q}} = (\tilde{q}_n)_{n \in \mathcal{N}}$ we denote its restriction to $\mathcal{N}_T$ by $\tilde{\boldsymbol{q}}_T = (\tilde{q}_n)_{n \in \mathcal{N}_T}$.

Theorem 3.7. *Suppose that the system (3.8), (3.9) and (3.10) satisfies the Slater constraint qualification, and also suppose that $\boldsymbol{p} \in \mathcal{P}_\rho$. If there is no ρ-arbitrage, there is a probability measure $\tilde{\boldsymbol{q}}$ such that*

1) $\tilde{\boldsymbol{q}}_T \in \mathcal{P}_\rho$,
2) $\tilde{q}_n > 0$ whenever $p_n > 0$ for $n \in \mathcal{N}_T$,
3) $\{ \boldsymbol{Z}_t \mid t \in [T] \}$ is a martingale process under $\tilde{\boldsymbol{q}}$.

Proof. Let us consider the optimization problem

$$(3.12) \quad \begin{array}{|ll}
\inf & -\sum_{n \in \mathcal{N}_T} p_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \\
\text{subject to} & \boldsymbol{z}_0^\top \boldsymbol{\theta}_0 = 0 \\
& \boldsymbol{z}_n^\top (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{a(n)}) = 0 \qquad (n \in \mathcal{N}_{-0}) \\
& \sum_{n \in \mathcal{N}_T} \pi_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \geq 0 \qquad (\boldsymbol{\pi} \in \mathcal{P}_\rho).
\end{array}$$

Since $\mathcal{P}_\rho$ is compact, we see that the set of coefficient vectors $\{ (\pi_n \boldsymbol{z}_n)_{n \in \mathcal{N}_T} \mid \boldsymbol{\pi} \in \mathcal{P}_\rho \}$ is compact. Suppose that there is no ρ-arbitrage, i.e., the optimal value of the problem is zero. Then by Lemma A.3 its dual problem is feasible. That is,

$$(3.13) \qquad \lambda_n \boldsymbol{z}_n = \sum_{m \in c(n)} \lambda_m \boldsymbol{z}_m \qquad (n \in \mathcal{N}_{-T}),$$

$$(3.14) \qquad p_n \boldsymbol{z}_n = \lambda_n \boldsymbol{z}_n - \sum_{i \in I} \mu_i \pi_n^{(i)} \boldsymbol{z}_n \quad (n \in \mathcal{N}_T)$$

hold for some finite number of probability measures $\boldsymbol{\pi}^{(i)} \in \mathcal{P}_\rho$ $(i \in I)$, λ_n $(n \in \mathcal{N})$ and $\mu_i \geq 0$ $(i \in I)$. Since $z_n^0 = 1$ for all $n \in \mathcal{N}$ it holds that

$$(3.15) \qquad \lambda_n = \sum_{m \in c(n)} \lambda_m \qquad (n \in \mathcal{N}_{-T}),$$

$$(3.16) \qquad \lambda_n = p_n + \sum_{i \in I} \mu_i \pi_n^{(i)} \quad (n \in \mathcal{N}_T).$$

By using (3.15) and (3.16) repeatedly we see that

$$\lambda_n > 0 \text{ when } \tilde{p}_n > 0,$$
$$\lambda_0 = \sum_{n \in \mathcal{N}_T} \lambda_n = \sum_{n \in \mathcal{N}_T} \left(p_n + \sum_{i \in I} \mu_i \pi_n^{(i)} \right) = 1 + \sum_{i \in I} \mu_i > 0.$$

Let $\tilde{q}_n$ be defined as

$$(3.17) \qquad \tilde{q}_n = \frac{\lambda_n}{\lambda_0} \quad (n \in \mathcal{N})$$

and we see that $\tilde{\boldsymbol{q}} = (\tilde{q}_n)_{n \in \mathcal{N}}$ has the desired properties. Clearly $\tilde{q}_n > 0$ whenever $\tilde{p}_n > 0$. By (3.13) and (3.15) we have

$$(3.18) \qquad \tilde{q}_n \boldsymbol{z}_n = \sum_{m \in c(n)} \tilde{q}_m \boldsymbol{z}_m \quad (n \in \mathcal{N}_{-T})$$

$$(3.19) \qquad \tilde{q}_n = \sum_{m \in c(n)} \tilde{q}_m \qquad (n \in \mathcal{N}_{-T}),$$

which implies that $\{ \boldsymbol{Z}_t \mid t \in [T] \}$ is a martingale process under $\tilde{\boldsymbol{q}}$.

For $n \in \mathcal{N}_T$ we have from (3.16) that

$$\tilde{q}_n = \frac{\lambda_n}{\lambda_0} = \frac{1}{1 + \sum_{i \in I} \mu_i} p_n + \sum_{i \in I} \frac{\mu_i}{1 + \sum_{i \in I} \mu_i} \pi_n^{(i)}.$$

Therefore $\tilde{\boldsymbol{q}}_T$ is a convex combination of $\boldsymbol{p}$ and $\boldsymbol{\pi}^{(i)}$'s. Since all of those are in $\mathcal{P}_\rho$, we have by the convexity of $\mathcal{P}_\rho$ that $\boldsymbol{q}$ lies in $\mathcal{P}_\rho$. This proves the theorem. $\qquad \square$

Theorem 3.8. *Suppose that $\mathcal{P}_\rho$ is a polytope and $\boldsymbol{p} \in \mathcal{P}_\rho$. If there is no ρ-arbitrage, there is a probability measure $\tilde{\boldsymbol{q}}$ satisfying the same conditions as in Theorem 3.7.*

Proof. When $\mathcal{P}_\rho$ is a polytope, the infinitely many inequality constraints of (3.12) are equivalent to a finite number of inequality constraints each corresponding to an extreme point of $\mathcal{P}_\rho$. Then the LP duality in Lemma A.3 holds. $\qquad \square$

In the following $\mathrm{ri}_\Pi(\mathcal{P}_\rho)$ denotes the interior of $\mathcal{P}_\rho$ relative to the hyperplane $\Pi := \{ \boldsymbol{\pi} \mid \boldsymbol{\pi} \in \mathbb{R}^{n_T}, \boldsymbol{e}^\top \boldsymbol{\pi} = 1 \}$.

Theorem 3.9. *If there is a martingale probability measure $\tilde{\boldsymbol{q}}$ such that $\tilde{\boldsymbol{q}}_T \in \mathrm{ri}_\Pi(\mathcal{P}_\rho)$, then there is no ρ-arbitrage.*

Proof. Since $\tilde{\boldsymbol{q}}_T \in \mathrm{ri}_\Pi(\mathcal{P}_\rho)$, $\tilde{q}_n > 0$ for all $n \in \mathcal{N}_T$ and there is a positive ε such that the intersection of the hyperplane Π and the ε-neighborhood of $\tilde{\boldsymbol{q}}_T$ is included in $\mathcal{P}_\rho$.

By Lemma A.4 we see that $(\alpha \tilde{\boldsymbol{q}}_T - \boldsymbol{p})/\|\alpha \tilde{\boldsymbol{q}}_T - \boldsymbol{p}\|_1$ lies in this neighborhood for a sufficiently large α, and hence in $\mathcal{P}_\rho$. Let $\tilde{\boldsymbol{r}} = (\alpha \tilde{\boldsymbol{q}} - \tilde{\boldsymbol{p}})/\|\alpha \tilde{\boldsymbol{q}} - \tilde{\boldsymbol{p}}\|_1$, $\boldsymbol{\lambda} = \alpha \tilde{\boldsymbol{q}}$ and $\mu = \|\alpha \tilde{\boldsymbol{q}} - \tilde{\boldsymbol{p}}\|_1$. Then $\tilde{\boldsymbol{r}}_T \in \mathcal{P}_\rho$,

$$(3.20) \qquad \boldsymbol{\lambda} + \mu(-\boldsymbol{r}) = \tilde{\boldsymbol{p}},$$

and

$$(3.21) \qquad \lambda_n \boldsymbol{z}_n - \sum_{m \in c(n)} \lambda_m \boldsymbol{z}_m = \boldsymbol{0} \quad (n \in \mathcal{N}_{-T})$$

hold since $\boldsymbol{\lambda}$ is a positive multiple of the martingale probability measure $\tilde{\boldsymbol{q}}$. The equation (3.20) implies that $\lambda_n \boldsymbol{z}_n + \mu(-\tilde{r}_n \boldsymbol{z}_n) = p_n \boldsymbol{z}_n$ for $n \in \mathcal{N}_T$. Combining this with (3.21) shows that the dual problem of problem (3.12) is feasible. Then by the weak duality theorem Lemma A.1 the optimal value of the problem is zero, meaning no ρ-arbitrage. $\qquad \square$

Note that Theorem 3.9 does not require $\boldsymbol{p}$ be in $\mathcal{P}_\rho$.

Let $\mathcal{M}$ be the set of probability measure under which $\{ \boldsymbol{Z}_t \mid t \in [T] \}$ is a martingale process. The following corollary states a generalized version of Theorem 2.3 under the condition that $\boldsymbol{p} \in \mathrm{ri}_\Pi(\mathcal{P}_\rho)$.

Corollary 3.10. *Suppose $\boldsymbol{p} \in ri_\Pi(\mathcal{P}_\rho)$. Suppose that the system (3.8), (3.9) and (3.10) satisfies the Slater constraint qualification or $\mathcal{P}_\rho$ is a polytope. Then there is no ρ-arbitrage if and only if there is $\tilde{\boldsymbol{q}} \in \mathcal{M}$ such that $\tilde{\boldsymbol{q}}_T \in ri_\Pi(\mathcal{P}_\rho)$.*

Proof. Note that the probability measure $\tilde{\boldsymbol{q}}$ whose existence was given by Theorem 3.7 satisfies $\tilde{\boldsymbol{q}}_T \in \mathrm{ri}_\Pi(\mathcal{P}_\rho)$ when $\boldsymbol{p} \in \mathrm{ri}_\Pi(\mathcal{P}_\rho)$. See the Accessibility Lemma (3.2.11) of Stoer and Witzgall [28] or Theorem 2.33 of Rockafellar and Wets [24]. Then the assertion is straightforward from the two theorems. $\qquad \square$

4. Pricing European style contingent claims

European style contingent claim is a claim which provides the holder with the right to receive payoff at maturity date T. Among examples is a *European call option* that gives the holder the right to buy the security at price K at maturity date T. The value of the option at T is then given by $\max\{0, S_T - K\}$ where S_T denotes the security price at T.

In this section we consider the pricing problem of European style contingent claims. We begin with considering the problem of hedging the risk arising from the claims.

Definition 4.1 (ρ-hedging). *Let y_n denote the value of a contingent claim Y at each terminal node $n \in \mathcal{N}_T$, and let c_n denote the relative value of y_n with respect to the numeraire s_n^0, i.e., $c_n := y_n/s_n^0$ for $n \in \mathcal{N}$. A self-financing trading strategy $\boldsymbol{\theta}$ is the ρ-hedging strategy of Y if*

1) *the expected value of the portfolio consisting of θ_n^j units of security j for $j = 0, \ldots, J$ and -1 unit of the claim is nonnegative, i.e., $\sum_{n \in \mathcal{N}_T} p_n(\boldsymbol{z}_n^\top \boldsymbol{\theta}_n - c_n) \geq 0$, and*

2) *the future wealth of the portfolio is acceptable with respect to the coherent risk measure ρ, i.e., $\rho((\boldsymbol{z}_n^\top \boldsymbol{\theta}_n - c_n)_{n \in \mathcal{N}_T}) \leq 0$.*

We see that the second condition $\rho((\boldsymbol{z}_n^\top \boldsymbol{\theta}_n - c_n)_{n \in \mathcal{N}_T}) \leq 0$ holds if and only if $\sum_{n \in \mathcal{N}_T} \pi_n(\boldsymbol{z}_n^\top \boldsymbol{\theta}_n - c_n) \geq 0$ for all $\boldsymbol{\pi} \in \mathcal{P}_\rho$.

4.1. Computation of upper and lower bounds of price.
We start this subsection with considering the seller's pricing problem of a European style contingent claim using the ρ-hedging.

The value c_n is the future cashflow that the seller of the contingent claim Y pays to the buyer at each terminal node $n \in \mathcal{N}_T$. An upper bound of the price of the claim is then the minimum cost for the seller to ρ-hedge this claim, which is mathematically given by the optimal value of the following problem.

$$(4.1) \quad \left|
\begin{aligned}
&\inf && \boldsymbol{z}_0^\top \boldsymbol{\theta}_0 \\
&\text{subject to} && \boldsymbol{z}_n^\top(\boldsymbol{\theta}_n - \boldsymbol{\theta}_{a(n)}) = 0 && (n \in \mathcal{N}_{-0}) \\
& && \sum_{n \in \mathcal{N}_T} \pi_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \geq \sum_{n \in \mathcal{N}_T} \pi_n c_n && (\boldsymbol{\pi} \in \mathcal{P}_\rho) \\
& && \sum_{n \in \mathcal{N}_T} p_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \geq \sum_{n \in \mathcal{N}_T} p_n c_n.
\end{aligned}
\right.$$

Throughout this section we assume that $\boldsymbol{p} \in \mathcal{P}_\rho$, then the last constraint is redundant and can be omitted. We assume that the Slater constraint qualification is satisfied. Then the uniform LP duality in Appendix holds with the dual problem

$$(4.2) \quad \left|
\begin{aligned}
&\sup && \sum_{\pi \in \mathcal{P}_\rho} \mu_\pi \left(\sum_{n \in \mathcal{N}_T} \pi_n c_n \right) \\
&\text{subject to} && \lambda_0 = 1 \\
& && \lambda_n \boldsymbol{z}_n = \sum_{m \in c(n)} \lambda_m \boldsymbol{z}_m && (n \in \mathcal{N}_{-T}) \\
& && \lambda_n \boldsymbol{z}_n = \sum_{\pi \in \mathcal{P}_\rho} \mu_\pi \pi_n \boldsymbol{z}_n && (n \in \mathcal{N}_T) \\
& && \mu_\pi \geq 0 && (\boldsymbol{\pi} \in \mathcal{P}_\rho) \\
& && |\{\boldsymbol{\pi} \in \mathcal{P}_\rho \mid \mu_\pi > 0\}| < \infty.
\end{aligned}
\right.$$

Note that the equality constraints imply that $\sum_{\pi \in \mathcal{P}_\rho} \mu_\pi = 1$ and $\lambda_n = \sum_{\pi \in \mathcal{P}_\rho} \mu_\pi \pi_n$ for $n \in \mathcal{N}_T$. Then letting

$$\tilde{q}_n := \begin{cases} \lambda_n & \text{for } n \in \mathcal{N}_{-T} \\ \sum_{\pi \in \mathcal{P}_\rho} \mu_\pi \pi_n & \text{for } n \in \mathcal{N}_T \end{cases}$$

and $\boldsymbol{c} = (c_n)_{n \in \mathcal{N}_T}$, and using the convexity of $\mathcal{P}_\rho$, the dual problem is equivalent to

$$
(4.3) \quad
\begin{vmatrix}
\sup & \tilde{\boldsymbol{q}}_T^\top \boldsymbol{c} \\
\text{subject to} & \tilde{q}_0 = 1 \\
& \tilde{q}_n \boldsymbol{z}_n = \sum_{m \in c(n)} \tilde{q}_m \boldsymbol{z}_m \quad (n \in \mathcal{N}_{-T}) \\
& \tilde{\boldsymbol{q}}_T \in \mathcal{P}_\rho,
\end{vmatrix}
$$

which is further rewritten as

$$
(4.4) \quad
\begin{vmatrix}
\sup & \mathbb{E}_{\tilde{\boldsymbol{q}}_T}(\boldsymbol{c}) \\
\text{subject to} & \tilde{\boldsymbol{q}}_T \in \mathcal{M}_T \cap \mathcal{P}_\rho,
\end{vmatrix}
$$

where $\mathcal{M}_T := \{\, \tilde{\boldsymbol{q}}_T \mid \tilde{\boldsymbol{q}} \in \mathcal{M} \,\}$.

Lemma 4.2. *Suppose $\boldsymbol{p} \in \mathcal{P}_\rho$ and that the constraints of (4.1) satisfy the Slater constraint qualification. Then the upper bound of claim Y's price is given by the optimal value of the problem (4.4).*

In the same way as in the previous section, we obtain the same result without assuming the Slater constraint qualification when $\mathcal{P}_\rho$ is a polytope.

On the other hand, the buyer pays y_0 in return for a promise of payments y_n at each terminal node $n \in \mathcal{N}_T$, and the exposure of buyer is then $-y_n$. Under the assumption of no ρ-arbitrage, the cost for ρ-hedging this position is no less than initial value of this position. Then the buyer's problem for obtaining the largest initial cost for ρ-hedging the risk may be modeled as the following linear program:

$$
(4.5) \quad
\begin{vmatrix}
\inf & \boldsymbol{z}_0^\top \boldsymbol{\theta}_0 \\
\text{subject to} & \boldsymbol{z}_n^\top (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{a(n)}) = 0 & (n \in \mathcal{N}_{-0}) \\
& \sum_{n \in \mathcal{N}_T} \pi_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \geq -\sum_{n \in \mathcal{N}_T} \pi_n c_n & (\boldsymbol{\pi} \in \mathcal{P}_\rho) \\
& \sum_{n \in \mathcal{N}_T} p_n \boldsymbol{z}_n^\top \boldsymbol{\theta}_n \geq -\sum_{n \in \mathcal{N}_T} p_n c_n.
\end{vmatrix}
$$

Again assuming the Slater constraint qualification we obtain the dual problem:

$$
(4.6) \quad
\begin{vmatrix}
\sup & \mathbb{E}_{\tilde{\boldsymbol{q}}_T}(-\boldsymbol{c}) \\
\text{subject to} & \tilde{\boldsymbol{q}}_T \in \mathcal{M}_T \cap \mathcal{P}_\rho,
\end{vmatrix}
$$

which is equivalent to

$$
(4.7) \quad
\begin{vmatrix}
-\inf & \mathbb{E}_{\tilde{\boldsymbol{q}}_T}(\boldsymbol{c}) \\
\text{subject to} & \tilde{\boldsymbol{q}}_T \in \mathcal{M}_T \cap \mathcal{P}_\rho.
\end{vmatrix}
$$

The optimal value of (4.7) gives the upper bound of initial value $-c_0$. Then we have the following lemma.

Lemma 4.3. *Suppose $\boldsymbol{p} \in \mathcal{P}_\rho$ and that the constraints of (4.5) satisfy the Slater constraint qualification. Then the lower bound of claim Y's price is given by the optimal value of the problem (4.7).*

Combining the two lemmas we obtain the theorem.

Theorem 4.4. *Suppose $\boldsymbol{p} \in \mathcal{P}_\rho$. Suppose that the constraints of (4.1) and (4.5) satisfy the Slater constraint qualification or $\mathcal{P}_\rho$ is a polytope. Then the lower and upper bounds of claim Y's price is given by*

$$
(4.8) \quad \inf\{\mathbb{E}_{\tilde{\boldsymbol{q}}_T}(\boldsymbol{c}) \mid \tilde{\boldsymbol{q}}_T \in \mathcal{M}_T \cap \mathcal{P}_\rho\} \text{ and } \sup\{\mathbb{E}_{\tilde{\boldsymbol{q}}_T}(\boldsymbol{c}) \mid \tilde{\boldsymbol{q}}_T \in \mathcal{M}_T \cap \mathcal{P}_\rho\}.
$$

5. NUMERICAL ILLUSTRATION

In this section, we provide some illustrative examples of pricing a European type contingent claim in a simple setting.

5.1. β-CVaR Pricing. Below are three well-known coherent risk measures $\rho(\boldsymbol{x})$ for net worth $\boldsymbol{x}$ and their corresponding set of probability measure $\mathcal{P}_\rho$. Here we write $\Pi_+ := \{\,\boldsymbol{\pi} \in \mathbb{R}^{n_T} \mid \boldsymbol{\pi} \geq \boldsymbol{0}, \boldsymbol{e}^\top \boldsymbol{\pi} = 1\,\}$ and $[x]^+ = \max\{x, 0\}$.

1) Maximum loss:
   $\rho(\boldsymbol{x}) = \max\{\,-x_n \mid n \in \mathcal{N}_T\,\}$
   $\mathcal{P}_\rho = \Pi_+$.
2) Mean absolute semi-deviation:
   $\rho(\boldsymbol{x}) = \sum_{n \in \mathcal{N}_T} p_n(-x_n) + \lambda \sum_{n \in \mathcal{N}_T} p_n[x_n - \sum_{n \in \mathcal{N}_T} p_n x_n]^+$ with $\lambda \in (0, 1/2)$
   $\mathcal{P}_\rho = \{\,\boldsymbol{\pi} \in \Pi_+ \mid (\lambda + 1)/(2\lambda + 1)\boldsymbol{p} \leq \boldsymbol{\pi} \leq (1 + \lambda)\boldsymbol{p}\,\}$.
3) Conditional value at risk (CVaR):
   $\rho(\boldsymbol{x}) = \min\{\,\alpha + 1/(1 - \beta) \sum_{n \in \mathcal{N}_T} p_n[-x_n - \alpha]^+ \mid \alpha \in \mathbb{R}\,\}$
   $\mathcal{P}_\rho = \{\,\boldsymbol{\pi} \in \Pi_+ \mid \boldsymbol{\pi} \leq \boldsymbol{p}/(1 - \beta)\,\}$.

Theorem 3.7 assumes $\boldsymbol{p} \in \mathcal{P}_\rho$, which may not hold for some coherent risk measures. However it holds for the above three measures. Furthermore all of the $\mathcal{P}_\rho$'s above are polytopes and hence Theorem 3.7 and Theorem 4.4 hold without the Slater constraint qualification. CVaR has a parameter β ranging in $[0, 1)$. The corresponding $\mathcal{P}_\rho$ coincides with Π_+ when $\beta \geq 1 - \min\{\,p_n \mid n \in \mathcal{N}_T\,\}$, and becomes smaller as β decreases. It is worth emphasizing here that the use of CVaR has advantage in that the bounding problems (4.8) result in linear program, for which a series of parametric solutions can be efficiently obtained. Therefore we use CVaR as the risk measure ρ in the following example .

Take a single period example where scenario tree has three terminal nodes (i.e., $T = 1$, $\mathcal{N}_T = \{1, 2, 3\}$). The sets of $\mathcal{M}_T$ and $\mathcal{P}_\rho$ are then shown as in Figure 1 where $\mathcal{M}_T$ is the line segment connecting A and B, $\mathcal{P}_\rho$ is the hexagon and $\mathcal{M}_T \cap \mathcal{P}_\rho$ is the line segment connecting C and D.
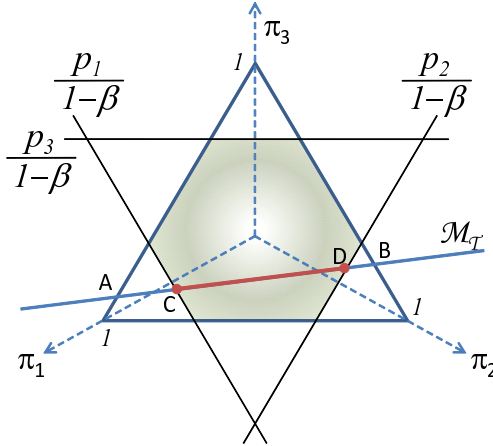

FIGURE 1. $\mathcal{M}_T \cap \mathcal{P}_\rho$ associated with β-CVaR

When $\beta \geq 1 - \min_{n \in \mathcal{N}_t} p_n$, one has $\mathcal{P}_\rho = \Pi_+$, the no β-CVaR arbitrage interval coincides with the no arbitrage interval that calculated by the fundamental theorem of asset price, i.e., Theorem 2.3. The set $\mathcal{P}_\rho$ becomes smaller as β decreases from 1 to 0 and the interval (4.8) becomes smaller as well.

In order to illustrate how the upper and lower bounds shrink, let us provide a numerical example. Figure 2 depicts an uncertain structure of single period market model, where three basic assets and a European call option are traded and four possible future states are considered
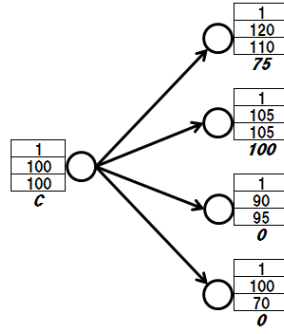
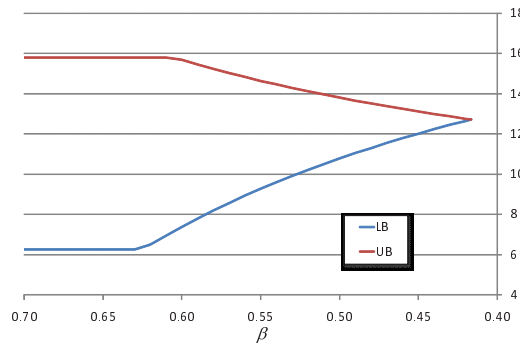FIGURE 2. An Example of Single Period Incomplete Market Model

FIGURE 3. Price Bound Shrinking via CVaR Model
Horizontal axis is the value of β in CVaR. Two bounds converge to 12.7232 at $\beta = 0.416667$.

at each branching. In the figure, the values of the three basic assets are indicated in the boxes attached with each circle which represents a state, while the value of the call option is indicated just below each box in italic letter. Figure 3 shows the ρ-arbitrage interval of the call option, calculated by the β-CVaR model for β from 0.70 to 0.416667, where UB stands for the upper bound, and LB stands for the lower one.

The price interval of the ordinary no-arbitrage theory is $(6.25, 15.7895)$, while that of the extended one shrinks as the value of β decreases. When $\beta = 0.416667$, the no ρ-arbitrage price is unique and 12.7232, which means that if the European call option is traded at 12.7232, the buyer and seller face with the same risk in terms of β-CVaR. In this sense, this value can be considered as a fair price of this option.

5.2. **Pricing with Non-Polyhedral Probability Sets and a Robust Optimization Viewpoint.** In addition to the the previous examples where the probability set $\mathcal{P}_\rho$ is a polytope, we give two non-polyhedral examples of $\mathcal{P}_\rho$.
Relative Entropy Restriction. Suppose $\boldsymbol{p} > \boldsymbol{0}$ and let $I(\boldsymbol{q}; \boldsymbol{p}) := \sum_{n \in \mathcal{N}_T} q_n \ln \frac{q_n}{p_n}$, which is the so-called relative entropy of $\boldsymbol{q}$ with respect to $\boldsymbol{p}$. Let the probability set $\mathcal{P}_\rho$ be given by

$$(5.1) \qquad \mathcal{P}_\rho := \{\, \boldsymbol{q} \in \Pi_+ \mid I(\boldsymbol{q}; \boldsymbol{p}) \leq C \,\},$$

where C is a positive constant. The corresponding problems (4.6) and (4.7) are still convex optimization problems because $I(\,\cdot\,; \boldsymbol{p})$ is a convex function on the unit simplex Π_+.
Ellipsoidal Distance Restriction. For a positive definite matrix G let $E_G(\boldsymbol{q}; \boldsymbol{p}) := (\boldsymbol{q} - \boldsymbol{p})^\top G(\boldsymbol{q} - \boldsymbol{p})$. This is the norm with respect to G as its metric and often denoted by $\|\boldsymbol{q} - \boldsymbol{p}\|_G$. Let

$$(5.2) \qquad \mathcal{P}_\rho := \{\, \boldsymbol{q} \in \Pi_+ \mid E_G(\boldsymbol{q}; \boldsymbol{p}) \leq C \,\}.$$
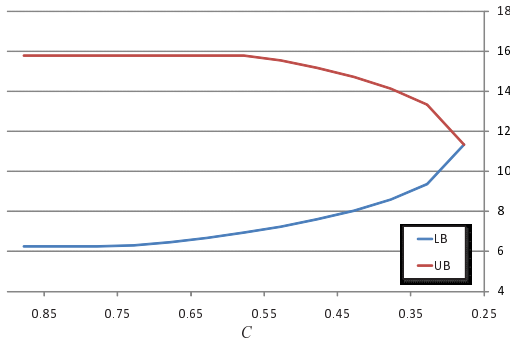
FIGURE 4. Price Bound Shrinking via Entropy Model

Horizontal axis is the value of C in (5.1). Two bounds converge to 11.3258 at $C = 0.277585$.

When G is a diagonal matrix of diagonal elements $(g_n)_{n \in \mathcal{N}_T}$, $E_G(\boldsymbol{q}; \boldsymbol{p}) = \sum_{n \in \mathcal{N}_T} g_n(q_n - p_n)^2$, and it is readily seen that the resulting optimization problem is a second order cone program.

In the above two examples, $\boldsymbol{p}$ lies in $\text{ri}_\Pi \mathcal{P}_\rho$, which ensures the generalized version of the fundamental theorem, Corollary 3.10. At the same time, this construction enables one to see the pricing as a robust optimization (see, e.g., [3]). Namely, the constraints such as $I(\boldsymbol{q}; \boldsymbol{p}) \leq C$ and $E_G(\boldsymbol{q}; \boldsymbol{p}) \leq 1$ prevent the probability $\boldsymbol{q}$ from deviating from the nominal probability $\boldsymbol{p}$, and define convex uncertainty sets. Employing such sets, the hedging problems (4.1) and (4.5) for sellers and buyers can be regarded as robust hedge problems, where the hedging portfolio would be robust against the uncertainty of probability under which expected cash flows of the claim are evaluated. Figure 4 illustrates the upper and lower bounds of the option price in Figure 2 for the case of $\mathcal{P}_\rho$ in (5.1). The two bounds are observed to converge to a single value, 11.3258, which is however different from the value given by CVaR.

6. CONCLUSION

In this paper, we investigate a mathematical programming approach for tightening the bounds of the price of European-type contingent claims in incomplete markets. The traditional fundamental theorem of finance has been described in the perfect hedge framework as in [17], whereas this paper examines how the martingale probability relates to the hedging strategy based on coherent risk measures through a duality theory of (semi-infinite) linear programming. As in the traditional theorem, the existence of a (restricted) martingale probability is shown to be equivalent to the non-existence of a generalized arbitrage through the duality. This kind of existence itself has been shown in the literature, but we fully enjoy the duality of mathematical programming, so that the interpretation would be clear to those who are not so familiar to probability theory.

Also, as discussed in the literature of discrete models, upper and lower bounds of a contingent claim price can be obtained as the optimal values of super-replicating problems for sellers and buyers, respectively. This formulation seems more practical than the perfect replication in the traditional setting since both sellers and buyers cannot escape from taking some risk of a positive loss in actual markets, which are obviously incomplete. In this paper, a numerical example is given where the price bounds shrink at a cost of taking a risk common to the sellers and the buyers. This zero-gap price can be a fair price in the sense that the sellers and the buyers share the same risk. Moreover, due to the duality representation of coherent measures, the proposed pricing can be considered as a robust hedging problem where the expected cash flow would be robust against the deviation of the evaluating probability from the original probability. This observation may provide an interesting link between option pricing in incomplete market and robust optimization.

When it comes to application of the proposed pricing method to real situation, the scenario tree should be constructed so that it fits well the real market. For further details of the tree

construction, see a series of works of Hoyland and Wallace, e.g., [14], where mathematical programming approaches are employed. Although the constructed tree can be of huge size, considering recent developments of algorithms for stochastic programming and computational environment, we believe that the mathematical programming approaches will play a role in evaluating fair prices of even more complicated contingent claims such as path-dependent options and American type options.

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APPENDIX A. LEMMAS

Let $\bar{S}$ be a finite subset of $\mathbb{R}^{n+1}$ and $\bar{T}$ be a compact subset of $\mathbb{R}^{n+1}$. A semi-infinite linear program defined by $\bar{S}$, $\bar{T}$ and $\boldsymbol{c} \in \mathbb{R}^n$ is the problem

$$(P) \quad \left| \begin{array}{ll} \inf & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{s}^\top \boldsymbol{x} = s_0 \quad ((\boldsymbol{s}, s_0) \in \bar{S}) \\ & \boldsymbol{t}^\top \boldsymbol{x} \geq t_0 \quad ((\boldsymbol{t}, t_0) \in \bar{T}). \end{array} \right.$$

The dual problem is

$$(D) \quad \left| \begin{array}{ll} \sup & \sum_{(s,s_0) \in \bar{S}} \lambda_{(s,s_0)} s_0 + \sum_{(t,t_0) \in \bar{T}} \mu_{(t,t_0)} t_0 \\ \text{subject to} & \sum_{(s,s_0) \in \bar{S}} \lambda_{(s,s_0)} \boldsymbol{s} + \sum_{(t,t_0) \in \bar{T}} \mu_{(t,t_0)} \boldsymbol{t} = \boldsymbol{c} \\ & \mu_{(t,t_0)} \geq 0 \quad ((\boldsymbol{t}, t_0) \in \bar{T}) \\ & |\{ (\boldsymbol{t}, t_0) \in \bar{T} \mid \mu_{(t,t_0)} > 0 \}| < \infty. \end{array} \right.$$

Let $v(P)$ and $v(D)$ denote the optimal objective function value of (P) and (D), respectively. Then the weak duality theorem holds. See Theorem (18) in Glashoff and Gustafson [10] for the proof.

Lemma A.1.
$$v(D) \leq v(P).$$

The pair (P) and (D) is said to yield *uniform LP duality* if for each $\boldsymbol{c} \in \mathbb{R}^n$ exactly one of the following cases holds:

(i) $v(P)) = -\infty$ and (D) is infeasible.
(ii) $v(D) = \infty$ and (P) is infeasible.
(iii) Both (P) and (D) are infeasible.
(iv) Both (P) and (D) are feasible and $v(P) = v(D)$, which is attained by some feasible solution of (D).

We say that primal problem (P) satisfies the *Slater constraint qualification* when there is a vector $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ such that

$$\boldsymbol{s}^\top \hat{\boldsymbol{x}} = s_0 \quad ((\boldsymbol{s}, s_0) \in \bar{S})$$
$$\boldsymbol{t}^\top \hat{\boldsymbol{x}} > t_0 \quad ((\boldsymbol{t}, t_0) \in \bar{T}).$$

Combining Theorem 2.2 and Theorem 3.2 of Duffin, Jeroslow and Karlovitz [6] we have the following lemma. See also Gehner [8].

Lemma A.2. *If the primal problem (P) satisfies the Slater constraint qualification, the problem pair (P) and (D) yields uniform LP duality.*

When $s_0 = 0$ for all $(\boldsymbol{s}, s_0) \in \bar{S}$ and $t_0 = 0$ for all $(\boldsymbol{t}, t_0) \in \bar{T}$, the primal and dual problems reduce to

$$(P_0) \quad \left| \begin{array}{ll} \inf & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{s}^\top \boldsymbol{x} = 0 \quad (\boldsymbol{s} \in S) \\ & \boldsymbol{t}^\top \boldsymbol{x} \geq 0 \quad (\boldsymbol{t} \in T) \end{array} \right.$$

and

$$(D_0) \quad \left| \begin{array}{ll} \sup & 0 \\[4pt] \text{subject to} & \sum_{s \in S} \lambda_s \boldsymbol{s} + \sum_{t \in T} \mu_t \boldsymbol{t} = \boldsymbol{c} \\[4pt] & \mu_t \geq 0 \qquad\qquad\qquad\quad (\boldsymbol{t} \in T) \\[4pt] & |\{\, \boldsymbol{t} \in T \mid \mu_t > 0 \,\}| < \infty, \end{array} \right.$$

where S is a finite subset of $\mathbb{R}^n$ and T is a compact subset of $\mathbb{R}^n$. The Slater constraint qualification is the existence of $\hat{\boldsymbol{x}}$ such that

$$\boldsymbol{s}^\top \hat{\boldsymbol{x}} = 0 \quad (\boldsymbol{s} \in S)$$
$$\boldsymbol{t}^\top \hat{\boldsymbol{x}} > 0 \quad (\boldsymbol{t} \in T).$$

By Lemma A.2 we see that the problem pair (P_0) and (D_0) yields uniform LP duality when (P_0) satisfies the Slater constraint qualification. Then we obtain the following lemma.

Lemma A.3. *Suppose (P_0) satisfies the Slater constraint qualification. Then $v(P_0) = 0$ if and only if (D_0) has a feasible solution.*

Lemma A.4. *Let $\boldsymbol{p} = (p_1, \ldots, p_m)^\top$ and $\boldsymbol{q} = (q_1, \ldots, q_m)^\top$ be nonnegative real vectors of $\mathbb{R}^m$ such that*
  1) $\boldsymbol{e}^\top \boldsymbol{p} = \boldsymbol{e}^\top \boldsymbol{q} = 1$,
  2) $q_n > 0$ whenever $p_n > 0$,
where $\boldsymbol{e} \in \mathbb{R}^m$ is a vector of ones. Then for any positive ε there is a positive α such that

$$\left\| \frac{\alpha \boldsymbol{q} - \boldsymbol{p}}{\|\alpha \boldsymbol{q} - \boldsymbol{p}\|_1} - \boldsymbol{q} \right\| < \varepsilon$$

holds, where $\|\cdot\|_1$ denotes the 1-norm and $\|\cdot\|$ denotes any norm of $\mathbb{R}^m$.

Proof. We will show that any real number α such that

$$\alpha > \max \left\{ \frac{\|\boldsymbol{p} - \boldsymbol{q}\|}{\varepsilon} + 1, \ \max \left\{ \frac{p_n}{q_n} \,\middle|\, p_n > 0 \right\} \right\}$$

meets the condition of the lemma. Note first that $\alpha q_n - p_n \geq 0$ since $\alpha > p_n / q_n$ whenever $p_n > 0$, and hence $\|\alpha \boldsymbol{q} - \boldsymbol{p}\|_1 = \boldsymbol{e}^\top (\alpha \boldsymbol{q} - \boldsymbol{p}) = \alpha - 1$. Therefore by the choice of α we see

$$\begin{aligned} \left\| \frac{\alpha \boldsymbol{q} - \boldsymbol{p}}{\|\alpha \boldsymbol{q} - \boldsymbol{p}\|_1} - \boldsymbol{q} \right\| &= \frac{1}{\|\alpha \boldsymbol{q} - \boldsymbol{p}\|_1} \left\| \alpha \boldsymbol{q} - \boldsymbol{p} - \|\alpha \boldsymbol{q} - \boldsymbol{p}\|_1 \boldsymbol{q} \right\| \\[6pt] &= \frac{1}{\alpha - 1} \left\| \alpha \boldsymbol{q} - \boldsymbol{p} - (\alpha - 1) \boldsymbol{q} \right\| \\[6pt] &= \frac{1}{\alpha - 1} \|\boldsymbol{q} - \boldsymbol{p}\| < \varepsilon. \end{aligned}$$

$\square$