

THE CENTRAL ASSIGNMENT GAME AND THE ASSIGNMENT MARKETS*

Mamoru KANEKO

University of Tsukuba, Ibaraki 305, Japan
Yale University, New Haven, CT 06520, USA

Received January 1981, revised version accepted November 1981

Initially this paper considers an assignment game without side payments and proves the non-emptiness of the core of it. Next, a market model with indivisible goods but without the transferable utility assumption is presented, and the non-emptiness of the core and the existence of a competitive equilibrium of the market model are shown, using the first result. Finally this paper presents a generalization of the market model and also shows the non-emptiness of the core and the existence of a competitive equilibrium using the results in the previous model.

1. Introduction

(1) The perfect divisibility of goods is assumed in most usual studies of market economies. If they consider economies where the numbers of units of goods consumed or produced are not small for any economic agents, this assumption would not be inadequate even though the goods are considered to be substantially indivisible, because models with perfect divisibility could be good approximations of such economies. However, if some goods have indivisible and large units and come in small number of units for some economic agents, this perfect divisibility assumption would not be so good. Therefore if we want to consider such a kind of market economy, we should give a model where such goods are treated as indivisible goods. A housing market is considered to be a typical example of such markets.

The model given by Shapley and Shubik (1972) is suited to the study of such a kind of market economy, though it is restrictive.¹ In their model the economic agents consist of sellers and buyers, and indivisible goods are traded for money which is considered to be a composite good. Each seller owns one unit of indivisible good initially and wants to sell it,

*The research in this paper was partially sponsored by the Office of Naval Research Contract no. N00014-77-C-0518.

¹von Böhm-Bawerk (1923) and von Neumann and Morgenstern (1953) already considered this kind of problem, though their models are very restrictive. Telser (1972) also considered it in the same line. Gale (1960) approached it from the dual price of assignment problem.

but never buys any indivisible goods. Each buyer wants to purchase at most one unit of the indivisible goods. This model is generalized by Kaneko (1976a) to the case where sellers may own more than one unit of indivisible goods as initial endowments.

The transferable utility assumption, however, is imposed upon these studies, which makes the analysis of them much easier. The transferable utility assumption does not allow any diminishing marginal utility of money and makes it constant.² In other words it does not permit any income effect. Since the goods have large units in the model, the proportions of the prices of indivisible goods to the initial endowments of money (the income levels) of some economic agents can not be negligible. Therefore it is inadequate to assume that the marginal utility of money is constant for such agents [e.g., buyers in the models of Shapley and Shubik (1972) and Kaneko (1976a)].

(2) The purpose of this paper is to extend the models of Shapley and Shubik (1972) and Kaneko (1976a) to cases without the transferable utility assumption.³ Section 2 provides an abstract generalization of Shapley-Shubik's model, which we call a central assignment game, and proves the non-emptiness of the core of it, showing the balancedness. Section 3 considers the market model given by Shapley and Shubik (1972) without making the transferable utility assumption, and proves the non-emptiness of the core of the market, using the non-emptiness of the core of the central assignment game. Furthermore, the equivalence of the core and the competitive equilibria is shown, which implies the existence of a competitive equilibrium. Section 4 extends the model of section 3 to a case where sellers may own more than one unit of indivisible goods as initial endowments but the transferable utility assumption is imposed upon the sellers. This model is a generalization of that of Kaneko (1976a).⁴ In this model the non-emptiness of the core and the existence of a competitive equilibrium are also shown, and the relationship between them is considered. The equivalence of them does not necessarily hold, but a sufficient condition for it is provided, though it is less general than that of Kaneko (1976a) in the case with the transferable utility assumption. Since the sufficient condition, however, is very weak, it teaches us that the competitive equilibria can be representatives of the core in the market models, which makes our further analysis much easier.⁵

²The transferable utility assumption is explained in Kaneko (1976b) and briefly in section 4 of this paper.

³Recently, Crawford and Knoer (1981) provided a generalization of Shapley-Shubik's model to the case without the transferable utility assumption. They proved the non-emptiness of the core of their model. We will discuss their results briefly in section 2.

⁴Exactly speaking, this market model is neither any generalization of that of section 3 nor that of Kaneko (1976a) in the usual mathematical sense.

⁵In Kaneko (1980), the theory of this paper is applied to a rental housing market, in which several properties of competitive rents are investigated.

2. The central assignment game

This section presents an abstract generalization of Shapley–Shubik’s assignment game and proves the non-emptiness of the core of it.

(3) Consider an $m+n$ -person game $(M \cup N, V) \equiv (\{1, \dots, m\} \cup \{1', \dots, n'\}, V)$ without side payments, where V is a characteristic function from the class of all coalitions to a class of subsets of $R^{M \cup N}$, i.e., $V(S) \subset R^{M \cup N}$ for all $S \subset M \cup N$. Here $R^{M \cup N}$ is the $m+n$ -dimensional Euclidean space whose coordinates are indexed by the members in $M \cup N$. We assume: for all non-empty $S \subset M \cup N$,

$$V(S) \text{ is a closed set in } R^{M \cup N}, \quad (1)$$

$$\text{if } x \in R^{M \cup N} \text{ and } y \in V(S) \text{ with } y_i \geq x_i \text{ for all } i \in S, \\ \text{then } x \in V(S), \quad (2)$$

$$\text{pro}_S \left[V(S) - \bigcup_{i \in S} (\text{interior of } V(\{i\})) \right] \text{ is bounded and non-empty.}^6 \quad (3)$$

We say that a non-empty coalition S can improve upon a vector $x \in V(M \cup N)$ iff there is a vector $y \in V(S)$ such that $y_i > x_i$ for all $i \in S$. The core of $(M \cup N, V)$ is the set of all vectors in $V(M \cup N)$ which cannot be improved upon by any non-empty coalition.

Let

$$\pi = \{S \subset M \cup N : |S| = 1 \text{ or } (|S| = 2, S \cap M \neq \phi, S \cap N \neq \phi)\},$$

where $|S|$ denotes the number of members in S . We call $p_S = \{T_1, \dots, T_k\}$ a π -partition of S iff p_S is a partition of S such that $T_t \in \pi$ for all $t = 1, \dots, k$. Let $P(S)$ be the set of all π -partitions of S .

We call $(M \cup N, V)$ a central assignment game iff V satisfies

$$V(S) = \bigcup_{p_S \in P(S)} \bigcap_{T \in p_S} V(T) \text{ for all } S \subset M \cup N. \quad (4)$$

Definition (4) means that every coalition is subdivided into a partition which consists of singleton sets and pairs of players in M and N . Thus the central assignment game has the same thought as Shapley–Shubik’s game. It is, however, not permitted in ours that any commodity (transferable utility or

⁶ $\text{pro}_S X = \{(x_i)_{i \in S} : x \in X\}$ for all $S \subset M \cup N$ and $X \subset R^{M \cup N}$

⁷Note that V satisfies the super additivity property.

other commodity) is transferred freely in coalitions with more than two members. This is the main difference from Shapley–Shubik’s. In fact, this difference is not important in considering the core concept. Here we explain this and the relations between our model and some related studies, briefly.

An *assignment game* $(M \cup N, v)$ with *side payments* is defined by

$$v(S) = \max_{p_S \in P(S)} \sum_{T \in p_S} v(T) \quad \text{for all } S \subset M \cup N.$$

This game is represented by the following game $(M \cup N, \bar{V})$ without side payments:

$$\bar{V}(S) = \left\{ x \in R^{M \cup N} : \sum_{i \in S} x_i \leq v(S) \right\} \quad \text{for all } S \subset M \cup N.$$

Then this $(M \cup N, \bar{V})$ is not a central assignment game, because transferable utility is transferred freely in coalitions with more than two members, but we can define a new central assignment game $(M \cup N, \bar{V}_0)$ by

$$\bar{V}_0(S) = \bigcup_{p_S \in P(S)} \bigcap_{T \in p_S} \bar{V}(T) \quad \text{for all } S \subset M \cup N.$$

Then the following lemma holds:

Lemma 1. *The core of $(M \cup N, \bar{V})$, i.e., the core of $(M \cup N, v)$, coincides with the core of $(M \cup N, \bar{V}_0)$.*

Proof. Obvious.

This implies that the central assignment game is a generalization of Shapley–Shubik’s game. Furthermore it is easily verified that Gale–Shapley’s (1962) marriage problem and Crawford–Knoer’s (1981) job matching problem are special cases of the central assignment game.

The main result of this section is the following theorem:

Theorem 1. *Every central assignment game has a non-empty core.*

(4) This subsection proves Theorem 1. It is necessary to prepare certain concepts. Let us call a family T of non-empty coalitions of $M \cup N$ *balanced* iff the system of equations

$$\sum_{S: S \ni j} \delta_S = 1 \quad \text{for all } j \in M \cup N, \quad (5)$$

has a non-negative solution with $\delta_S = 0$ for all $S \notin T$. The numbers $\{\delta_S\}$ are

called *balanced weights* for T . A game $(M \cup N, V)$ in characteristic function form is said to be *balanced* iff the following inclusion statement,

$$\bigcap_{S \in T} V(S) \subset V(M \cup N), \tag{6}$$

holds for all balanced families T . The fundamental theorem of Scarf (1967) states that the core of a balanced game $(M \cup N, V)$ with (1), (2) and (3) is non-empty. Hence, it is sufficient to show that every central assignment game $(M \cup N, V)$ is balanced.

For any non-empty coalition S , if $(m+n)$ by $(m+n)$ zero-one matrix $A_S = (a_{S:ij})$, all rows and columns of which: are indexed by members in $M \cup N$, satisfies

$$\begin{aligned} \sum_{i \in M \cup N} a_{S:ij} &= 1 & \text{if } j \in S, & & \sum_{j \in M \cup N} a_{S:ij} &= 1 & \text{if } i \in S, \\ &= 0 & \text{if } i \notin S, & & &= 0 & \text{if } j \notin S, \end{aligned} \tag{7}$$

then we call A_S an S -permutation matrix. We define $D_S(b) = (d_{S:ij}(b))$ for each b in $R^{M \cup N}$ and each non-empty coalition S as follows:

$$\begin{aligned} d_{S:ij}(b) &= 1 & \text{if } i \in S \cap M, j \in N, b \in V(\{i, j\}), \\ &= 1 & \text{if } i \in S \cap M, j \in M, b \in V(\{i\}), \\ &= 1 & \text{if } i \in S \cap N, j \in N, b \in V(\{j\}), \\ &= 1 & \text{if } i \in S \cap N, j \in M, \\ &= 0 & \text{otherwise.} \end{aligned} \tag{8}$$

Lemma 2. $V(S) = \{b \in R^{M \cup N} : D_S(b) \geq A_S \text{ for some } S\text{-permutation matrix } A_S\}$ for all $S \subset M \cup N$.

Proof. See the appendix.

Now we are in a position to prove that $(M \cup N, V)$ is a balanced game. The following proof is almost the same as that of the theorem of Shapley and Scarf (1974) that the core is non-empty in the market model where only indivisible goods are exchanged. But as our game is more complicated, we give the proof of it for mathematical completeness.

Proof of the Balancedness. Let T be an arbitrary balanced family of coalitions, and let $b \in \bigcap_{S \in T} V(S)$. Let $\{\delta_S\}$ be balancing weights for T . Then it

holds that

$$D_{M \cup N}(b) = \sum_{S \in T} \delta_S D_S(b).$$

For, if $d_{M \cup N:ij}(b) = 1$, then $d_{S:ij}(b) = 1$ if $i \in S$ and $d_{S:ij}(b) = 0$ if $i \notin S$ by (8), which implies

$$\sum_{S \in T} \delta_S d_{S:ij}(b) = \sum_{\substack{S \in T \\ S \ni i}} \delta_S = 1,$$

and if $d_{M \cup N:ij}(b) = 0$, then $d_{S:ij}(b) = 0$ for all $S \in T$, which implies

$$\sum_{S \in T} \delta_S d_{S:ij}(b) = 0.$$

Since $b \in V(S)$ for each $S \in T$, there is an S -permutation matrix A_S by Lemma 2 such that $D_S(b) \geq A_S$, and so

$$\sum_{S \in T} \delta_S D_S(b) \geq \sum_{S \in T} \delta_S A_S.$$

Call the matrix on the right B ; then we have

$$D_{M \cup N}(b) \geq B.$$

The crucial fact about B is that it is doubly stochastic, i.e., it is non-negative and has all row- and column-sums equal to 1. This follows directly from the definition of balancing weights; in fact, the j th column sum is

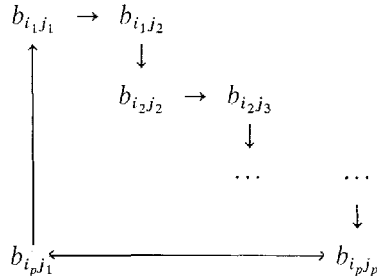
$$\begin{aligned} \sum_{i \in M \cup N} \sum_{S \in T} \delta_S a_{S:ij} &= \sum_{S \in T} \delta_S \sum_{i \in M \cup N} a_{S:ij} \\ &= \sum_{S \in T} \delta_S = 1 \quad \text{if } j \in S, \\ &= 0 \quad \text{if } j \notin S, \\ &= \sum_{\substack{S \in T \\ S \ni j}} \delta_S = 1, \end{aligned}$$

and the argument for the row-sum is the same.

The next step is to change B into an $M \cup N$ -permutation matrix $A_{M \cup N}$, i.e., to eliminate any fractional entries without changing the row- or column-sums and to do so without disturbing any entries which are already 0 or 1.

Since all entries of $D_{M \cup N}(b)$ are 0 or 1, we will thereby ensure that $D_{M \cup N}(b) \geq A_{M \cup N}$, which implies $b \in V(M \cup N)$.

Since a fraction cannot occur alone in any row- or column, either B is already an $M \cup N$ -permutation matrix or there is a closed loop of fractional entries:



Alternately adding and subtracting a fixed number $\varepsilon > 0$ to the elements of this loop will clearly preserve row- and column-sums. If ε is too large, the negative entries will be created, but making ε as large as possible consistent with non-negativity will produce a new doubly stochastic matrix B' that has at least one more zero than B , and hence fewer fractional entries. If B' is not yet an $M \cup N$ -permutation matrix, we repeat the same operation. Eventually we can obtain what we want — an $M \cup N$ -permutation matrix $A_{M \cup N}$ such that $D_{M \cup N}(b) \geq A_{M \cup N}$. Q.E.D.

3. The assignment market

This section reformulates the market model of Shapley and Subik (1972) without making the transferable utility assumption and shows that there exist a non-empty core and a competitive equilibrium in the market.

(5) Consider a market consisting of players $M = \{1, \dots, m\}$ and $N = \{1', \dots, n'\}$. A player in M may be called a *seller* and one in N a *buyer* in the following. In this market s -kinds of *indivisible goods* are exchanged for *money*. A seller owns exactly one unit of indivisible good before trade. Hence M can be subdivided into $M_1 \cup M_2 \cup \dots \cup M_s$, where $i \in M_t$ ($t = 1, \dots, s$) owns one unit of the t th indivisible good. Without loss of generality, we can assume that $M_t \neq \emptyset$ for all $t = 1, \dots, s$, and $M_t = \{m_{t-1} + 1, \dots, m_t\}$ for all $t = 2, \dots, s$, where $0 < m_1 < m_2 < \dots < m_s = m$. No buyer owns any indivisible goods before trade. Each player $i \in M \cup N$ owns $I_i > 0$ amount of money initially, where money is perfectly divisible and should be interpreted as a composite good. That is, player i 's *initial endowment* is (e^i, I_i) if $i \in M_t$ and

$(0, I_i)$ if $i \in N$, where e^t is the s -dimensional unit vector with $e_t^t = 1$. For simplicity, we denote i 's initial endowment by (a^i, I_i) ($i \in M \cup N$).

Each player $i \in M \cup N$ has a *preference relation* R_i on the *consumption set* $X \equiv I_+^s \times R_+$, where I_+ is the set of all non-negative integers, I_+^s the direct product of s number of I_+ and R_+ the set of all non-negative real numbers. $(x, m) \in X$ means amounts of the indivisible goods and money to be consumed. We assume that every R_i ($i \in M \cup N$) is a weak ordering. $(x, m_1) R_i (y, m_2)$ means that player i prefers (x, m_1) to (y, m_2) or is indifferent between them. We define i 's *strict preference* P_i by $(x, m_1) P_i (y, m_2) \Leftrightarrow \text{not } (y, m_2) R_i (x, m_1)$ and i 's *indifference relation* Q_i by $(x, m_1) Q_i (y, m_2) \Leftrightarrow (x, m_1) R_i (y, m_2) \ \& \ (y, m_2) R_i (x, m_1)$. We assume — for all $i \in M \cup N$:

- (A) *Monotonicity with respect to money.* If $m_1 > m_2$, then $(x, m_1) P_i (x, m_2)$ for all $x \in I_+^s$.
- (B) *Archimedean property.* If $(x, m_1) P_i (y, m_2)$, then there is an m_3 such that $(x, m_1) Q_i (y, m_3)$.

It would not be necessary to explain the meanings of these assumptions.

Lemma 3. If $(x, m) R_i (0, 0)$ for all $(x, m) \in X$, then there is a continuous utility function U^i on X , i.e., $(x, m_1) R_i (y, m_2)$ iff $U^i(x, m_1) \geq U^i(y, m_2)$, where the relative topology of the $(s+1)$ -dimensional Euclidean space R^{s+1} is introduced into $X = I_+^s \times R_+$.

Proof. See the appendix.

We make the following assumptions separately on the sellers and the buyers — for all sellers $i \in M_t$ ($t = 1, \dots, s$):

- (CS) *Satiation.* For each $(x, m) \in X$, if $x_t = 0$, then $(x, m) Q_i (0, m)$ and if $x_t \geq 1$, then $(x, m) Q_i (e^t, m) P_i (0, m)$.

And for all $i \in N$:

- (CB) *Satiation.* For each $(x, m) \in X$, if $x_t \geq 1$ for some t and $(e^t, m) R_i (e^k, m)$ for all k with $x_k \geq 1$, then $(x, m) Q_i (e^t, m) P_i (0, m)$.

The last assumption is:

- (D) For all $i \in N$, $(0, I_i) P_i (x, 0)$ for all $x \in I_+^s$.

It is noted that assumptions (CS) and (CB) are stronger than the supposition of Lemma 3, which implies that Lemma 3 holds under these assumptions.

Assumption (CS) means that even if seller $i \in M_i$ has indivisible goods other than the t th good or more than one unit of the t th good, then his utility does not rise more than that of the initial state. It follows from this assumption that sellers never buy any indivisible goods. Assumption (CB) means that even if buyer i consumes more than one unit of indivisible goods, then his utility does not rise more than that of having one unit of the most preferred one in the goods. More precisely, if the buyer purchases one unit of a more preferred good as the second unit, his utility increase, but it is indifferent from this to purchase the good as the first unit. This assumption implies that buyers never purchase more than one unit if all the prices of the goods are positive. Assumption (D) means that any buyer purchases no indivisible good by paying all his income. When the marginal utility of money at $(x, 0)$ is very large and the income I_i is not too small, this assumption is satisfied. Since assumption (CS) implies that no seller buys any indivisible goods, i.e., lower his amount of money, it is not necessary to consider vectors $(x, 0)$ in the case of sellers. Therefore assumption (D) is imposed only upon the buyers.

These assumptions (CS) and (CB) seem to look strange at a first glance and are unfamiliar to one having the knowledge of the standard equilibrium theory. Then we should justify these assumptions. Consider an example of a market with s kinds of houses for rent, and consider two cases where a buyer (household) i rents one unit of k th house and where he rents two units of the house.⁸ These states are represented as (e^k, m_1) and $(2e^k, m_2)$, where m_1 and m_2 are the amounts of money after paying the rents for the house(s), respectively. It is natural to suppose that there is a non-negative real number ε such that

$$(e^k, m_1) Q_i (2e^k, m_1 - \varepsilon).$$

This ε is the maximal amount that i pays for the 2nd unit of the house when he has rented already one unit of it. When the k th house is not small to live in for him, the real number ε could be considered to be small relative to the rent of the house. Then it is reasonable and convenient to assume that the number ε is zero. This is an assumption for idealization. Next consider the case where $(e^t, m_1) R_i (e^k, m_2)$ and $x = e^t + e^k$. Then it is natural to assume that $(x, m_1) R_i (e^t, m_1)$, and that $(x, m_1 - \varepsilon) Q_i (e^t, m_1)$ for some $\varepsilon \geq 0$. By the same reasoning as the above, this ε could be assumed to be zero. We also can explain the meaning of (CS) in almost the same way.

Assumption (D) is justified in the context of the above rental housing market as follows. Our market model is a partial equilibrium model, which focus our consideration on the indivisible goods (rental housings) in question, and 'money' is the composite good of all other commodities which are not considered explicitly in our model. Buyer i 's consumption vector $(x^i, 0)$ means

⁸A rental housing market of this type is considered in Kaneko (1980).

that he rents x^i units of houses but does not consume any commodities. Conversely, $(0, I_i)$ means that he does not rent any house in question but can consume some commodities, using his income I_i . This means that he can rent a house (an apartment) or can live in a hotel which are not included in our market model. Thus $(0, I_i)$ is a normal state but $(x^i, 0)$ an abnormal one for an individual life. Therefore it is innocuous to assume (D).

The above formulation is a reformulation of Shapley–Shubik’s market model without making the transferable utility assumption. We call the above market model an *assignment market*.

(6) Now we are in a position to discuss the main theme of this section.

For any non-empty coalition S , we call $(x^S, m^S) = ((x^i, m^i))_{i \in S}$ an *S-allocation* iff

$$(x^i, m^i) \in X \quad \text{for all } i \in S, \quad (9)$$

$$\sum_{i \in S} (x^i, m^i) = \sum_{i \in S} (a^i, I_i). \quad (10)$$

We call an $M \cup N$ -allocation simply an *allocation*. We say that a non-empty coalition S can improve upon an allocation $(x^{M \cup N}, m^{M \cup N})$ iff there is an S -allocation (y^S, m_1^S) such that

$$(y^i, m_1^i) P_i(x^i, m^i) \quad \text{for all } i \in S. \quad (11)$$

The *core* of the assignment market is the set of all allocations which can not be improved upon by any non-empty coalition.

The characteristic function V of the assignment market is defined in the usual manner as follows — for all $S \subset M \cup N$:

$$V(S) = \{b \in R^{M \cup N} : \text{for some } S\text{-allocation } (x^S, m^S), \\ b_i \leq U^i(x^i, m^i) \text{ for all } i \in S\}. \quad (12)$$

It is easily verified that for any b in the core of $(M \cup N, V)$, there is an allocation $(x^{M \cup N}, m^{M \cup N})$ in the core of the assignment market such that $U^i(x^i, m^i) \geq b_i$ for all $i \in M \cup N$, and conversely that $(U^i(x^i, m^i))_{i \in M \cup N}$ belongs to the core of $(M \cup N, V)$ for any allocation $(x^{M \cup N}, m^{M \cup N})$ in the core of the assignment market. Hence the non-emptiness of the core of $(M \cup N, V)$ is equivalent to that of the core of the assignment market. So, we show the non-emptiness of the core of $(M \cup N, V)$.

We define another characteristic function V_0 using V as follows — for all non-empty $S \subset M \cup N$:

$$V_0(S) = \bigcup_{P_S \in P(S)} \bigcap_{T \in P_S} V(T). \quad (13)$$

Of course the new game $(M \cup N, V_0)$ is a central assignment game. Hence the core of $(M \cup N, V_0)$ is non-empty by Theorem 1.⁹ Although V_0 is different from V , the following relations hold:

$$\begin{aligned} V_0(S) &\subset V(S) \text{ for all } S \subset M \cup N, \\ &= V(S) \text{ for all } S \in \pi. \end{aligned} \tag{14}$$

Moreover we can prove:

Theorem 2. The core of $(M \cup N, V_0)$ coincides with the core of $(M \cup N, V)$.

From Theorems 1 and 2 and the above remark, we get:

Theorem 3. The cores of $(M \cup N, V)$ and the assignment market are non-empty.

Next let us consider the relationship between the core and the competitive equilibria in the assignment market. We call a pair $(p, (x^{M \cup N}, m^{M \cup N}))$ of a price vector $p = (p_1, \dots, p_s) \in R_+^s$ and an allocation $(x^{M \cup N}, m^{M \cup N})$ a *competitive equilibrium* iff

$$\begin{aligned} &\text{for all } i \in M \cup N, (x^i, m^i) R_i (y^i, m_1^i) \text{ for all } (y^i, m_1^i) \in X \\ &\text{such that} \quad py^i + m_1^i \leq I_i + pa^i, \end{aligned} \tag{15}$$

$$\text{for all } i \in M \cup N, px^i + m^i = I_i + pa^i. \tag{16}$$

We call $(x^{M \cup N}, m^{M \cup N})$ a *competitive allocation* iff there is a price vector such that $(p, (x^{M \cup N}, m^{M \cup N}))$ is a competitive equilibrium, and p a *competitive price vector*.

Shapley and Shubik (1972) show that the core always coincides with the set of all competitive allocations in the assignment market with the transferable utility assumption. This theorem is true even in the assignment market without the transferable utility assumption.

Theorem 4. The core coincides with the set of all competitive allocations in the assignment market.

From this theorem and Theorem 3, we get:

Theorem 5. There exists a competitive equilibrium in the assignment market.

⁹It is clear that V_0 satisfies conditions (2) and (3). It follows from Lemma 3 that V_0 satisfies condition (1).

Let us summarize the results which have been demonstrated:

- (1) Every central assignment game has a non-empty core (Theorem 1).
- (2) The core of the assignment market is equivalent to the core of $(M \cup N, V)$ in the sense mentioned in this subsection.
- (3) The core of $(M \cup N, V_0)$ coincides with the core of $(M \cup N, V)$ (Theorem 2).
- (5) The core coincides with the set of all competitive allocations in the assignment market (Theorem 4).

Therefore we have:

- (6) There exists a competitive equilibrium in the assignment market (Theorem 5).

The logical relations are illustrated in fig. 1.

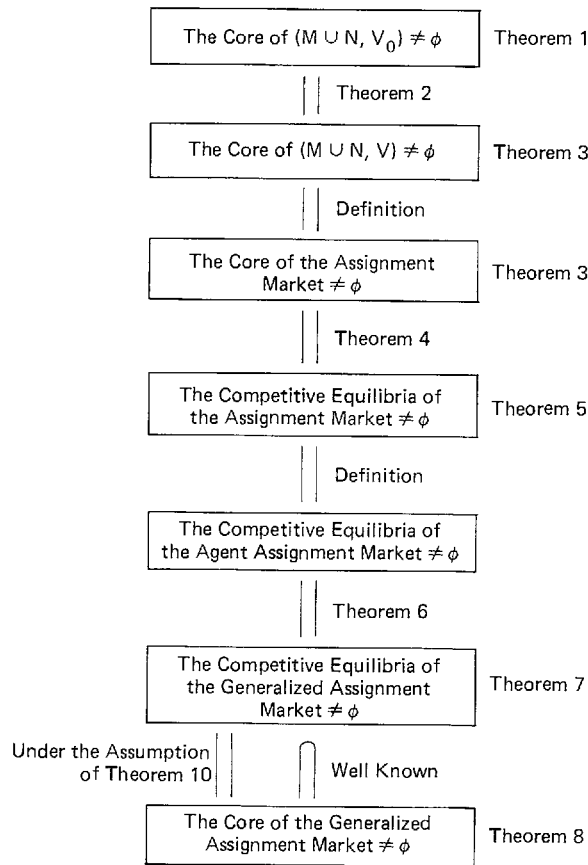


Fig. 1

A distinctive feature of the assignment market is bilateral exchange between sellers and buyers. The core concept represents this feature very well (see also Corollary 7), but the competitive equilibrium does not. Furthermore we do not make any assumption of 'largeness' on the market but allow commodity differentiation. Therefore the competitive equilibrium could not be justified in the standard way. However, we have shown the equivalence of the core and the competitive equilibria. One practical advantage of the competitive equilibrium is that the definition of it is much easier and more manageable than that of the core. Therefore our result implies that for a practical purpose it is enough to consider the competitive equilibrium in the assignment market.

(7) This subsection proves Theorems 2 and 4. The following lemmas are necessary to prove Theorem 2. The proofs of them are given in the appendix.

Lemma 4. For any allocation $(x^{M \cup N}, m^{M \cup N})$, if a non-empty coalition S satisfies

$$\sum_{i \in S} (a^i, I_i) \geq \sum_{i \in S} (x^i, m^i) \quad \text{and} \quad m^i > 0 \quad \text{for all } i \in S, \quad (17)$$

then S can improve upon the allocation $(x^{M \cup N}, m^{M \cup N})$. Here $x \geq y$ means $x \geq y$ but $x \neq y$.

Lemma 5. Let $(x^{M \cup N}, m^{M \cup N})$ belong to the core of the assignment market. Then there are partitions of M and N such that $M = \{i_1, \dots, i_k\} \cup M_0$, $N = \{j_1, \dots, j_k\} \cup N_0$ and

$$(x^{i_t}, m^{i_t}) + (x^{j_t}, m^{j_t}) = (a^{i_t}, I_{i_t}) + (a^{j_t}, I_{j_t}) \quad \text{and} \\ x^{i_t} = 0 \quad \text{for all } t = 1, \dots, k, \quad (18)$$

$$(x^i, m^i) = (a^i, I_i) \quad \text{for all } i \in M_0 \cup N_0. \quad (19)$$

Note that $x^{j_t} = a^{j_t}$ for all $t = 1, \dots, k$ by (18).

The proof of Lemma 5 implies the following corollary:

Corollary 6. Under the supposition of Lemma 5, for all $i \in M_t \cap \{i_1, \dots, i_k\}$ and $j \in \{j_1, \dots, j_k\}$ with $x^j = e^j$, it holds that

$$(x^i, m^i) + (x^j, m^j) = (a^i, I_i) + (a^j, I_j).$$

Proof of Theorem 2. Let b be in the core of $(M \cup N, V)$. Since b can not be

improved upon in $(M \cup N, V_0)$ by (14), it is sufficient to show that b is in $V_0(M \cup N)$. There is an allocation $(x^{M \cup N}, m^{M \cup N})$ such that $U^i(x^i, m^i) \geq b_i$ for all $i \in M \cup N$. Note that this allocation is in the core of the assignment market. Let the partitions of M and N given in Lemma 5 be $M = \{i_1, \dots, i_k\} \cup M_0$ and $N = \{j_1, \dots, j_k\} \cup N_0$. We put $M_0 = \{i(1), \dots, i(g)\}$ and $N_0 = \{j(1), \dots, j(h)\}$. We define a π -partition $p_{M \cup N} = \{T_1, \dots, T_f\}$ ($f = k + g + h$) as follows:

$$T_t = \{i_t, j_t\} \quad \text{for all } t = 1, \dots, k,$$

$$T_{k+t} = \{i(t)\} \quad \text{for all } t = 1, \dots, g,$$

$$T_{k+g+t} = \{j(t)\} \quad \text{for all } t = 1, \dots, h.$$

By Lemma 5 it holds that $b \in V(T)$ for all $T \in p_{M \cup N}$, i.e., $b \in \bigcap_{T \in p_{M \cup N}} V(T)$. This means that b is in $V_0(M \cup N)$.

Conversely, we show that any b in the core of $(M \cup N, V_0)$ belongs to the core of $(M \cup N, V)$. Suppose b is in the core of $(M \cup N, V_0)$. By (14) b is in $V(M \cup N)$. We prove that if b can be improved upon by a non-empty coalition in the game $(M \cup N, V)$, then b can be also improved upon by some non-empty coalition in the game $(M \cup N, V_0)$. So, we will complete the proof.

Since b is in $V_0(M \cup N)$, there is a π -partition $p_{M \cup N}$ of $M \cup N$ such that $b \in \bigcap_{T \in p_{M \cup N}} V(T)$. We put the set of pairs

$$\{T: |T|=2 \text{ and } T \in p_{M \cup N}\} = \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\},$$

and

$$M_0 = M - \{i_1, \dots, i_k\} \quad \text{and} \quad N_0 = N - \{j_1, \dots, j_k\}.$$

Then it holds that

$$b \in V(\{i_t, j_t\}) \quad \text{for all } t = 1, \dots, k,$$

$$\in V(\{i\}) \quad \text{for all } i \in M_0 \cup N_0.$$

Hence there is an allocation $(x^{M \cup N}, m^{M \cup N})$ such that

$$(x^{i_t}, m^{i_t}) + (x^{j_t}, m^{j_t}) = (a^{i_t}, I_{i_t}) + (a^{j_t}, I_{j_t}) \quad \text{for all } t = 1, \dots, k,$$

$$(x^i, m^i) = (a^i, I_i) \quad \text{for all } i \in M_0 \cup N_0,$$

$$U^i(x^i, m^i) \geq b_i \quad \text{for all } i \in M \cup N.$$

Suppose that $U^i(x^i, m^i) > b_i$ for some $i \in M \cup N$. If $i \in M_0 \cup N_0$, then

$$b_i < U^i(x^i, m^i) = \sup \text{pro}_{(i)} V(\{i\}) = \sup \text{pro}_{(i)} V_0(\{i\}),$$

which is a contradiction to the supposition that b is in the core of $(M \cup N, V_0)$. Let $i = j_t$ ($t \leq k$). If $m^h = 0$, then

$$U^j(x^j, m^h) < U^j(a^j, I_j) = \sup \text{pro}_{(j)} V_0(\{j_t\}),$$

by assumption (D), which is a contradiction. Then we have $m^h > 0$. So, we can choose a positive real number ε by Lemma 3 and assumption (A) such that

$$U^h(x^h, m^h - \varepsilon) > b_{j_t} \quad \text{and} \quad U^i(x^i, m^i + \varepsilon) > b_{i_t}.$$

Since $(U^i(x^i, m^i + \varepsilon), U^j(x^j, m^h - \varepsilon)) \times (b_j)_{j \neq i, j_t}$ belongs to $V(\{i, j_t\}) = V_0(\{i, j_t\})$, $\{i, j_t\}$ can improve upon b in the game $(M \cup N, V_0)$, which is a contradiction. Let $i = i_t$ ($t \leq k$). Then $U^i(x^i, m^i) \geq U^i(a^i, I_i)$ together with assumptions (A) and (CS) implies $m^i \geq I_i > 0$. So, we can get a contradiction similarly with the above if $U^i(x^i, m^i) > b_{i_t}$. Hence it holds that

$$U^i(x^i, m^i) = b_i \quad \text{for all } i \in M \cup N.$$

Suppose that a non-empty coalition S can improve this allocation $(x^{M \cup N}, m^{M \cup N})$ in the assignment market, which is equivalent to that S can improve upon b in the game $(M \cup N, V)$. This means that there is an S -allocation (y^S, m_1^S) such that

$$U^i(y, m_1) > U^i(x^i, m^i) \quad \text{for all } i \in S. \quad (20)$$

We choose a buyer $j \in S$ such that $y_j^j \geq 1$ and $U^j(e^j, m_1^j) = U^j(y^j, m_1^j)$ for some $f \leq s$, which implies that there is a seller $i \in S \cap M_f$ with $y_f^i = 0$. This choice is always possible because if not, (20) can not hold. Of course, it holds that $y^i + y^j \geq a^i + a^j$. If $m_1^i + m_1^j \leq I_i + I_j$, then the coalition $\{i, j\}$ can improve upon $(x^{M \cup N}, m^{M \cup N})$ by (20) and the choice of i , which is a contradiction to the supposition that b is in the core of $(M \cup N, V_0)$. Hence we have $m_1^i + m_1^j > I_i + I_j$, which implies

$$\sum_{t \in S - \{i, j\}} y^t \leq \sum_{t \in S - \{i, j\}} a^t \quad \text{and} \quad \sum_{t \in S - \{i, j\}} m_1^t < \sum_{t \in S - \{i, j\}} I_t.$$

It follows that if $S - \{i, j\} \neq \emptyset$, $S - \{i, j\}$ can improve upon $(x^{M \cup N}, m^{M \cup N})$, i.e., there is an $(S - \{i, j\})$ -allocation $(z^{S - \{i, j\}}, m_2^{S - \{i, j\}})$ such that

$$U^t(z^t, m_2^t) > U^t(y^t, m_1^t) > U^t(x^t, m^t) \quad \text{for all } t \in S - \{i, j\}.$$

This is (20) for the case of $S - \{i, j\}$ and $S - \{i, j\}$ is two members less than S . If $S = \{i, j\}$, then $\{i, j\}$ can improve upon b in the game $(M \cup N, V_0)$. If $S \neq \{i, j\}$, then, repeating the same argument as the above, we can reach a T -allocation (w^T, m_0^T) such that $|T| \leq 2$ and $U^t(w^t, m_0^t) > U^t(x^t, m^t)$ for all $t \in T$. Hence b can be improved upon in the game $(M \cup N, V_0)$. Q.E.D.

The latter part of the proof of Theorem 2 implies the following corollary:

Corollary 7. An allocation $(x^{M \cup N}, m^{M \cup N})$ is in the core of the assignment market iff $(x^{M \cup N}, m^{M \cup N})$ can not be improved upon by any coalition S in π .

It should be noted that since any allocation in the core of the assignment market is the sum of partial allocations as shown in Lemma 5, we do not need to consider the grand coalition $M \cup N$.

This corollary has an important implication. It is often said that the core-theory neglects costs of coalition-formations and that as costs of forming large coalitions are very large, it is implausible to assume to be able to bargain freely. This author readily agrees on this criticism in general. However, this corollary implies that this criticism is not persuasive at least in the assignment market because the same result is derived even if only the coalitions in π are permitted and the coalitions in π are very small. Therefore the core of the assignment market is free from this criticism.

Proof of Theorem 4. It can be shown in the well-known manner that the competitive allocations are included by the core. Hence we need to show the converse inclusion.

Let $(x^{M \cup N}, m^{M \cup N})$ be in the core. Let us consider the partition given by Lemma 5. If there is a seller $i \in \{i_1, \dots, i_k\} \cap M_f$ and a buyer $j \in \{j_1, \dots, j_k\}$ such that $x^j = e^f$, $m^j = I_j - r$, $m^i = I_i + r'$ and $r \neq r'$, then we have $m^i + m^j \neq I_i + I_j$, which is a contradiction to Corollary 6. Hence it holds that if there is a buyer $j \in \{j_1, \dots, j_k\}$ with $x^j = e^f$, there is a real number r_f such that

$$\begin{aligned} m^i &= I_i + r_f \quad \text{for all } i \in M_f \cap \{i_1, \dots, i_k\}, \\ m^j &= I_j - r_f \quad \text{for all } j \in \{j_1, \dots, j_k\} \quad \text{with } x^j = e^f. \end{aligned}$$

If $r_f \leq 0$, then $U^i(e^f, I_i) > U^i(0, I_i + r_f)$ for all $i \in M_f \cap \{i_1, \dots, i_k\}$ by assumption (CS), which is a contradiction. Hence we have $r_f > 0$.

We define $p = (p_1, \dots, p_s)$ by

$$\begin{aligned} p_f &= r_f && \text{if there is a buyer } j \in \{j_1, \dots, j_k\} \quad \text{with } x^j = e^f, \\ &= \min_{i \in M_f} q_f(i) && \text{otherwise,} \end{aligned} \quad (21)$$

where $q_f(i)$ is defined by $U^i(e^f, I_i) = U^i(0, I_i + q_f(i))$ for all $i \in M_f$ ($f = 1, \dots, s$). The existence and uniqueness of $q_f(i)$ are ensured by assumptions (A) and (B). It is clear by assumption (CS) that $q_f(i) > 0$ for all $i \in M_f$. Hence $p_f > 0$ for all $f \leq s$. It can easily be verified that the budget constraint (16) follows from (21) and Corollary 6. Hence we need only to show (15).

Suppose that there is a $(y^i, m_1^i) \in X$ for some $i \in M \cup N$ such that $U^i(y^i, m_1^i) > U^i(x^i, m^i)$ and $py^i + m_1^i \leq I_i + pa^i$. First, let $i \in M_f$. In this case it is sufficient to assume $y^i = 0$ by assumption (CS) and the individual rationality $U^i(x^i, m^i) \geq U^i(a^i, I_i)$. Then we have $p_f > \min_{i \in M_f} q_f(i)$, because otherwise, $U^i(0, m_1^i) = U^i(0, I_i + p_f) \leq U^i(a^i, I_i) \leq U^i(x^i, m^i)$, which is a contradiction. If $i \in M_f \cap \{i_1, \dots, i_k\}$, then the existence of such a (y^i, m_1^i) is impossible by (CS), (21) and Corollary 6. So, $i \in M_0 \cap M_f$. Since $p_f > \min_{i \in M_f} q_f(i)$, there is a buyer $j \in \{j_1, \dots, j_k\}$ with $x^j = e^f$. For a sufficiently small real number $\varepsilon > 0$, it holds by assumption (A) that

$$U^i(0, I_i + p_f - \varepsilon) > U^i(x^i, m^i),$$

$$U^j(x^j, I_j - p_f + \varepsilon) > U^j(x^j, m^j).$$

That is, $\{i, j\}$ can improve upon $(x^{M \cup N}, m^{M \cup N})$, which is a contradiction. Second, let $i \in N$. Then we can assume $(y^i, m_1^i) = (e^f, I_i - p_f)$ for some $f \leq s$. Since $U^i(e^f, I_i - p_f) > U^i(x^i, m^i) \geq U^i(a^i, I_i)$, we have $I_i - p_f > 0$ by assumption (D). Hence if $M_f \cap \{i_1, \dots, i_k\} \neq \emptyset$, then for any $j \in M_f \cap \{i_1, \dots, i_k\}$, it holds that for some $\varepsilon > 0$

$$U^i(e^f, I_i - p_f - \varepsilon) > U^i(x^i, m^i),$$

$$U^j(0, I_j + p_f + \varepsilon) > U^j(0, I_j + p_f) = U^j(x^j, m^j),$$

which is a contradiction. Next, if $M_f \cap \{i_1, \dots, i_k\} = \emptyset$, then for a seller j with $q_f(j) = \min_{t \in M_f} q_f(t)$,

$$U^i(e^f, I_i - p_f - \varepsilon) > U^i(x^i, m^i),$$

$$U^j(0, I_j + p_j + \varepsilon) > U^j(0, I_j + q_f(j)) = U^j(a^j, I_j) = U^j(x^j, m^j),$$

because $p_f = q_f(t)$ and $(a^j, I_j) = (x^j, m^j)$ by the assumption that $M_f \cap \{i_1, \dots, i_k\} = \emptyset$, where ε is a sufficiently small positive number. This is a contradiction. Q.E.D.

4. The generalized assignment market

(8) This section considers a generalization of the assignment market, in which each seller $i \in M_k$ ($k = 1, \dots, s$) is permitted to own more than one unit

of the k th individual good initially. That is, his initial endowment of the individual goods is $w_i e^k$ in the model of this section, where w_i may be any positive integer. We assume — for all $i \in M_k$ ($k=1, \dots, s$):

(CS') *Satiation.* If $(x, m) Q_i(x_k e^k, m)$ for all $(x, m) \in X$ and if $x_k \geq w_i$ then

$$(x, m) Q_i(w_i e^k, m) P_i((w_i - 1)e^k, m) P_i \dots P_i(e^k, m) P_i(0, m).$$

We can justify this assumption by the similar reasoning to that of assumption (CB).

We impose the transferable utility assumption on the sellers $i \in M$:

(E) *Constant marginal utility of money.* If $(x, m_1) Q_i(y, m_2)$, then

$$(x, m_1 + \delta) Q_i(y, m_2 + \delta) \quad \text{for all } \delta > 0.$$

Kaneko (1976b) shows that assumptions (A), (B) and (E) imply that

there is a real-valued function $u^i(x)$ on I^s such that

$$(x, m_1) R_i(y, m_2) \quad \text{iff} \quad u^i(x) + m_1 \geq u^i(y) + m_2. \quad (22)$$

Assumption (E) would not necessarily be strong for the sellers. This assumption states that there is no income effect in the sellers' utility functions. It would not be inadequate to assume no income effect in the domain where the amount of money is not small. Each seller's amount of money can be only increased by selling his initial endowment. If I_i ($i \in M$) is not small, then we need to consider only the domain where the income effect can be negligible. Furthermore the sellers are considered to be firms in our market model. Then it would be natural to assume that each firm's objective is to maximize its profits, which is equivalent to assumption (E). Note that condition (22) does not mean the marginal utility of money is constant in the sense of von Neumann–Morgenstern's utility (i.e., risk neutral). That is, any monotone transformation of $u^i(x) + m$ is allowed. Finally note that assumption (E) should not be made on the buyers because each buyer's amount of money is decreased by purchasing an indivisible good and the proportion of the payment to the initial income is not negligible.

In Kaneko (1976a) it is permitted that a seller owns more than one kind of goods, i.e., $a_f^i > 0$ for $i \in M_k$ ($f \neq k$), but it is also assumed that $u^i(x)$ satisfies $u^i(x) = \sum_{f=1}^s u^i(x_f e^f)$. When each $u^i(x_f e^f)$ is not constant return to scale, i.e., $u^i(x_f e^f) = x_f u^i(e^f)$, this assumption is inadequate, but rather it is plausible to assume that $u^i(x) = u^i(\sum_{f=1}^s x_f e^f)$. Because the indivisible goods are permitted

to be different but they are not substantially different goods, which is an implication of (CS') or the reasoning of (CB). By this reason we do not generalize the assignment market to such a form. But we note that the following results can be gained without any essential change in this case.

We put $a_i(g) = u^i(g e^k) - u^i((g-1)e^k)$ for all $i \in M_k$ ($k=1, \dots, s$) and $g \leq w_i$. We can put $u^i(0) = 0$ without loss of generality. For convenience sake, we put $a_i(g) = 0$ for all $g > w_i$ ($i \in M$). Then we have

$$u^i(x) = \sum_{g=1}^{x_k} a_i(g) \quad \text{for all } x \in I_+^s. \quad (23)$$

Assumption (CS') implies that $a_i(g) > 0$ for all $g \leq w_i$ ($i \in M$). Further we assume that the marginal utility of the k th indivisible good is non-increasing for all $i \in M_k$ ($k=1, \dots, s$), that is:

(F) *Non-increasing marginal utility of the indivisible good.* $a_i(g) \geq a_i(g+1)$ for all g .

We call this market model (M, N) a *generalized assignment market*. The market models of Shapley and Shubik (1972) and Kaneko (1976a) are special cases of the generalized assignment market.¹⁰

In order to characterize this market model, we shall define another market model (M^*, N) , which we call the *agent assignment market* of a generalized assignment market (M, N) .¹¹ The buyers N are the same as the buyers N of the generalized assignment market (M, N) . The sellers M^* consist of M_1^*, \dots, M_s^* , i.e., $M^* = M_1^* \cup M_2^* \cup \dots \cup M_s^*$ such that

$$M_k^* = \bigcup_{i \in M_k} \{i(1), \dots, i(w_i)\} \quad \text{for all } k=1, \dots, s. \quad (24)$$

We assume that each seller $i(g) \in M_k^*$ ($k=1, \dots, s$) owns one unit of the k th indivisible good and $I_{i(g)} > 0$ amount of money initially, i.e., $(a^{i(g)}, I_{i(g)}) = (e^k, I_{i(g)})$. Seller $i(g)$'s utility function is given as

$$U^{i(g)}(x, m) = u^{i(g)}(x) + m, \quad (25)$$

$$\begin{aligned} u^{i(g)}(x) &= a_i(g) \quad \text{if } x_k \geq 1, \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (26)$$

¹⁰Exactly, it is slightly different from a generalization of that of Kaneko (1976a) as remarked above.

¹¹The same procedure was employed in Kaneko (1976a).

Each seller i attaches the label $i(g)$ showing the name of proprietor i and the number of unit g to each unit and employs an agent acting as a seller of the unit who has the valuation $a_i(g)$ of the unit given by the proprietor i . Of course, the agent assignment market is an assignment market given in the previous section.

We can show that there is a one-to-one mapping from the set of all competitive allocations in the generalized assignment market to that in the agent assignment market.

Theorem 6. If $(p, (x^{M \cup N}, m^{M \cup N}))$ is a competitive equilibrium in the generalized assignment market (M, N) , then $(p, (x^{M^* \cup N}, m^{M^* \cup N}))$ defined by

$$\begin{aligned} x^{i(g)} &= e^k \quad \text{if } x_k^i \geq g \quad \text{and } i(g) \in M_k^*, \\ &= 0 \quad \text{otherwise,} \end{aligned} \quad (27)$$

$$\begin{aligned} m^{i(g)} &= p_k + I_{i(g)} \quad \text{if } x^{i(g)} = 0 \quad \text{and } i(g) \in M_k^*, \\ &= I_{i(g)} \quad \text{otherwise,} \end{aligned} \quad (28)$$

is a competitive equilibrium in the agent assignment market (M^*, N) .¹² Conversely if $(p, (x^{M^* \cup N}, m^{M^* \cup N}))$ is a competitive equilibrium in the agent assignment market, then $(p, (x^{M \cup N}, m^{M \cup N}))$ defined by

$$x^i = \sum_{g=1}^{w_i} x^{i(g)} \quad \text{for all } i \in M, \quad (29)$$

$$m^i = I_i + \sum_{g=1}^{w_i} (m^{i(g)} - I_{i(g)}) \quad \text{for all } i \in M, \quad (30)$$

is a competitive equilibrium in the generalized assignment market (M, N) .

From Theorems 5 and 6, we get:

Theorem 7. There exists a competitive equilibrium in the generalized assignment market.

Since the core includes the competitive allocations, we get:

Theorem 8. The core of the generalized assignment market is non-empty.

¹² p , x^j and m^j ($j \in N$) in $(p, (x^{M^* \cup N}, m^{M^* \cup N}))$ are the same as those in $(p, (x^{M \cup N}, m^{M \cup N}))$.

Proof of Theorem 6. Let $(p, (x^{M \cup N}, m^{M \cup N}))$ be a competitive equilibrium in the generalized assignment market. Then it is easily verified that $x^i = x_k^i e^k \leq w_k^i e^k$ for all $i \in M_k$ ($k=1, \dots, s$) and $x^j=0$ or e^k for all $j \in N$. Since $(x^{M \cup N}, m^{M \cup N})$ is an allocation,

$$\sum_{i \in M \cup N} (x^i, m^i) = \sum_{i \in M \cup N} (a^i, I_i).$$

Let $(p, (x^{M^* \cup N}, m^{M^* \cup N}))$ be given by (27) and (28). Clearly the budget constraint (16) holds for all $i \in M \cup N$. First, we show that $(x^{M^* \cup N}, m^{M^* \cup N})$ is an allocation. By (27) we have

$$\sum_{i \in M} x_k^i = \sum_{i \in M_k} x_k^i = \sum_{i \in M_k} \sum_{g=1}^{x_k^i} x_k^{i(g)} = \sum_{i(g) \in M_k^*} x_k^{i(g)} = \sum_{i(g) \in M^*} x_k^{i(g)},$$

which implies

$$\sum_{i \in M^* \cup N} x^i = \sum_{i \in M^* \cup N} a^i.$$

It is clear that

$$\sum_{i \in M} m^i = \sum_{i \in M} I_i + \sum_{k=1}^s \sum_{i \in M_k} (w_i - x_k^i) p_k,$$

which implies

$$\sum_{j \in N} m^j = \sum_{j \in N} I_j - \sum_{k=1}^s \sum_{i \in M_k} (w_i - x_k^i) p_k.$$

Since by (28)

$$\sum_{i(g) \in M^*} m^{i(g)} = \sum_{i(g) \in M} I_{i(g)} + \sum_{k=1}^s \sum_{i \in M_k} (w_i - x_k^i) p_k,$$

it holds that

$$\sum_{i(g) \in M^*} m^{i(g)} + \sum_{j \in N} m^j = \sum_{i(g) \in M^*} I_{i(g)} + \sum_{j \in N} I_j.$$

Clearly the utility maximization (15) is true for all $i \in N$. Hence we need to show that (15) is true for all $i \in M^*$. Suppose that there is an $i(g) \in M_k^*$ for whom (15) is not true. This means that if $x^{i(g)} = e^k$, then

$$p_k + I_{i(g)} > a_i(g) + I_{i(g)} = u^{i(g)}(x^{i(g)}) + m^{i(g)},$$

i.e., $p_k > a_i(g)$ and if $x^{i(g)} = 0$, then

$$a_i(g) + I_{i(g)} > p_k + I_{i(g)} = u^{i(g)}(x^{i(g)}) + m^{i(g)},$$

i.e., $a_i(g) > p_k$. Since $(p, (x^{M \cup N}, m^{M \cup N}))$ is a competitive equilibrium, it holds that

$$\sum_{g=1}^{x_k^i} a_i(g) + p_k(w_i - x_k^i) + I_i \geq \sum_{g=1}^h a_i(g) + p_k(w_i - h) + I_i \quad \text{for all } h.$$

By this inequality and assumption (F) we have $p_k \leq a_i(g)$ for all $g \leq x_k^i$ and $p_k \geq a_i(g)$ for all $g > x_k^i$. This contradicts the above fact, i.e., that if $x^{i(g)} = e^k$, i.e., $x_k^i \leq g$ by (27), then $p_k > a_i(g)$ and if $x^{i(g)} = 0$, i.e., $x_k^i < g$, then $p_k < a_i(g)$. Hence (15) is true for all $i(g) \in M^*$.

Let $(p, (x^{M^* \cup N}, m^{M^* \cup N}))$ be a competitive equilibrium in the agent assignment market, and let $(p, (x^{M \cup N}, m^{M \cup N}))$ be given by (29) and (30). Similarly with the above, we can show the budget constraint (16) for all $i \in M$ and that $(x^{M \cup N}, m^{M \cup N})$ is an allocation. We show the utility maximization (16) for all $i \in M$. Suppose that there is a seller $i \in M_k$ such that for some y_k^i ,

$$\begin{aligned} u^i(x^i) + m^i &= \sum_{g=1}^{x_k^i} a_i(g) + p_k(w_i - x_k^i) + I_i \\ &< \sum_{g=1}^{y_k^i} a_i(g) + p_k(w_i - y_k^i) + I_i = u^i(y_k^i e^k) + m^i. \end{aligned}$$

If $x_k^i > y_k^i$, then we have

$$\sum_{g=y_k^i+1}^{x_k^i} a_i(g) < p(x^i - y^i),$$

which implies that there is an $i(g) \in M_k^*$ such that $p_k > a_i(g)$ and $g \leq x_k^i$. If $x^{i(g)} = 0$, then by (29), $x^{i(g')} = e^k$ for some $g' > g$. For this g' , we have $p_k > a_i(g) \geq a_i(g')$ by assumption (F). Hence we can assume that $p_k > a_i(g)$ and $x^{i(g)} = e^k$. Thus we have

$$p_k + I_{i(g)} > a_i(g) + I_{i(g)} = u^{i(g)}(x^{i(g)}) + m^{i(g)},$$

which is a contradiction to the supposition that $(p, (x^{M^* \cup N}, m^{M^* \cup N}))$ is a competitive equilibrium. If $x_k^i < y_k^i$, then we have

$$\sum_{g=x_k^i+1}^{y_k^i} a_i(g) > p_k(y_k^i - x_k^i),$$

which implies that there is an $i(g) \in M_k^*$ such that $a_i(g) > p_k$ and $x^{i(g)} = 0$. Hence we have

$$a_i(g) + I_{i(g)} = u^{i(g)}(a^{i(g)}) + I_{i(g)} > p_k + I_{i(g)} = u^{i(g)}(x^{i(g)}) + m^{i(g)},$$

which is a contradiction. Q.E.D.

(9) In an assignment market, the core always coincides with the set of all competitive allocations. It is, however, not necessarily true in a generalized assignment market, but a weak condition for the equivalence can be given.

In Kaneko (1976a) the following theorems are given as more general versions in a generalized assignment market with the transferable utility assumption. As the proofs of them are almost the same as Lemma 3 and Theorem II in Kaneko (1976a) we do not give the proofs in this paper.

If two sellers i_1 and i_2 in M have the same preference orderings, i.e., the same utility functions and the same initial endowments, the sellers are called the *same type*.

Theorem 9. *If for each $i \in M_k$, there is at least one seller $i' \in M_k$ ($i \neq i'$) who is the same type as i , then the k th indivisible good has a common price in each allocation in the core, i.e., there is a p_k such that*

$$m^i = I_i + p_k(w_i - x^i) \quad \text{for all } i \in M_k, \quad (31)$$

$$m^i = I_i - p_k \quad \text{for all } i \in N \quad \text{with } x^i = e^k. \quad (32)$$

Theorem 10. *If for each $i \in M$, there is at least one seller i' ($i \neq i'$) who is the same type as i , then the core coincides with the set of all competitive allocations.*

In Kaneko (1976a), the core is considered in the case where the supposition of Theorem 10 is not true, i.e., a seller becomes a monopolist in a certain sense. It says that the core permits price discrimination. This result is also true in the generalized assignment market without the transferable utility assumption, but as it can be gained in almost the same way, we give an explanation in a diagram.

Let $s = 1$. Then the supply curve and the demand curve are drawn as fig. 2. Let us consider the case where the supposition of Theorem 9 is not true. Let $a_M = \min \{a_i(g) : i \in M - \{1\}, g = 1, \dots, w_i\}$. If a_M is greater than the intersection of the supply and demand curves, e.g., a_M in fig. 2, then the core permits price discrimination in any allocation in the core, i.e., seller 1 trades the good at different prices not more than a_M with buyers. Seller 1 is a monopolist in this sense. If a_M is not greater than the intersection, the core coincides with the set of all competitive equilibria, because one seller becomes a competitor with seller 1. Of course, the good is traded at a common price in the intersection. This is precisely a more sufficient condition than Theorem 10 for the equivalence of the core and the set of all competitive allocations, which is corresponding to the supposition of Theorem II of Kaneko (1976a).

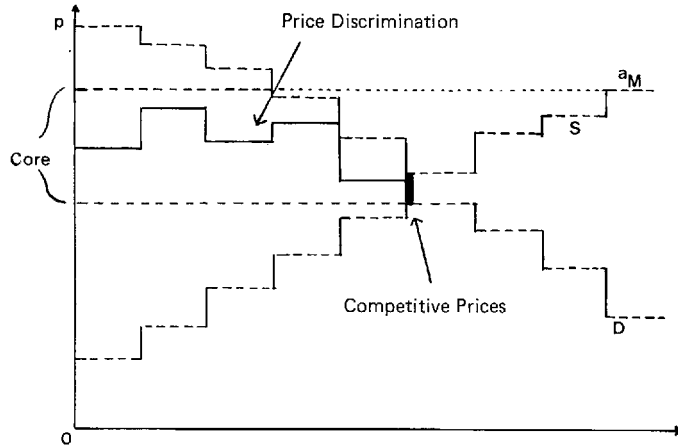


Fig. 2

When $s \geq 2$, the similar price discrimination occurs in the case where the supposition of Theorem 9 is not true. But since the indivisible goods are not substantially different, any seller could regard the sellers owning the other goods even beside the ones owning the same good as competitors with him. This fact makes the price discrimination in the core narrow.

Clearly the sufficient condition for the equivalence given by Theorem 10 is weak. Further even if the equivalence does not hold, the price discrimination in the core is not too wide. Then we get the conclusion that in most classes of generalized assignment markets, the competitive equilibria could be representatives of the core.

Finally, we note that permissible coalitions can be constrained to a subclass of that of all the coalitions in the generalized assignment market for the equivalence of the core and the competitive equilibria similarly to Corollary 7, which is shown in the case with the transferable utility assumption in Lemma 2 of Kaneko (1976a).

Appendix

Proof of Lemma 2. Suppose that there is an S -permutation matrix A_S such that $D_S(b) \geq A_S$. Let $\{(i, j) : a_{S,ij} = 1, i \in M \text{ and } j \in N\} = \{(i_1, i_2), \dots, (i_k, j_k)\}$. Note that $i_t \in S$ and $j_t \in S$ for all $t = 1, \dots, k$ by (7). Let $M \cap S - \{i_1, \dots, i_k\} = \{i(1), \dots, i(g)\}$ and $N \cap S - \{j_1, \dots, j_k\} = \{j(1), \dots, j(h)\}$. Of course $S = \{i_1, \dots, i_k, i(1), \dots, i(g), j_1, \dots, j_k, j(1), \dots, j(h)\}$. We define a π -partition $p_S = \{T_1, \dots, T_f\}$ ($f = k + g + h$) by

$$T_t = \{i_t, j_t\} \quad \text{for all } t = 1, \dots, k,$$

$$T_{k+t} = \{i(t)\} \quad \text{for all } t = 1, \dots, g,$$

$$T_{k+g+t} = \{j(t)\} \quad \text{for all } t = 1, \dots, h.$$

Since $d_{S:i_t j_t}(b) = a_{S:i_t j_t} = 1$, $i_t \in M \cap S$ and $j_t \in N \cap S$, we have $b \in V(\{i_t, j_t\})$ by (8). Since $a_{S:i(t)i} = 1$ for some $i \in M \cap S$ and $a_{S:j(t)j}(b) = 1$ for some $j \in N \cap S$ by (7) and the definition of $i(t)$ and $j(t)$, we have $d_{S:i(t)i}(b) = 1$ and $d_{S:j(t)j}(b) = 1$, which implies $b \in V(\{i(t)\})$ and $b \in V(\{j(t)\})$ by (8). Thus we have shown that $b \in V(T_t)$ for all $t = 1, \dots, f$, i.e., $b \in \bigcap_{T \in p_S} V(T)$.

Let $b \in \bigcap_{T \in p_S} V(T)$ for some π -partition p_S . Let

$$\{T \in p_S : |T| = 2\} = \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\},$$

$$\{T \in p_S : |T| = 1 \text{ and } T \subset M\} = \{\{i(1)\}, \dots, \{i(g)\}\},$$

$$\{T \in p_S : |T| = 1 \text{ and } T \subset N\} = \{\{j(1)\}, \dots, \{j(h)\}\}.$$

We assume $g \leq h$ in the following, but we can prove the following similarly if $g > h$. We define an S -permutation matrix $A_S = (a_{S:ij})$ as follows — for all i_t ($t = 1, \dots, k$):

$$\begin{aligned} a_{S:i_t j} &= 1 & \text{if } j = j_t, \\ &= 0 & \text{otherwise,} \end{aligned}$$

for all $i(t)$ ($t = 1, \dots, g$):

$$\begin{aligned} a_{S:i(t)j} &= 1 & \text{if } j = j(t), \\ &= 0 & \text{otherwise,} \end{aligned}$$

for all $i \in M - S$:

$$a_{S:ij} = 0 \quad \text{for all } j \in M \cup N,$$

for all $i \in N$:

$$\begin{aligned} a_{S:ij} &= a_{S:ji} & \text{if } i \in \{j_1, \dots, j_k, j(1), \dots, j(g)\} \text{ and } j \in M, \\ &= 1 & \text{if } i \in \{j(g+1), \dots, j(h)\} \text{ and } i = j, \\ &= 0 & \text{otherwise.} \end{aligned}$$

It is easily verified that this A_S is an S -permutation matrix.

Now we show that $D_S(b) \geq A_S$. When $a_{S:i_j} = 0$, it is always true that $d_{S:i_j}(b) \geq a_{S:i_j}$. So, suppose $a_{S:i_j} = 1$. If $(i, j) = (i_t, j_t)$ for some $t \leq k$, then b is in $V(\{i_t, j_t\})$ because $b \in \bigcap_{T \in \mathcal{P}_S} V(T)$. This implies $d_{S:i_j}(b) = 1$. Let $(i, j) = (i(t), j(t))$ for some $t \leq g$. Then $b \in V(\{i(t)\}) \cap V(\{j(t)\}) \subset V(\{i(t), j(t)\})$ by the supposition, $b \in \bigcap_{T \in \mathcal{P}_S} V(T)$, and the superadditivity of V . This implies $s_{S:i_j}(b) = 1$. When $i = j(t)$ and $j = j(t)$ ($g+1 \leq t \leq h$) it is also true by the same reason that $b \in V(\{j(t)\})$, which implies $d_{S:i_j}(b) = 1$. When $i \in N \cap S$ and $j \in S \cap M$, we have always $d_{S:i_j}(b) = 1$ by (8). We have shown $D_S(b) \geq A_S$. Q.E.D.

Proof of Lemma 3. For any $(x, m) \in X$ there is a unique real number d by assumptions (A) and (B) such that $(x, m) Q_i(0, d)$. We define $U^i(x, m)$ by

$$U^i(x, m) = d. \quad (\text{A.1})$$

It can be easily verified that this U^i satisfies $(x, m_1) R_i(y, m_2)$ iff $U^i(x, m_1) \geq U^i(y, m_2)$. We show that U^i is a continuous function. Let $\{(x^\alpha, m^\alpha)\}$ be a sequence in X which converges to $(x, m) \in X$. Since I_+^s has the discrete topology, there is an integer k such that $x^\alpha = x$ for all $\alpha \geq k$. Since $\{(x^\alpha, m^\alpha)\}$ is a converging sequence, there is a m_1 such that $0 \leq m^\alpha \leq m_1$ for all α , which implies $0 \leq U^i(x, m^\alpha) \leq U^i(x, m_1)$ for all $\alpha \geq k$. Hence the sequence $\{U^i(x, m^\alpha)\}$ has at least one limit point in R_+ . Let the sequence $\{U^i(x, m^\alpha)\}$ have a limit point not equal to $U^i(x, m)$ and let $\{U^i(x, m^{\alpha^\beta})\}$ be a converging subsequence such that $\lim U^i(x, m^{\alpha^\beta}) = u \neq U^i(x, m)$. Let ε be a sufficiently small positive number. Let $u > U^i(x, m)$. Then we have $(0, U^i(x, m) + \varepsilon) P_i(x, m)$ by assumptions (A) and (A.1). By assumptions (A) and (B) there is a $\delta > 0$ such that $(0, U^i(x, m) + \varepsilon) Q_i(x, m + \delta)$, i.e., $U^i(x, m) + \varepsilon = U^i(x, m + \delta)$. But it holds that $m^{\alpha^\beta} < m + \delta$ for all $\beta \geq$ some β_0 , which implies $U^i(x, m^{\alpha^\beta}) \leq U^i(x, m + \delta) = U^i(x, m) + \varepsilon \leq u$. This is a contradiction. Next let $u < U^i(x, m)$. Since $U^i(x, m^\alpha) \geq U^i(x, 0)$ by assumption (A), we have $u \geq U^i(x, 0)$. Hence there is an m_0 such that $(0, u + \varepsilon) Q_i(x, m_0)$, i.e., $u + \varepsilon = U^i(x, m_0)$. Since ε is sufficiently small, we have $m_0 < m$. But since there is a β_0 such that $U^i(x, m^{\alpha^\beta}) < u + \varepsilon = U^i(x, m_0)$ for all $\beta \geq \beta_0$ by $u = \lim U^i(x, m^{\alpha^\beta})$, we have $m_0 > m^{\alpha^\beta}$ by assumption (A). This contradicts that $\lim m^{\alpha^\beta} = m$. Hence we have shown that any limit point coincides with $U^i(x, m)$. Q.E.D.

Proof of Lemma 4. If $\sum_{i \in S} I_i > \sum_{i \in S} m^i$, then S can improve upon $(x^{M \cup N}, m^{M \cup N})$ by the S -allocation (y^S, m_1^S) such that $y^i = x^i$ for all $i \in S$ and $m_1^i = m^i + (\sum_{i \in S} (I_i - m^i)) / |S|$ for all $i \in S$. Let $\sum_{i \in S} I_i = \sum_{i \in S} m^i$. Then there is an $f \leq s$ such that $\sum_{i \in S} x_{f_j}^i < \sum_{i \in S} a_{f_j}^i$, which implies that $x_{f_j}^{i_0} = 0$ for some $i_0 \in S \cap M_f$. By assumption (CB) we have $U^{i_0}(e^f, m^{i_0}) > U^{i_0}(x^{i_0}, m^{i_0}) = U^{i_0}(0, m^{i_0})$. If ε is a sufficiently small positive number, then $U^{i_0}(e^f, m^{i_0} - \varepsilon) > U^{i_0}(x^{i_0}, m^{i_0})$ by Lemma 3. Hence it holds that $U^i(x^i, m^i + \varepsilon / (|S| - 1)) > U^i(x^i, m^i)$ for all $i \in S - \{i_0\}$, which implies that S can improve upon $(x^{M \cup N}, m^{M \cup N})$. Q.E.D.

Proof of Lemma 5. If $m^i=0$ for some $i \in N$, then $U^i(x^i, m^i) < U^i(a^i, I_i)$ by assumption (D), which contradicts that $(x^{M \cup N}, m^{M \cup N})$ is in the core. If $m^i=0$ for some $i \in M_f$, then $U^i(0, m^i) < U^i(e^f, 0) < U^i(e^f, I_i)$, which is a contradiction. Hence we have $m^i > 0$ for all $i \in M \cup N$.

Suppose that there is a buyer j with $x^j \neq 0$. Then by assumption (CB) there is an $f \leq s$ such that $x_f^j > 0$ and $U^j(e^f, m^j) = U^j(x^j, m^j)$. Then there is a seller $i \in M_f$ with $x_f^i = 0$. We show that $m^i + m^j = I_i + I_j$, $x^i = 0$ and $x^j = e^f$. If $m^i + m^j > I_i + I_j$, then the coalition $(M - \{i\}) \cup (N - \{j\})$ can improve upon $(x^{M \cup N}, m^{M \cup N})$, because $\sum_{t \neq i, j} a^t \geq \sum_{t \neq i, j} x^t$ and $\sum_{t \neq i, j} I_t > \sum_{t \neq i, j} m_t$. If $m^i + m^j < I_i + I_j$, then the coalition $\{i, j\}$ can improve upon $(x^{M \cup N}, m^{M \cup N})$ by the allocation $((y^t, m_t^i))_{t=i, j}$ such that

$$y^i = 0, \quad m_1^i = m^i + (I_i + I_j - m^i - m^j)/2,$$

$$y^j = e^f, \quad m_1^j = m^j + (I_i + I_j - m^i - m^j)/2.$$

Hence it holds that $m^i + m^j = I_i + I_j$. If $x_{f'}^i = 1$ for some $f' \neq f$ or $x^j \geq e^f$, then the coalition $(M - \{i\}) \cup (N - \{j\})$ can improve upon $(x^{M \cup N}, m^{M \cup N})$ by Lemma 4 because $\sum_{t \neq i, j} a^t \geq \sum_{t \neq i, j} x^t$ and $\sum_{t \neq i, j} m_t = \sum_{t \neq i, j} I_t$. Hence we have $x^i = 0$ and $x^j = e^f$.

Repeating this argument, we choose all such pairs and denote the set of them by $\{(i_1, j_1), \dots, (i_k, j_k)\}$. We put $M_0 = M - \{i_1, \dots, i_k\}$ and $N_0 = N - \{j_1, \dots, j_k\}$. Clearly (18) holds for these $\{(i_1, j_1), \dots, (i_k, j_k)\}$. Of course, no buyer $j \in N_0$ has any indivisible good, i.e., $x^j = 0$ for all $i \in N_0$. For, otherwise, we can find another pair (i_{k+1}, j_{k+1}) for which (18) holds by the above argument. Hence any sellers in M_0 do not trade indivisible goods with any buyers in N_0 . Further even when two sellers in M_0 trade indivisible goods with each other, they have no profits, or decrease their utilities. So, all sellers in M_0 do not trade at all. The similar argument is valid for buyers in N_0 . By this reason (19) holds for all sellers in M_0 and buyers in N_0 . Q.E.D.

References

- Crawford, V.P. and E.M. Knoer, 1981, Job matching with heterogeneous firms and workers, *Econometrica* 49, 437-450.
 Gale, D., 1960, The theory of linear economic models (McGraw-Hill, New York).
 Gale, D. and L.S. Shapley, 1962, College admissions and the stability of marriage, *American Mathematical Monthly* 69, 9-15.
 Kaneko, M., 1976a, On the core and competitive equilibria of a market with indivisible goods, *Naval Research Logistics Quarterly* 23, 321-337.
 Kaneko, M., 1976b, Note on transferable utility, *International Journal of Game Theory* 5, 183-185.
 Kaneko, M., 1980, Housing markets with indivisibilities, Cowles Foundation discussion paper no. 571, *Journal of Urban Economics*, forthcoming.
 Scarf, H.E., 1967, The core of an n -person game, *Econometrica* 35, 50-69.

- Shapley, L.S. and H.E. Scarf, 1974, On cores and indivisibility, *Journal of Mathematical Economics* 1, 23–37.
- Shapley, L.S. and M. Shubik, 1972, The assignment game I: The core, *International Journal of Game Theory* 1, 111–130.
- Telser, L.S., 1972, *Competition, collusion and game theory* (Aldine, New York).
- Von Böhm-Bawerk, E., 1923, *Positive theory of capital*, translated by W. Smart, Original publication 1891 (Books for Libraries, New York).
- Von Neumann, J. and O. Morgenstern, 1953, *Theory of games and economic behavior*, 3rd ed. (Princeton University Press, Princeton, NJ).