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COMMON KNOWLEDGE LOGIC AND GAME LOGIC

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Abstract. We show the faithful embedding of common knowledge logic CKL into game logic GL, that is, CKL is embedded into GL and GL is a conservative extension of the fragment obtained by this embedding. Then many results in GL are available in CKL, and *vice versa*. For example, an epistemic consideration of Nash equilibrium for a game with pure strategies in GL is carried over to CKL. Another important application is to obtain a Gentzen-style sequent calculus formulation of CKL and its cut-elimination. The faithful embedding theorem is proved for the KD4–type propositional CKL and GL, but it holds for some variants of them.

§1. Introduction. *Common knowledge logic* CKL is an epistemic propositional logic with one knowledge (belief) operator for each player and a common knowledge operator. Syntactical axiomatizations of various types of CKL are provided (Halpern-Moses [2] and Lismont-Mongin [11]). Common knowledge logic has been developed from the model theoretic side, particularly, soundness and completeness have been proved to show that the intended notion of common knowledge is well captured in these axiomatizations. There is another approach to similar problems, which Kaneko-Nagashima [7], [9] call *game logic* GL (GL(H) in the Hilbert style and GL(G) in the Gentzen style). In GL, a richer, first-order language in which infinitary conjunctions and disjunctions are allowed is adopted to formulate common knowledge directly as a conjunctive formula. Game logic has been developed from the proof theoretic side together with game theoretic applications. Although these approaches can treat similar problems, their explicit relationship has not yet been investigated. We carry out such investigations in this paper.

Since game logic GL has a richer language, it may be expected that GL is actually stronger than common knowledge logic CKL. It is, however, more essential to ask whether GL is, in a sense, a conservative extension of CKL. In this paper, we prove that CKL is faithfully embedded into (the propositional fragment of) GL, that is, CKL is embedded into GL and GL is a conservative extension of the fragment obtained by this embedding.

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The faithful embedding result enables us not only to see the relationship between CKL and GL, but also to convert many results from one side to the other. For example, the epistemic consideration of Nash equilibrium, object theorems as well as metatheorems, in a finite game with pure strategies in Kaneko [5] can be converted to CKL. Also, our analysis provides a Gentzen style sequent calculus of CKL and its cut-elimination theorem. In the other direction, we obtain model theory for the fragment of GL obtained by the embedding, and its decidability.

There are several variants of CKL as well as GL depending upon choices of various epistemic axioms. We present the faithful embedding result for common knowledge and game logics based on KD4. We will give comments on other variants in Section 7.

In GL common knowledge is described as an infinitary conjunctive formula $C(A)$, while in CKL it is described as $C_0(A)$ with a certain additional axiom and an inference rule for C_0, where C_0 is a common knowledge operator symbol. In the literature of epistemic logic, the definition of common knowledge in GL is called the *iterative* definition, and the one in CKL is called the *fixed-point* definition (cf., Barwise [1]). Our faithful embedding theorem implies that these definitions are equivalent in CKL and GL.

Game logic GL is an infinitary extension, KD4$^{\omega}$, of finitary multi-modal KD4 together with an additional axiom called the *C-Barcan*:

$$\text{GL} = \text{KD4}^{\omega} + \text{C-Barcan}.$$

Axiom C-Barcan is introduced to allow the fixed point property $C(A) \supset K_i C(A)$ to be provable for all $i = 1, \ldots, n$, where K_i is the knowledge operator of player i. The additional C-Barcan axiom is needed to have the faithful embedding theorem of CKL into GL. In KD4$^{\omega}$ without the C-Barcan axiom, the iterative definition of common knowledge still makes sense, but would lose the fixed point property. The fixed point property is indispensable for the full epistemic consideration of Nash equilibrium.

We prove our faithful embedding result for the propositional part. The proof relies upon the cut-elimination theorem for GL(G) obtained in Kaneko-Nagashima [9] as well as upon the soundness-completeness theorem for CKL proved in Halpern-Moses [2] and Lismont-Mongin [11]. We prove one lemma – Lemma 4.4 – using the soundness-completeness theorem for CKL. So far, completeness is available only for *propositional* CKL. If Lemma 4.4 could be proved for *predicate* common knowledge logic, the faithful embedding theorem would be obtained for predicate CKL and GL. This remains open.

The structure of this paper is as follows: Section 2 formulates finitary and infinitary epistemic logics KD4 and KD4$^{\omega}$ in the Hilbert style. In Section 3, we define common knowledge logic CKL as well as game logic GL in the Hilbert style. Then we state the faithful embedding theorem. The embedding part is immediately proved, but the faithfulness part needs game logic GL in the Gentzen style sequent calculus and its cut-elimination, which is the subject of Section 4. Section 5 formulates CKL directly as a sequent calculus, whose cut-elimination is proved from the results of Section 4. Section 6 discusses game theoretical applications. Section 7 gives some remarks.

§2. Epistemic logic KD4 and its infinitary extension KD4$^\omega$. We use the two sets, $\mathscr{P}^{\mathrm{f}}$ and $\mathscr{P}^\omega$, of formulae for common knowledge and game logics. The following is the list of primitive symbols: *Propositional variables*: $\mathbf{p}_0, \mathbf{p}_1, \ldots$; *Knowledge operators*: $K_1, \ldots, K_n$; *Common knowledge operator*: C_0; *Logical connective*: $\neg$ (*not*), $\supset$ (*implies*), $\bigwedge$ (*and*), $\bigvee$ (*or*) (where $\bigwedge$ and $\bigvee$ may be applied to infinitely many formulae); *Parentheses*: (,). The indices $1, \ldots, n$ of $K_1, \ldots, K_n$ are the names of players.

Let $\mathscr{P}^{\mathrm{f}}$ be the set of all formulae generated by the finitary inductive definition with respect to $\neg, \supset, \bigwedge, \bigvee, K_1, \ldots, K_n, C_0$ from the propositional variables, i.e., (i) each propositional variable is in $\mathscr{P}^{\mathrm{f}}$, (ii) if A, B are in $\mathscr{P}^{\mathrm{f}}$, so are $(\neg A), (A \supset B), K_1(A), \ldots, K_n(A), C_0(A)$, and (iii) if Φ is a nonempty finite subset of $\mathscr{P}^{\mathrm{f}}$, then $(\bigwedge \Phi), (\bigvee \Phi)$ are in $\mathscr{P}^{\mathrm{f}}$.

We define the set $\mathscr{P}^\omega$ of infinitary formulae using induction twice. We denote $\mathscr{P}^{\mathrm{f}}$ by $\mathscr{P}^0$. Suppose that $\mathscr{P}^0, \mathscr{P}^1, \ldots, \mathscr{P}^k$ are already defined $(k < \omega)$. Then we allow the expressions $(\bigwedge \Phi)$ and $(\bigvee \Phi)$ for any nonempty *countable* subset Φ of $\mathscr{P}^k$. Now from the union $\mathscr{P}^k \cup \{(\bigwedge \Phi), (\bigvee \Phi) : \Phi \text{ is a countable subset of } \mathscr{P}^k\}$, we obtain the space $\mathscr{P}^{k+1}$ of formulae by the standard finitary inductive definition with respect to $\neg, \supset, \bigwedge, \bigvee, K_1, \ldots, K_n$ and C_0. We denote $\bigcup_{k<\omega} \mathscr{P}^k$ by $\mathscr{P}^\omega$. An expression in $\mathscr{P}^\omega$ is called simply a *formula.*[1] We abbreviate $\bigwedge\{A, B\}$ and $\bigvee\{A, B\}$ as $A \bigwedge B$ and $A \bigvee B$, etc.

The primary reason to adopt the infinitary language for game logic is to express common knowledge explicitly as a conjunctive formula. The *common knowledge of a formula* A is defined as follows: For any $m \geq 0$, we denote the set $\{K_{i_1}K_{i_2}\ldots K_{i_m}$: each K_{i_t} is one of $K_1, \ldots, K_n$ and $i_t \neq i_{t+1}$ for $t = 1, .., m-1\}$ by $\mathsf{K}(m)$. For $m = 0$, $K_{i_1}K_{i_2}\ldots K_{i_m}$ is interpreted as the null symbol. We define the *common knowledge formula* of A by

$$\bigwedge\{K(A) : K \in \textstyle\bigcup_{m<\omega}\mathsf{K}(m)\}, \tag{1}$$

which we denote by $C(A)$. Note that if A is in $\mathscr{P}^m$, the set $\{K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}$ is a countable subset of $\mathscr{P}^m$ and its conjunction, $C(A)$, is in $\mathscr{P}^{m+1}$. Hence the space $\mathscr{P}^\omega$ is closed with respect to the operation $C(\cdot)$.

The infinitary language $\mathscr{P}^\omega$ permits to express common knowledge as a conjunctive formula $C(A)$. This is often called the *iterative definition* of common knowledge. One remark is that unless some logical structure is given, the common knowledge formula $C(A)$ would be meaningless. In the subsequent sections, we specify the logical structure. In the finitary language $\mathscr{P}^{\mathrm{f}}$, $C(A)$ is not permitted. Therefore we prepare the common knowledge operator symbol C_0 to define common knowledge in terms of this symbol C_0 together with some axiom and inference rule, which will be called the fixed point definition. This will be discussed in Subsection 3.1.

We give the following five axiom schemata and three inference rules: For any formulae A, B, C, and set Φ of formulae,

(L1): $A \supset (B \supset A)$;

(L2): $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

[1]Our language is a propositional (relatively small) fragment, including additional knowledge operators, of the infinitary language $\mathrm{L}_{\omega_1\omega}$ of Karp [10]. Particularly, we note that $\bigwedge \Phi$ and $\bigvee \Phi$ may not be in $\mathscr{P}^\omega$ for some countable subset Φ of $\mathscr{P}^\omega$. For our purpose of discussing common knowledge, however, the space $\mathscr{P}^\omega$ is large enough.

(L3): $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;
(L4): $\bigwedge \Phi \supset A$, where $A \in \Phi$;
(L5): $A \supset \bigvee \Phi$, where $A \in \Phi$;
and

$$\frac{A \supset B \quad A}{B} \text{ (MP)}$$

$$\frac{\{A \supset B : B \in \Phi\}}{A \supset \bigwedge \Phi} \text{ ($\bigwedge$-Rule)} \qquad \frac{\{A \supset B : A \in \Phi\}}{\bigvee \Phi \supset B} \text{ ($\bigvee$-Rule)}.$$

The above logical axioms and inference rules form classical (finitary and infinitary) logic (when we adopt $\mathscr{P}^{\mathrm{f}}$ and $\mathscr{P}^{\omega}$, respectively).

The following are axioms and inference rule for operators K_i for $i = 1, \dots, n$:

(MP_i): $K_i(A \supset B) \bigwedge K_i(A) \supset K_i(B)$;

($\perp_i$): $\neg K_i(\neg A \bigwedge A)$;

(PI_i): $K_i(A) \supset K_i K_i(A)$;

and

(Necessitation): $\dfrac{A}{K_i(A)}$.

Axioms MP_i, $\perp_i$ and PI_i are called K, D and 4 in the modal logic literature. Thus we call this logic KD4 when we restrict all formulae occurring in the above axioms and inferences to ones in $\mathscr{P}^{\mathrm{f}}$. When we allow formulae in $\mathscr{P}^{\omega}$, this logic is denoted by $\mathrm{KD4}^{\omega}$.[2]

A *proof* in KD4 is a finite tree with the following properties: (i) a formula in $\mathscr{P}^{\mathrm{f}}$ is associated with each node and the formula associated with each leaf is an instance of the above axioms; and (ii) adjoining nodes together with their associated formulae form an instance of the inference rules. A *proof* of A is one whose root A is associated with. If there is a proof of A, then A is said to be *provable in* KD4.

Since $\mathrm{KD4}^{\omega}$ is infinitary, the definition of a proof in KD4 should be slightly extended in $\mathrm{KD4}^{\omega}$. A *proof* in $\mathrm{KD4}^{\omega}$ is a countable tree with the following properties: (i) every path from the root is finite, (ii) a formula in $\mathscr{P}^{\omega}$ is associated with each node and the formula associated with each leaf is an instance of the axioms; and (iii) adjoining nodes together with their associated formulae form an instance of the inference rules.

Of course, if A is provable in KD4, then it is provable in $\mathrm{KD4}^{\omega}$.

§3. Common Knowledge Logic CKL and Game Logic GL(H).

3.1. Common Knowledge Logic CKL. Common knowledge logic CKL is defined by adding the following axiom CA and inference rule CI to KD4:

(CA): $C_0(A) \supset A \bigwedge K_1 C_0(A) \bigwedge \dots \bigwedge K_n C_0(A)$;

(CI): $\dfrac{B \supset A \bigwedge K_1(B) \bigwedge \dots \bigwedge K_n(B)}{B \supset C_0(A)}$.

[2]According to the literature of epistemic logic, our "knowledge" should be called "belief" since we do not assume assume (T): $K_i(A) \supset A$. On the other hand, since $C(A)$ includes A as a conjunct, it is common "knowledge". In fact, all of our results remain true even when we use common "belief".

A *proof* in CKL allows CA and CI in addition to the axioms and inference rules for KD4. We denote $\vdash_C A$ iff there is a proof P of A in CKL.

Axiom CA states that if A is common knowledge (in the sense of $C_0(A)$), then A holds and each player knows the common knowledge of A. Inference Rule CI states that if any formula B has this property, it contains (deductively) the common knowledge of A. Thus these require $C_0(A)$ to be a fixed-point with respect to the property of CA. In this sense, this is called the *fixed-point* definition of common knowledge.

To obtain the faithful embedding result of CKL into GL(H), we make some semantical consideration of CKL. Two types of semantics have been considered in literature: the Kripke and neighborhood semantics. For example, completeness (and soundness) theorem was given by Halpern-Moses [2] for various common knowledge logics with respect to Kripke semantics, and by Lismont-Mongin [11], [12] with respect to both neighborhood and Kripke semantics. Here we use the Kripke semantics for CKL.

A *Kripke frame* is given as $\mathscr{M} = (W; R_1, \ldots, R_n)$, where W is an arbitrary set of worlds and R_i is a serial, transitive relation over $W \times W$ for $i = 1, \ldots, n$. Let σ be an assignment, i.e., a function from $W \times \{\mathbf{p}_0, \mathbf{p}_1, \ldots\}$ to $\{\top, \bot\}$. Given $\mathscr{M}$, $w \in W$ and σ, we define the *valuation relation* $(\mathscr{M}, w) \models_\sigma$ in the standard manner with the following additional definitions:

(K1): $(\mathscr{M}, w) \models_\sigma K_i(A) \Leftrightarrow (\mathscr{M}, v) \models_\sigma A$ for any world v with $(w, v) \in R_i$;
(K2): $(\mathscr{M}, w) \models_\sigma C_0(A) \Leftrightarrow (\mathscr{M}, w) \models_\sigma K(A)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$.

We write $\mathscr{M} \models A$ iff $(\mathscr{M}, w) \models_\sigma A$ for all $w \in W$ and all assignments σ, and write $\models A$ iff $\mathscr{M} \models A$ for all (serial and transitive) frames $\mathscr{M}$. Then the following theorem holds (cf., Halpern-Moses [2] and Lismont-Mongin [11],[12]).[3]

THEOREM 3.1 (Completeness of CKL). *For any A in $\mathscr{P}^{\mathrm{f}}$, $\vdash_C A$ if and only if $\models A$.*

3.2. Game Logic GL(H) in Hilbert-style. We denote game logic in Hilbert-style by GL(H) to distinguish from game logic in Gentzen-style, which will be denoted by GL(G).

We call a formula A in $\mathscr{P}^\omega$ a *cc-formula* iff (i) it contains no infinitary disjunction; (ii) if it contains an infinitary conjunction, the infinitary conjunctive formula is $C(B)$ for some B; and (iii) it contains no C_0. A finitary formula is a cc-formula, and any subformula of a cc-formula A is a cc-formula, too. Any cc-formula A contains only a finite number of common knowledge subformulae.

We define GL(H) by adding the following axiom to KD4$^\omega$:

(C-Barcan): $\bigwedge\{K_i K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\} \supset K_i C(A)$,

where A is a cc-formula and $i = 1, \ldots, n$. The provability relation $\vdash_G$ of GL(H) is defined from the provability in KD4$^\omega$ by adding C-Barcan as a logical axiom. The converse of C-Barcan, $K_i C(A) \supset \bigwedge\{K_i K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}$, is provable in KD4$^\omega$ for any formula A in $\mathscr{P}^\omega$. The point of C-Barcan is that the outermost $\bigwedge$ and K_i are interchangeable if A is a cc-formula.

[3] Halpern-Moses [2] proved soundness and completeness for various common knowledge logics based on such as K, KD45, S4, S5, not including KD4, with respect to Kripke semantics. Lismont-Mongin [11] proved their results for those including some logics weaker than K as well as for KD4, with respect to neighborhood semantics. Nevertheless, Theorem 3.1 can be proved directly by modifying the proof of Theorem 4.3 of Halpern-Moses [2].

Kaneko-Nagashima [7] adopted a stronger form of C-Barcan, i.e., for any countable infinite subset Φ of $\mathscr{P}^k$ for any $k < \omega$,

$$\bigwedge\{K_i(B) : B \in \Phi\} \supset K_i(\bigwedge\Phi).$$

Game logic GL(H) presented here is weaker than the corresponding logic of [7]. Nevertheless, the restricted one is sufficient for practical purposes, i.e., game theoretical applications such as in Section 6.

In GL(H), the properties corresponding to axiom CA and inference CI hold, where C-Barcan is used only to prove $\vdash_G C(A) \supset K_i C(A)$ for all $i = 1, \ldots, n$.

LEMMA 3.2. **(1)**: *$\vdash_G C(A) \supset A \bigwedge K_1 C(A) \bigwedge \cdots \bigwedge K_n C(A)$ for any cc-formula A;*

(2): *if $\vdash_G B \supset A \bigwedge K_1(B) \bigwedge \cdots \bigwedge K_n(B)$, then $\vdash_G B \supset C(A)$.*

Thus, logic GL(H) may be regarded as an extension of the common logic CKL by interpreting $C_0(A)$ in CKL as $C(A)$ in GL(H). To make this interpretation explicit, we introduce a translator ψ from $\mathscr{P}^{\mathrm{f}}$ to $\mathscr{P}^{\omega}$. Once this translator is given, we can talk about the converse of the above interpretation.

The *translator* ψ is defined to be the mapping $\mathscr{P}^{\mathrm{f}}$ to $\mathscr{P}^{\omega}$ assigning to each A in $\mathscr{P}^{\mathrm{f}}$ the formula $\psi(A)$ in $\mathscr{P}^{\omega}$ which is obtained, by induction on the structure of A, by substituting $C(\psi(B))$ for any subformula $C_0(B)$ in A. Then the following lemma holds for ψ.

LEMMA 3.3. *ψ is a bijection from $\mathscr{P}^{\mathrm{f}}$ to the set of all cc-formulae.*

PROOF. By definition, ψ is a mapping from $\mathscr{P}^{\mathrm{f}}$ to the set of all cc-formulae. We can prove by induction on the structures of formulae in $\mathscr{P}^{\mathrm{f}}$ that ψ is injective. Next, we define a function $\hat{\psi}$ from the set of cc-formulae to $\mathscr{P}^{\mathrm{f}}$ inductively by replacing $C(B)$ by $C_0(\hat{\psi}(B))$. By induction, we can verify that $\hat{\psi}$ is an injective mapping from the set of cc-formulae to $\mathscr{P}^{\mathrm{f}}$, and also that $\hat{\psi}(\psi(A)) = A$ for all $A \in \mathscr{P}^{\mathrm{f}}$, i.e., $\hat{\psi}$ is the inverse mapping of ψ. Thus, ψ is a bijection from $\mathscr{P}^{\mathrm{f}}$ to the set of cc-formulae. ⊣

Now we can state our first result. The *only-if* part will be proved in the end of this section and the *if* part will be proved in Section 4.

THEOREM 3.4 (Faithful Embedding I). *For any A in $\mathscr{P}^{\mathrm{f}}$, $\vdash_C A$ if and only if $\vdash_G \psi(A)$.*

This theorem clarifies the relationship between CKL and GL(H), and has also the implication that the iterative and fixed-point definitions of common knowledge are equivalent. By this theorem, we would obtain some important results converted from CKL to the fragment $\psi(\mathscr{P}^{\mathrm{f}})$ of GL(H) and *vice versa*. One example is a soundness-completeness theorem for the fragment $\psi(\mathscr{P}^{\mathrm{f}})$ of GL(H). Other applications will be discussed in Sections 5 and 6.

PROOF OF THE ONLY-IF PART OF THEOREM 3.4. Suppose $\vdash_C A$. Then there is a proof P of A in CKL. We translate every formula occurring in P by ψ. The translation $\psi(B)$ of an instance B of an axiom in CKL other than CA is an instance of the corresponding axiom in GL(H). Hence $\vdash_G \psi(B)$. The translated instance $\psi(B)$ of CA is provable in GL(H) by Lemmas 3.2.(1) and Lemma 3.3. Every instance of MP, $\bigwedge$–Rule, $\bigvee$–Rule and Necessitation remain legitimate with the translation ψ in GL(H). Every instance, translated by ψ, of the inference CI is also legitimate by Lemma 3.2.(2). ⊣

Since C-Barcan already deviates from cc-formulae, a proof of the latter $\psi(A)$ of Theorem 3.4 in GL(H) would not be in $\psi(\mathscr{P}^{\mathrm{f}})$ if C-Barcan occurs in the proof. We can, however, change C-Barcan into

$$\frac{\{B \supset K_i K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}}{B \supset K_i C(A)} \text{ (C-Barcan}^*\text{)},$$

where A is a cc-formula and B is any formula. GL(H) is equivalent to $\mathrm{KD4}^{\omega}+$ C-Barcan*, in which a proof of A in $\mathscr{P}^{\mathrm{f}}$ can be translated into a proof in $\psi(\mathscr{P}^{\mathrm{f}})$.

§4. Game Logic GL(G) in sequent calculus and the proof of the Faithful Embedding Theorem. Game logic GL(G) is equivalent to GL(H) with respect to their deducibilities, but cut-elimination holds for GL(G). Cut-elimination has a lot of applications, and the faithful embedding theorem is also its application.

To define GL(G), we prepare another symbol $\rightarrow$. Let Γ, Θ be finite subsets of $\mathscr{P}^{\omega}$. The expression $\Gamma \rightarrow \Theta$ is called a *sequent*. We abbreviate the set-theoretic brackets, e.g., $\{A\} \cup \Gamma \rightarrow \Theta \cup \{B\}$ is denoted as $A, \Gamma \rightarrow \Theta, B$. The counterpart of $\Gamma \rightarrow \Theta$ in GL(H) is $\bigwedge \Gamma \supset \bigvee \Theta$, where $\bigwedge \Gamma$ and $\bigvee \Theta$ are $\neg \mathbf{p}_0 \bigvee \mathbf{p}_0$ and $\neg \mathbf{p}_0 \bigwedge \mathbf{p}_0$, respectively, if Γ and Θ are empty.

Game logic GL(G) is defined by one axiom schema and various inference rules.

Initial Sequents: An initial sequent is of the form $A \rightarrow A$, where A is any formula.

Inference Rules: We have three kinds of inference rules: structural, operational and K-inference rules.

Structural Inferences:

$$\frac{\Gamma \rightarrow \Theta}{\Delta, \Gamma \rightarrow \Theta, \Lambda} \text{ (th)}$$

$$\frac{\Gamma \rightarrow \Theta, M \qquad M, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda} \text{ } (M) \text{ (cut)},$$

where M is called the *cut-formula.*

Operational Inferences:

$$\frac{\Gamma \rightarrow \Theta, A}{\neg A, \Gamma \rightarrow \Theta} (\neg \rightarrow) \qquad \frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A} (\rightarrow \neg)$$

$$\frac{\Gamma \rightarrow \Theta, A \qquad B, \Gamma \rightarrow \Theta}{A \supset B, \Gamma \rightarrow \Theta} (\supset\rightarrow) \qquad \frac{A, \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \supset B} (\rightarrow\supset)$$

$$\frac{A, \Gamma \rightarrow \Theta}{\bigwedge\Phi, \Gamma \rightarrow \Theta} (\bigwedge \rightarrow) \ (A \in \Phi) \qquad \frac{\{\Gamma \rightarrow \Theta, A : A \in \Phi\}}{\Gamma \rightarrow \Theta, \bigwedge\Phi} (\rightarrow \bigwedge)$$

$$\frac{\{A, \Gamma \rightarrow \Theta : A \in \Phi\}}{\bigvee\Phi, \Gamma \rightarrow \Theta} (\bigvee \rightarrow) \qquad \frac{\Gamma \rightarrow \Theta, A}{\Gamma \rightarrow \Theta, \bigvee\Phi} (\rightarrow \bigvee) \ (A \in \Phi),$$

K-Inferences:

$$\frac{\Gamma, K_i(\Delta) \rightarrow \Theta}{K_i(\Gamma, \Delta) \rightarrow K_i(\Theta)} \text{ } (K \rightarrow K),$$

where Θ consists of at most one formula, $K_i(\Gamma)$ denotes the set $\{K_i(A) : A \in \Gamma\}$ and $K_i(\Gamma, \Delta)$ is $K_i(\Gamma \cup \Delta)$. The last inference rule is the CK-Barcan inference:

$$\frac{\{\Gamma \to \Theta, K_i K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\} \qquad K_i C(A), \Gamma \to \Theta}{\Gamma \to \Theta} \text{ (CK-B)},$$

where A is a cc-formula.

In an operational inference, the formula newly created in the lower sequent is called the *principal formula*. The principal formulae of (CK-B) are $K_i K(A)$, $K \in \bigcup_{m<\omega} \mathsf{K}(m)$, and $K_i C(A)$.

In a similar manner to in Section 3, a *proof* in GL(G) is defined to be a countable tree with the following properties: (i) every path from the root is finite; (ii) a sequent is associated with each node, and the sequent associated with each leaf is an initial sequent; and (iii) adjoining nodes together with the associated sequents form an instance of the above inference rules. A sequent $\Gamma \to \Theta$ is said to be *provable* in GL(G), denoted by $\vdash_G \Gamma \to \Theta$, iff there is a proof P such that $\Gamma \to \Theta$ is associated with the root of P.

The relationship between GL(H) and GL(G) is as follows, which was stated in Kaneko-Nagashima [9].

THEOREM 4.1 (Equivalence of GL(H) and GL(G)). *For any A in $\mathscr{P}^{\omega}$,*
(1): *if* $\vdash_G A$, *then* $\vdash_G \ \to A$;
(2): *if* $\vdash_G \Gamma \to \Theta$, *then* $\vdash_G \bigwedge \Gamma \supset \bigvee \Theta$.

SKETCH OF THE PROOF OF THEOREM 4.1. For (1), it suffices to show that every instance B of the axioms of GL(H) is provable in GL(G), i.e., $\vdash_G \ \to B$, and that every instance of the inference rules of GL(H) is legitimate in GL(G). For (2), it suffices to prove $\vdash_G A \supset A$ and that every instance of the inference rules of GL(G) is legitimate in GL(H). E.g., for (CK-B), it suffices to show that if $\vdash_G$ $\bigwedge \Gamma \supset (\bigvee \Theta) \bigvee K_i K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$ and $\vdash_G K_i C(A) \bigwedge (\bigwedge \Gamma) \supset \bigvee \Theta$, then $\vdash_G \bigwedge \Gamma \supset \bigvee \Theta$. ⊣

To prove the faithful embedding theorem, we need the cut-elimination theorem for GL(G), which was proved in Kaneko-Nagashima [9].

THEOREM 4.2 (Cut-Elimination for GL(G)). *If* $\vdash_G \Gamma \to \Theta$, *then there is a cut-free proof P of* $\Gamma \to \Theta$.

The inference rule (CK-B) is a restriction of the $\bigwedge$-Barcan inference used in Kaneko-Nagashima [9]:

$$\frac{\{\Gamma \to \Theta, K_i(B) : B \in \Phi\} \qquad K_i(\bigwedge \Phi), \Gamma \to \Theta}{\Gamma \to \Theta} \ (\bigwedge\text{-B}),$$

where Φ is a countable subset of $\mathscr{P}^k$ for some $k < \omega$. The above (CK-B) is the restriction of this ($\bigwedge$-B) to Φ having the form $\{K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}$ with a cc-formula A. Cut-elimination is typically sensitive to changes in inference rules, but is not with the restriction on the principal formulae of the above Barcan inference.

An important consequence of Theorem 4.2 is the subformula property. However, (CK-B) does not fully enjoy this property. Hence a cut-free proof having (CK-B), in fact, violates the full subformula property that any formula occurring in a cut-free proof is a subformula of some formula in the endsequent of the proof. Nevertheless, since the principal formulae of (CK-B) are cc-formulae, it holds that if a non-cc-formula A occurs in a cut-free proof, it is also a subformula in the endsequent. This implies

(2) if the endsequent of a cut-free proof P consists of cc-formulae only, all the formulae occurring in P are cc-formulae.

Since the inverse images of those formulae by ψ are in $\mathscr{P}^{\mathrm{f}}$, we can consider this proof $\psi^{-1}(P)$ from the viewpoint of deducibility $\vdash_C$. Now we can state the main result, which will be proved in the end of this section.

THEOREM 4.3 (Faithful Embedding II). *For any A in $\mathscr{P}^{\mathrm{f}}$, $\vdash_C A$ if and only if $\vdash_G \ \rightarrow \psi(A)$.*

When we restrict $\mathscr{P}^{\omega}$ to the space of finitary formulae, the above logic becomes sequent calculus KD4, since (CK-B) is not allowed. Sequent calculus KD4 admits cut-elimination and its cut-free proof satisfies the *full* subformula property. Since CKL is a conservative extension of KD4 by Theorem 3.1, GL(G) is also a conservative extension of KD4 by Theorem 4.3. A concrete application of this observation will be given in Section 6.

The *only-if* part of Theorem 4.3 follows from the *only-if* part of Theorem 3.4 and Theorem 4.1.(1). Hence we prove the *if* part of Theorem 4.3. Once this is proved, the *if* part of Theorem 3.4 follows from this and Theorem 4.1.(1).

To prove the *if* part of Theorem 4.3, first, we prepare the following lemma.

LEMMA 4.4. *For any A, B in $\mathscr{P}^{\mathrm{f}}$,*
(1): *if $\vdash_C A \supset K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$, then $\vdash_C A \supset C_0(B)$;*
(2): *if $\vdash_C A \supset K_i K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$, then $\vdash_C A \supset K_i C_0(B)$.*

PROOF. We prove only (2). Suppose $\vdash_C A \supset K_i K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$. For any frame $\mathscr{M} = (W; R_1, \dots, R_n)$, world $w \in W$ and assignment σ, by the soundness part of Theorem 3.1, $(\mathscr{M}, w) \models_\sigma A \supset K_i K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$. Let $(\mathscr{M}, w) \models_\sigma A$. Then $(\mathscr{M}, w) \models_\sigma K_i K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$. This implies that for any v with $(w, v) \in R_i$, $(\mathscr{M}, v) \models_\sigma K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$. Thus for all v with $(w, v) \in R_i$, $(\mathscr{M}, v) \models_\sigma C_0(B)$ by K2. Thus $(\mathscr{M}, w) \models_\sigma K_i C_0(B)$ by K1. This implies that for all $\mathscr{M}$, $w \in W$ and σ, $(\mathscr{M}, w) \models_\sigma A \supset K_i C_0(B)$. Thus $\vdash_C A \supset K_i C_0(B)$ by the completeness part of Theorem 3.1. ⊣

Since ψ is a bijection from $\mathscr{P}^{\mathrm{f}}$ to the set of all cc-formulae by Lemma 3.3, we use the notational convention that for a formula B in $\mathscr{P}^{\mathrm{f}}$ and a finite subset Γ of $\mathscr{P}^{\mathrm{f}}$, the images $\psi(B)$ and $\psi(\Gamma)$ are denoted by B' and Γ'.

Now suppose that $\vdash_G \ \rightarrow A'$ and A' is a cc-formula. Then there is a cut-free proof P' of $\rightarrow A'$ by Theorem 4.2. Since this proof P' satisfies (2), we can translate each sequent $\Gamma' \rightarrow \Theta'$ in P' by ψ^{-1} into $\Gamma \rightarrow \Theta$ in $\mathscr{P}^{\mathrm{f}}$. Then we prove by induction on the tree structure of P' from its leaves that for every sequent $\Gamma' \rightarrow \Theta'$ in P', $\vdash_C \bigwedge \Gamma \supset \bigvee \Theta$, where Γ and Θ are $\psi^{-1}(\Gamma')$ and $\psi^{-1}(\Theta')$. This inductive proof is essentially the same as the proof of Theorem 4.2.(2), except $(\bigwedge \rightarrow)$, $(\rightarrow \bigwedge)$ with the infinitary principal formulae and (CK-B). We show that the translations of these inferences by ψ^{-1} are legitimate in CKL.

In the case of $(\bigwedge \rightarrow)$,

$$\frac{K(B'), \Gamma' \rightarrow \Theta'}{C(B'), \Gamma' \rightarrow \Theta'}, \text{ where } K \in \bigcup_{m<\omega} \mathsf{K}(m).$$

The induction hypothesis is that $\vdash_C K(B) \bigwedge (\bigwedge \Gamma) \supset \bigvee \Theta$. Since $\vdash_C C(B) \supset K(B)$, we have $\vdash_C C(B) \bigwedge (\bigwedge \Gamma) \supset \bigvee \Theta$.

In the case of $(\rightarrow \bigwedge)$,

$$\frac{\{\Gamma' \rightarrow \Theta', K(B') : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}}{\Gamma' \rightarrow \Theta', C(B')}.$$

The induction hypothesis is that $\vdash_C \bigwedge \Gamma \supset (\bigvee \Theta) \bigvee K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$. This implies $\vdash_C (\bigwedge \Gamma) \bigwedge (\neg \bigvee \Theta) \supset K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$. By Lemma 4.4.(1), we have $\vdash_C (\bigwedge \Gamma) \bigwedge (\neg \bigvee \Theta) \supset C_0(B)$, i.e., $\vdash_C \bigwedge \Gamma \supset (\bigvee \Theta) \bigvee C_0(B)$.

In the case of (CK-B),

$$\frac{\{\Gamma' \rightarrow \Theta', K_i K(B') : K \in \bigcup_{m<\omega} \mathsf{K}(m)\} \quad K_i C(B'), \Gamma' \rightarrow \Theta'}{\Gamma' \rightarrow \Theta'}.$$

The inductive hypothesis is that $\vdash_C (\bigwedge \Gamma) \bigwedge (\neg \bigvee \Theta) \supset K_i K(B)$ for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$ and $\vdash_C K_i C_0(B) \supset (\neg \bigwedge \Gamma) \bigvee (\bigvee \Theta)$. By Lemma 4.4.(2), we have $\vdash_C (\bigwedge \Gamma) \bigwedge (\neg \bigvee \Theta) \supset K_i C_0(B)$. Hence we have $\vdash_C (\bigwedge \Gamma) \bigwedge (\neg \bigvee \Theta) \supset (\neg \bigwedge \Gamma) \bigvee (\bigvee \Theta)$, i.e., $\vdash_C (\bigwedge \Gamma) \supset (\bigvee \Theta)$.

§5. Common Knowledge Logic CKL(G) in Sequent Calculus. As an application of the theorems of Section 4, we obtain common knowledge logic CKL(G) in sequent calculus admitting cut-elimination.

Sequent calculus CKL(G) is obtained from GL(G) by restricting the language to $\mathscr{P}^{\mathrm{f}}$, replacing (CK-B) by

$$\frac{\{\Gamma \rightarrow \Theta, K_i K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\} \quad K_i C_0(A), \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta} \ (\mathrm{C_0K\text{-}B})$$

and adding two other inference rules:

$$\frac{K(A), \Gamma \rightarrow \Theta}{C_0(A), \Gamma \rightarrow \Theta} \ (\mathrm{C_0} \rightarrow) \ (K \in \bigcup_{m<\omega} \mathsf{K}(m))$$

$$\frac{\{\Gamma \rightarrow \Theta, K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}}{\Gamma \rightarrow \Theta, C_0(A)} \ (\rightarrow \mathrm{C_0}).$$

Since the principal formulae of $(\rightarrow \bigwedge)$ and $(\bigvee \rightarrow)$ are finitary, they have only finite numbers of upper sequents. Only $(\mathrm{C_0K\text{-}B})$ and $(\rightarrow \mathrm{C_0})$ have infinite numbers of upper sequents. The rules $(\mathrm{C_0} \rightarrow)$ and $(\rightarrow \mathrm{C_0})$ correspond to special cases of $(\bigwedge \rightarrow)$ and $(\rightarrow \bigwedge)$ in GL(G).

We use the same symbol $\vdash_C$ to denote provability relation in CKL(G), which is defined in the same way as in Section 4 (as having a countable proof). We present two theorems as applications of Theorems 4.1, 4.2 and 4.3. Before it, we state one lemma.

LEMMA 5.1. **(1)**: *If P is a proof of $\Gamma \rightarrow \Theta$ in CKL(G), then $\psi(P)$ is a proof of $\psi(\Gamma) \rightarrow \psi(\Theta)$ in GL(G).*

(2): *If P' is a cut-free proof of $\Gamma' \rightarrow \Theta'$ in GL(G) and Γ', Θ' consist of cc-formulae, then $\psi^{-1}(P')$ is a cut-free proof of $\psi^{-1}(\Gamma') \rightarrow \psi^{-1}(\Theta')$ in CKL(G).*

PROOF. (1) For this, we have to verify only that the translations of ($C_0 \to$), ($\to C_0$) and (C_0K-B) by ψ are legitimate in GL(G). Indeed, these translations are instances of ($\bigwedge \to$), ($\to \bigwedge$) and (CK-B). For example, ($\to C_0$) is translated into

$$\frac{\{\psi(\Delta) \to \psi(\Lambda), K(\psi(A)) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}}{\psi(\Delta) \to \psi(\Lambda), C(\psi(A))} \ (\to \bigwedge).$$

Thus $\psi(P)$ is a proof of $\psi(\Gamma) \to \psi(\Theta)$ in GL(G).

(2) Let P' be a cut-free proof of $\Gamma' \to \Theta'$ in GL(G). Since Γ' and Θ' consist of cc-formulae, all formulae occurring in P' are cc-formulae by (2). Hence the inverse image $\psi^{-1}(P')$ of P' is well defined by Lemma 3.3. Since P' may have infinitary ($\bigwedge \to$), ($\to \bigwedge$) and (CK-B), the translations of them by ψ^{-1} are instances of ($C_0 \to$), ($\to C_0$) and (C_0K-B), respectively. Hence $\psi^{-1}(P')$ is a proof of $\psi^{-1}(\Gamma') \to \psi^{-1}(\Theta')$ and has no (cut). ⊣

This lemma states that CKL(G) is essentially equivalent to the fragment of GL(G) defined by the translator ψ. Hence it follows from Theorems 4.1 and 4.3 that CKL(G) is deductively equivalent to CKL.

THEOREM 5.2 (Equivalence between CKL and CKL(G)). **(1)** *If* $\vdash_C A$, *then* $\vdash_C \to A$;

(2) *if* $\vdash_C \Gamma \to \Theta$, *then* $\vdash_C \bigwedge \Gamma \supset \bigvee \Theta$.

Cut-elimination for CKL(G) follows from Lemma 4.4.(1), (2) and Theorem 4.2.

THEOREM 5.3 (Cut-Elimination for CKL(G)). *If* $\vdash_C \Gamma \to \Theta$, *then there is a cut-free proof* P *of* $\Gamma \to \Theta$ *in CKL(G).*

§6. Game theoretical applications.

Here we briefly look at the epistemic axiomatization of Nash equilibrium in Kaneko [5] in game logic GL, while considering conversions of this axiomatization to CKL and some merits obtained from the faithful embedding theorems.

Consider individual *ex ante* decision making in a finite game $g = (g_1, \dots, g_n)$ with pure strategies, where each player i has his *strategy set* $\Sigma_i = \{\mathrm{s}_{i1}, \dots, \mathrm{s}_{i\ell_i}\}$ and *his payoff function* g_i is a real-valued function on $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ for $i = 1, \dots, n$. Tables 6.1 and 6.2 are 2-person games with $\ell_1 = 2, \ell_2 = 3$ and with $\ell_1 = \ell_2 = 2$, respectively.

	s_{21}	s_{22}	s_{23}
s_{11}	5, 5	0, 0	5, 3
s_{12}	6, 0	2, 2	1, 1

Table 6.1

	s_{21}	s_{22}
s_{11}	5, 5	1, 6
s_{12}	6, 1	3, 3

Table 6.2
Prisoner's Dilemma

We prepare $2n$-ary predicate symbol $R_i(\cdot : \cdot)$ and n-ary predicate symbol $D_i(\cdot)$ for each $i = 1, \dots, n$. We define $R_i(a_1, \dots, a_n : b_1, \dots, b_n)$ and $D_i(a_1, \dots, a_n)$ to be *atomic* formulae, where $i = 1, \dots, n$ and $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \Sigma$. The spaces $\mathscr{P}^\omega$ and $\mathscr{P}^{\mathrm{f}}$ of formulae are defined based on these atomic formulae as propositional variables. The game theoretical intents of $R_i(a_1, \dots, a_n : b_1, \dots, b_n)$ and $D_i(a_1, \dots, a_n)$ are, respectively, that player i *weakly prefers* strategy profile

$(a_1, \ldots, a_n)$ to another $(b_1, \ldots, b_n)$, and that player i *predicts* $(a_1, \ldots, a_n)$ to be chosen by the players as their decisions. We determine the predictions $D_i(\cdot)$ by certain axioms, while $R_i(a_1, \ldots, a_n : b_1, \ldots, b_n)$'s are given primitives.

Each payoff function g_i is described in our formal language as the formula:

$$(\bigwedge\{R_i(x : y) : g_i(x) \geq g_i(y) \text{ and } x, y \in \Sigma\}) \\ \bigwedge(\bigwedge\{\neg R_i(x : y) : \; g_i(x) < g_i(y) \text{ and } x, y \in \Sigma\}),$$

which we denote by G_i. For example, the common knowledge of the payoff functions is described by $C(\bigwedge_i \mathrm{G}_i)$, where i varies over the players.

Nash equilibrium is described as the formula $\bigwedge_i \bigwedge_{y_i} R_i(a : y_i, a_{-i})$, which is denoted by $\mathrm{Nash}(a)$. Here

$$a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n), (y_i, a_{-i}) = (a_1, \ldots, a_{i-1}, y_i, a_{i+1}, \ldots, a_n),$$

and y_i varies over Σ_i. It means that each player i maximizes his payoff under predicted strategies $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. In both games of Tables 6.1 and 6.2, only $(\mathrm{s}_{12}, \mathrm{s}_{22})$ is a Nash equilibrium.

The following are base axioms for $D_1(\cdot), \ldots, D_n(\cdot)$: for each $i = 1, \ldots, n$,

Axiom $\mathrm{D1}_i^0 : \bigwedge_x (D_i(x) \supset \bigwedge_{y_i} R_i(x : y_i, x_{-i}))$;

Axiom $\mathrm{D2}_i^0 : \bigwedge_x \bigwedge_j (D_i(x) \supset D_j(x))$;

Axiom $\mathrm{D3}_i^0 : \bigwedge_x (D_i(x) \supset K_i(D_i(x)))$;

Axiom $\mathrm{D4}_i^0 : \bigwedge_x \bigwedge_y \bigwedge_j (D_i(x) \bigwedge D_i(y) \supset D_i(x_j, y_{-j}))$,

where x and y vary over the strategy profiles Σ. Each axiom is described as follows.

$\mathrm{D1}_i^0$: (*Best Response to Predicted Decisions*): When player i predicts final decisions $x_1, \ldots, x_n$ for the players, his own decision x_i maximizes his payoff against his prediction x_{-i}, that is, x_i is a best response to x_{-i}.

$\mathrm{D2}_i^0$: (*Identical Predictions*): The other players reach the same predictions as player i's.

$\mathrm{D3}_i^0$: (*Knowledge of Predictions*): Player i knows his own predictions.

$\mathrm{D4}_i^0$: (*Interchangeability*): Player i's predictions are interchangeable, which is a requirement for independent decision making.

We assume that player i himself knows these axioms as his behavioral postulate. Thus, the axiom for him is $(\mathrm{D1}_i^0 \bigwedge \cdots \bigwedge \mathrm{D4}_i^0) \bigwedge K_i(\mathrm{D1}_i^0 \bigwedge \cdots \bigwedge \mathrm{D4}_i^0)$, which we denote by $\mathrm{D}_i(1\text{–}4)$. We denote $\bigwedge_i \mathrm{D}_i(1\text{–}4)$ by $\mathrm{D}(1\text{–}4)$.

Since $\mathrm{D}(1\text{–}4)$ is a nonlogical axiom, we do not allow it to be an initial formula in a proof, for our logics do not satisfy the deduction theorem. We treat nonlogical axioms as follows: For a set Γ of formulae and a formula A, we write $\Gamma \vdash_G A$ iff $\vdash_G \bigwedge \Phi \supset A$ for some finite subset Φ of Γ. Similarly, $\Gamma \vdash_C A$ is defined.

Axiom $\mathrm{D}(1\text{–}4)$ is far from being sufficient to determine $D_1(\cdot), \ldots, D_n(\cdot)$. $\mathrm{D}_i(1\text{–}4)$ requires player i to know the axioms, $\bigwedge_{j \neq i} \mathrm{D}_j(1\text{–}4)$, for the other players. Thus, we assume that all the players know $\mathrm{D}(1\text{–}4)$, i.e., $\bigwedge_i K_i(\mathrm{D}(1\text{–}4))$. However, this addition does not solve the problem: under this addition, we have

(3) $$\mathrm{D}(1\text{–}4), \bigwedge_i K_i(\mathrm{D}(1\text{–}4)) \vdash_G D_i(a) \supset K_j K_k(D_i(a)).$$

This requires the imaginary player k of the mind of player j to know the behavioral postulate, $\mathrm{D}_i(1\text{–}4)$, for player i, for otherwise, consequence (3) would not make sense for k. This suggests to add another formula $\bigwedge_j \bigwedge_i K_j K_i(\mathrm{D}(1\text{–}4))$. Under this addition, however, we meet the same difficulty as that in (3) of the depth of one more degree, and need to go to the next step. This process is an infinite regress of adding the knowledge of D(1–4) of any finite depth. Thus the infinite regress leads to the set:

(4) $$\{K(\mathrm{D}(1\text{–}4)) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}.$$

The conjunction of this set is the common knowledge, $C(\mathrm{D}(1\text{–}4))$, of D(1–4). We adopt this common knowledge as an axiom for $D_i(\cdot), i = 1, \dots, n$. Then:

LEMMA 6.1. $C(D(1\text{–}4)) \vdash_\omega \bigwedge_x (D_i(x) \supset C(\mathrm{Nash}(x))$.

For the converse, we need the condition of *interchangeability* on game g :

(5) if a and b are Nash equilibria, then so is (b_i, a_{-i}) for all $i = 1, \dots, n$.

Both games of Tables 6.1 and 6.2 satisfy (5), since each has a unique Nash equilibrium. Then $\{C(\mathrm{Nash}(x)) : x \in \Sigma\}$ satisfies $C(\mathrm{D}(1\text{–}4))$ under the common knowledge of $\bigwedge_i \mathrm{G}_i$. That is, if every occurrence of $D_i(a)$ in $C(\mathrm{D}(1\text{–}4))$ is replaced by $C(\mathrm{Nash}(a))$ for any $a \in \Sigma$ and $i = 1, \dots, n$, which is denoted by $C(\mathrm{D}(1\text{–}4))[C(\mathrm{Nash})]$, then:

LEMMA 6.2. *If game g satisfies* (5), *then* $C(\bigwedge_i G_i) \vdash_G C(D(1\text{–}4))[C(Nash)]$.

Thus, $C(\mathrm{Nash}(\cdot))$ is a solution of $C(\mathrm{D}(1\text{–}4))$, and Lemma 6.1 states that it is the deductively weakest. Hence $C(\mathrm{Nash}(\cdot))$ can be regarded as what $C(\mathrm{D}(1\text{–}4))$ determines. To formulate this claim explicitly, we introduce one more axiom schema:

WD: $C(\mathrm{D}(1\text{–}4))[\mathscr{A}] \supset \bigwedge_i \bigwedge_x (A_i(x) \supset D_i(x))$,

where $\mathscr{A}$ is a family $\{A_i(x) : x \in \Sigma \text{ and } i = 1, \dots, n\}$ of formulae, and $C(\mathrm{D}(1\text{–}4))[\mathscr{A}]$ is obtained from $C(\mathrm{D}(1\text{–}4))$ by replacing all occurrences of $D_i(a)$ by $A_i(a)$ for all $a \in \Sigma$ and $i = 1, \dots, n$. Lemmas 6.1 and 6.2 together with WD imply the following theorem.

THEOREM 6.3. *Let g be a game satisfying* (5). *Then* $C(D(1\text{–}4)), C(\bigwedge_i G_i), WD \vdash_G \bigwedge_x (D_i(x) \equiv C(Nash(x)))$.

For a game g not satisfying (5), we need further assumptions in order to have a result parallel to Theorem 6.3 (see Kaneko [5]). Also for the game of Table 6.2, the above axiomatization requiring the common knowledge of various formulae may be regarded as too stringent. In this game, each player can make a decision to maximize his payoff by using only the knowledge of his own payoff function. In fact, we can weaken the above axiomatization for such games. However, for the game of Table 6.1, the above axiomatization is unavoidable. See Kaneko [3].

By the faithful embedding theorems, the above axiomatization can be converted to CKL. We note that WD is a schema and has more formulae in GL than in CKL but only $\{C(\mathrm{Nash}(x)) : x \in \Sigma\}$ is used as $\mathscr{A}$ in Theorem 6.3. Since these are cc-formulae, no difficulty arises with the conversion of Theorem 6.3 to CKL.

Now we consider some metatheorems related to the above axiomatization. The infinite regress heuristically discussed above D(1–4) can be evaluated by the depth lemma given by Kaneko-Nagashima [8]. They proved the following lemma, using the concept of K-depth. The *K-depth* $\delta_i(A)$ ($i = 1, \dots, n, A \in \mathscr{P}^{\mathrm{f}}$) is the maximum number of nesting occurrences of K_i's in A, ignoring the occurrences of the same K_i in the immediate scope of K_i, i.e., it is defined inductively as follows: $\delta_i(A) = 0$ if A is atomic, $\delta_i(\neg A) = \delta_i(A)$, $\delta_i(A \supset B) = \max(\delta_i(A), \delta_i(B))$, $\delta_i(\bigwedge \Phi) = \delta_i(\bigvee \Phi) = \max_{A \in \Phi} \delta_i(A)$, $\delta_i(C_0(A)) = 0$, $\delta_i(K_j(A)) = 0$ if $i \neq j$ and $\delta_i(K_j(A)) = \max(\delta_i(A), \max_{k \neq i} \delta_k(A) + 1)$ if $j = i$. Let $\delta(A) = \max_i \delta_i(A)$. Then:

LEMMA 6.4 (Depth Lemma). *Let $K \in \mathsf{K}(m)$, and A, B formulae in $\mathscr{P}^{\mathrm{f}}$. If $\vdash_{KD4} B \supset K(A)$ and $\delta(B) < m$, then $\vdash_{KD4} \neg B$ or $\vdash_{KD4} A$.*

In [8], this lemma is proved using the cut-elimination theorem for S4 in sequent calculus, which can be modified into a proof in KD4. In GL(G), (CK-B) is an obstacle to prove directly this lemma, since it violates the subformula property. However, as was stated after Theorem 4.3, GL(G) is a conservative extension of KD4. Hence the depth lemma holds for GL(G), GL(H) as well as CKL.

It follows from Lemma 6.4 that the common knowledge of D(1–4) is needed for Lemma 6.1, i.e., if the K-depth of the antecedent of Lemma 6.1 is finite, we could not derive $C(\mathrm{Nash}(\cdot))$. Also, Lemma 6.4 justifies that (3) requires $K_j K_k(\mathrm{D}(1\text{–}4))$. Indeed, it follows from Lemma 6.4 that $K_j K_k(\mathrm{D}(1\text{–}4))$ is not derived from $\mathrm{D}(1\text{–}4), \bigwedge_i K_i(\mathrm{D}(1\text{–}4))$. See Kaneko [5] for further applications.

The last comment is on the undecidability result obtained in Kaneko-Nagashima [7] for a game with mixed strategies. Even when mixed strategies are allowed, we could obtain the above axiomatization with no essential changes, though a predicate extension of GL as well as the language of an ordered field theory are required. For a game with pure strategies, we cannot guarantee a game to have a Nash equilibrium, but it is the basic theorem by Nash [13] that every finite game has a Nash equilibrium in mixed strategies. It follows from this existence result and Tarski's completeness theorem on the real closed field theory that $C(\Phi_{\mathrm{rcf}}), C(\bigwedge_i \mathrm{G}) \vdash_\omega C(\exists \vec{x} \mathrm{Nash}(\vec{x}))$, where Φ_{rcf} is the set of real closed field axioms in the ordered field language based on the constants $\mathbf{0}, \mathbf{1}$, function symbols $+, -, \cdot, /$, and predicates $\geq, =$. Note that the equality axioms are included in Φ_{rcf}. However, the playability of a game is formulated, based on the counterpart of Theorem 6.3, as whether $C(\Phi_{\mathrm{rcf}}), C(\bigwedge_i \mathrm{G}) \vdash_\omega \exists \vec{x} C(\mathrm{Nash}(\vec{x}))$ or not. That is, the pure knowledge of the existence is not sufficient, but the specific knowledge is required. In fact, Kaneko-Nagashima [7] prove that there is a 3-person game with two pure strategies for each player such that it has a unique Nash equilibrium but

$$\text{(6)} \qquad \begin{array}{l} \text{neither } C(\Phi_{\mathrm{rcf}}), C(\bigwedge_i \mathrm{G}_i) \vdash_\omega \exists \vec{x} C(\mathrm{Nash}(\vec{x})) \\ \text{nor } C(\Phi_{\mathrm{rcf}}), C(\bigwedge_i \mathrm{G}_i) \vdash_\omega \neg \exists \vec{x} C(\mathrm{Nash}(\vec{x})). \end{array}$$

This means that any player can neither reach a decision nor he can tell he cannot reach a decision. Hence he cannot play a game. See Kaneko [6] for more detailed discussions on this subject.

The above undecidability result itself can be obtained in predicate CKL by the embedding theorem of predicate CKL to predicate GL. However, the undecidability is based on a *term-existence* theorem proved in [9], and this would not be converted to predicate CKL without proving faithfulness. So far, the faithful embedding result is available only for the propositional CKL and GL, since completeness for the predicate CKL is not yet proved. If the completeness theorem for predicate

CKL is proved, the results of this paper could be extended to predicate CKL and GL without difficulty.

The propositional counterpart of the term-existence theorem is the disjunctive property:

$$\text{if } \Gamma, \Theta \text{ do not include } K_i, i = 1, \dots, n, \text{ and if } C(\Gamma) \vdash_\omega \bigvee C(\Theta), \\ \text{then } C(\Gamma) \vdash_\omega C(A) \text{ for some } A \in \Theta,$$

which can be converted to CKL. Using this disjunctive property, we can evaluate some playability of a game with pure strategies. See Kaneko [5].

§7. Concluding remarks. (1) The iterative definition, $C(A)$, of common knowledge makes sense in KD4$^\omega$, though it loses the fixed point property $C(A) \supset K_i C(A)$ for $i = 1, \dots, n$. However, KD4$^\omega$ is not sufficient from the game theoretical as well as semantical points of view. Lemma 6.2, *a fortiori*, Theorem 6.3, *cannot* be obtained in KD4$^\omega$, and also, it is proved in [4] that KD4$^\omega$ is Kripke-incomplete.

(2) Halpern-Moses [2] and Lismont-Mongin [11] gave sound-completeness results for various common knowledge logics, including the K-, K4-, KD45-, S4-, S5-types as well as logics weaker than the K-type. Our faithful embedding theorems could be available (with appropriate modifications) as far as the cut-elimination for game logic in question and the completeness theorem for the corresponding common knowledge logic are available. So far, cut-elimination holds for the K-, K4-, KD- as well as S4-type game logics, but fails for the sequent calculus KD45, S5 (cf., Ohnishi-Matsumoto [14])[4] and is unknown for logics weaker than K. Therefore we could obtain the parallel results from the K-type to S4-type game and common knowledge logics.

(3) We have proved the faithful embedding of CKL into the propositional fragment of GL. Predicate GL without the $\forall$-Barcan axiom, $\forall x K_i(A(x)) \supset K_i(\forall x A(x))$, is a conservative extension of the propositional GL under the choice of an appropriate language. Hence propositional CKL is faithfully embedded into predicate GL.

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[4]Ohnishi-Matsumoto [14] gave a counterexample for cut-elimination in their formulation of sequent calculus S5. However, Takano [15] proved in this system that for any proof, there is another proof of the same endsequent enjoying the subformula property. There are some other formulations of S5 admitting cut-elimination and/or the subformula property. However, they have quite delicate structures and are difficult for some applications. See also Takano [15] for related references. The development of S5-type game logic and the faithful embedding of the S5-type common knowledge logic to it remain open.

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