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EPISTEMIC MODELS OF SHALLOW DEPTHS AND DECISION MAKING IN GAMES: HORTICULTURE

MAMORU KANEKO AND NOBU-YUKI SUZUKI

Abstract. Kaneko-Suzuki developed epistemic logics of shallow depths with multiple players for investigations of game theoretical problems. By shallow depth, we mean that nested occurrences of belief operators of players in formulae are restricted, typically to be of finite depths, by a given *epistemic structure*. In this paper, we develop various methods of surgical operations (cut and paste) of epistemic world models. An example is a bouquet-making, i.e. tying several models into a bouquet. Another example is to engraft a model to some branches of another model. By these methods, we obtain various meta-theorems on semantics and syntax on epistemic logics. To illustrate possible uses of our meta-theorems, we present one game theoretical theorem, which is also a meta-theorem in the sense of logic.

§1. Introduction. To study decision making in games, common knowledge has been regarded as important, and various common knowledge extensions of multi-agent epistemic logics have been discussed.¹ Nevertheless, common knowledge is an idealized limit concept and does not help us to consider less ideal game theoretical situations. Kaneko-Suzuki [9], [10] and [11] have changed the direction of research into finite and bounded problems, and have developed epistemic logics of shallow depths, where shallow depths mean that nested occurrences of belief operators B_1, B_2, \dots, B_n are bounded. Their developments include both proof theory and model theory as well as applications to game theory. In [9], definitions and general theorems such as completeness are provided. In [10], proof theoretical considerations are given, and in [11], game theoretical problems are discussed.

In this paper, we develop various methods of surgical operations, “cut and paste”, of (semantical) models. These methods enable us to construct more complex models from simpler ones and *vice versa*. Thus, we develop a *horticulture* of models. By these operations, we obtain various semantical results. Such semantical results are translated into syntactical meta-theorems by the completeness theorem given in Kaneko-Suzuki [9]. We illustrate the uses of these meta-theorems by proving one meta-theorem on decision making in a simple 2-person game.

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¹For a recent survey of common knowledge logics, see Kaneko et al. [8]. For recent general developments of epistemic logics, see Fagin et al. [3] and Meyer-van der Hoek [12].

Specifically, we develop the following methods of surgical operations:

- (1) *bouquet making* by tying various models at their roots;
- (2) *cutting* a part of a model;
- (3) *cane extension* of a model;
- (4) *engrafting* a model into another model.

These are model-theoretic operations to combine various models, to cut some part of a model and/or change a model to another one. By these operations, the truth valuations of formulae in models are compared with those of the corresponding formulae in the constructed models.²

Then we consider the corresponding proof-theoretic counterpart of truth valuations. These operations give meta-theorems in the syntactical sense. Examples of syntactical results obtained by our methods are the *Epistemic Separation Theorem*, *Epistemic Disjunction Theorem* and *Depth Lemma*, which are refinements of the results proved proof-theoretically in Kaneko-Nagashima [7] and [6]. Thus, we give a systematic way of proving these results as well as presenting more meta-theorems.

As mentioned above, the development of our epistemic logics of shallow depths and their semantics are undertaken for game theoretical applications. The objective of our horticulture is also the game theoretical uses of these operations and of meta-theorems. To illustrate the uses of the meta-theorems, we will give one theorem which is on game theoretical decision making and is also a meta-theorem in the sense of logic. Here, we will prove it, step by step by applications of our meta-theorems, though the game itself is kept simple. See Kaneko-Suzuki [11] for extensive discussions of applications to game theory.

Throughout the paper, we refer to the following two games to illustrate various basic concepts as well as our meta-theorems. Consider the following 2-person games of Tables 1.1 (*Prisoner's Dilemma*) and Table 1.2.

	s_{21}	s_{22}
s_{11}	(5, 5)	(1, 6)
s_{12}	(6, 1)	(3, 3)

TABLE 1.1. $g^1 = (g_1^1, g_2^1)$.

	s_{21}	s_{22}
s_{11}	(5, 5)	(1, 2)
s_{12}	(6, 1)	(3, 3)

TABLE 1.2. $g^2 = (g_1^2, g_2^2)$.

In each game, player $i = 1, 2$ has two pure strategies s_{i1}, s_{i2} , and chooses one simultaneously and independently of the other player. The entries of each matrix are vectors of payoffs to the players, e.g., if 1 and 2 choose s_{12} and s_{21} , they would receive 6 and 1, respectively. Table 1.2 is a modification of Table 1.1, where only player 2's payoff 6 for (s_{11}, s_{22}) is changed to 2.

In Table 1.1, the second strategy, s_{12} , for player 1 gives a better payoff whatever player 2 chooses, and the symmetric argument holds for the strategy s_{22} of player 2. In this sense, the second strategy of each player is called a *dominant* strategy. Here, it is sufficient to assume that each player believes that his own payoff function is described by the game of Table 1.1, to obtain the result that each player chooses a

²In the literature of modal logic, surgical operations of models have been used in various manners. For example, some surgical operations comparable with some of ours are found in Chellas [2] and Hughes-Cresswell [4]. Different types of operations of models are found in Bull-Seegerberg [1].

dominant strategy. We use the notation $B_1(\hat{g}_1^1)$ and $B_2(\hat{g}_2^1)$, to mean that each $i = 1, 2$ believes that his payoff function is g_i^1 . Here \hat{g}_i^1 is the set of formulae describing payoff function g_i^1 , which will be defined in Section 2.1. We need only formulae without nested occurrences of $B_1(\cdot)$ and $B_2(\cdot)$, to describe the decision making by dominant strategies in the game g^1 .

In Table 1.2, player 2 has no dominant strategy, while 1 still has the same dominant strategy s_{12} as in Table 1.1. Thus, 2 cannot choose a dominant strategy under the assumption $B_2(\hat{g}_2^2)$. One possible way to have a decision for player 2 is to predict what 1 would choose. For example, player 2 might choose his second strategy s_{22} under the prediction that 1 would choose the dominant strategy s_{12} . In this case, we need to add $B_2B_1(\hat{g}_1^2)$ to $B_1(\hat{g}_1^2)$ and $B_2(\hat{g}_2^2)$. Thus, we have nested occurrences of belief operators only of the form $B_1B_2(\cdot)$ to describe the process of decision making of players.

The purpose of the above examples is to point out that it may suffice to consider bounded nested occurrences of belief operators. This boundedness will be captured by the concept of an *epistemic structure*, which will be defined in Section 2. Both proof- and model-theories are developed relative to a given epistemic structure, which will be provided in Section 2. The main contributions, i.e., the horticulture of models, will come after Section 2.

In this paper, we adopt the KD-type epistemic logics of shallow depths and their semantics. We have various reasons for this choice. We treat logics of beliefs rather than those of knowledge, since we would like to allow false beliefs. Accordingly, we drop the Axiom of Truthfulness (Axiom T). We drop the Axiom of Positive Introspection, too, since it involves implicitly an infinite structure and deviates from our original motivation for epistemic logics of shallow depths. Nevertheless, it is possible to include the Axiom of Positive Introspection with appropriate modifications, which will be discussed in Section 6. In the final section (Section 6), we will give some remarks on the choice of epistemic axioms, and on some other extensions (restrictions) of our structures.

§2. Epistemic logics GL_E and epistemic world semantics. In Sections 2.1 to 2.3, we give a brief survey of the development of epistemic logics of shallow depths and their semantics from Kaneko-Suzuki [9], [10] and [11]. In Section 2.4, we present one theorem on decision making in the game of Table 1.2. This theorem illustrates the uses of meta-theorems given in the subsequent sections.

Since we adopt a KD-type epistemic logic of shallow depths, we refer to the standard multi-modal KD^n in various places. The exact relationship of our logic to KD^n is briefly discussed in Section 2.3.

2.1. Epistemic structures and the epistemic depths of formulae. To define the epistemic logic GL_E , we start with the following list of primitive symbols:

Propositional variables: $\mathbf{p}_0, \mathbf{p}_1, \dots$;

Logical connectives: \neg (not), \supset (implies), \wedge (and), \vee (or);

Unary Belief operators: B_1, \dots, B_n ; and *parentheses:* $(,)$,

where the subscripts $1, \dots, n$ of $B_1(\cdot), \dots, B_n(\cdot)$ are the names of *players*. We denote the set of players $\{1, \dots, n\}$ by N . The set $PV := \{\mathbf{p}_0, \mathbf{p}_1, \dots\}$ is assumed to be a nonempty countable set.

The entire set \mathcal{S} of formulae is defined from PV inductively: (i) each $p \in PV$ is a formula, (ii) if A, B are formulae, so are $(\neg A), (A \supset B), B_1(A), \dots, B_n(A)$, and (iii) if Φ is a nonempty finite set of formulae, then $(\bigwedge \Phi), (\bigvee \Phi)$ are also formulae.³ We write $\{B_i(A) : A \in \Phi\}$ as $B_i(\Phi)$, and follow the standard way of abbreviating parentheses. We also abbreviate $\bigwedge\{A, B\}, \bigvee\{A, B, C\}$ as $A \wedge B, A \vee B \vee C$, etc. We say that a formula A is *nonepistemic* if and only if no B_i occurs in A for any $i \in N$.

To give a restriction on the set \mathcal{S} in terms of the depth of nesting occurrences of B_i ($i \in N$), we, first, introduce the notion of an epistemic structure and, second, describe the epistemic depth of each formula. These concepts reflect *interpersonal* epistemic aspects as well as *intrapersonal* ones, that is, a player's thinking about other players, as well as a player's thinking about himself.

Let $N^{<\omega}$ be the set of all finite sequences (i_1, \dots, i_m) consisting of players in N ($m \geq 0$). When $m = 0$, (i_1, \dots, i_m) is stipulated to be the *null symbol* ε . We call each $e = (i_1, \dots, i_m)$ in $N^{<\omega}$ an *epistemic status*. Note that we allow repetitive occurrences, i.e., $i_k = i_{k+1}$, in $e = (i_1, \dots, i_m)$. By an epistemic status $e = (i_1, \dots, i_m)$, we express the nested structure of the scopes, i.e., i_m is the player imagined by i_{m-1} , who is imaginary also in the mind of i_{m-2}, \dots , etc. For example, $B_{i_1}B_{i_2} \dots B_{i_m}(A)$ means that player i_1 believes that i_2 believes that \dots that i_m believes A . We abbreviate $B_{i_1}B_{i_2} \dots B_{i_m}(A)$ as $B_e(A)$. When e is the null symbol ε , $B_e(A)$ is stipulated to be A itself.

We use the standard *concatenation*: $e * e' = (i_1, \dots, i_m, j_1, \dots, j_k)$ for any $e = (i_1, \dots, i_m)$ and $e' = (j_1, \dots, j_k) \in N^{<\omega}$. We stipulate $\varepsilon * e = e * \varepsilon = e$. When $e' = (j)$, we write simply $e * j$ for $e * (j)$. Then $B_{e*j}(A) = B_e B_j(A)$. For any subset F of $N^{<\omega}$, we write $e * F = \{e * e' : e' \in F\}$.

We say that a nonempty subset E of $N^{<\omega}$ is an *epistemic structure* if and only if for any $(i_1, \dots, i_m) \in N^{<\omega}$ with $m \geq 1$,

$$(1) \quad (i_1, \dots, i_m) \in E \quad \text{implies} \quad (i_1, \dots, i_{m-1}) \in E.$$

This implies $\varepsilon \in E$. The sets $N^{<\omega}$ and $\{\varepsilon\}$ are trivial examples for epistemic structures. A less trivial example is $E^{(2)} = \{\varepsilon, (1), (2), (2, 1)\}$ described as the tree of Figure 2.1, which is the epistemic structure required for the decision making of player 2 in the game of Table 1.2. That is, player 2 thinks about 1's thinking, which is expressed by epistemic status $(2, 1)$ and then, based on this, 2 thinks about his own decision making, which is expressed by (2) . At the null symbol ε , the investigator's thinking is expressed. Thus, by an epistemic structure E , we express the interpersonal (as well as intrapersonal) structure of how the players think about others' thinking.

For any set F of epistemic statuses in $N^{<\omega}$, the minimal epistemic structure including F is uniquely defined, which we denote by F^* . For example, for $\ell = (i_1, \dots, i_l) \in N^{<\omega}$, $\{\ell\}$ is singleton, but $\{\ell\}^* = \{(i_1, \dots, i_k) : k = 0, \dots, l\}$ has $l + 1$ epistemic statuses, which is depicted by Figure 2.2. This will be used to define a cane extension in Section 4.

The following are properties owned by epistemic structures.

³This formulation slightly deviates from the standard one, but we adopt this to facilitate certain game theoretical considerations.

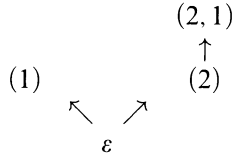


FIGURE 2.1.

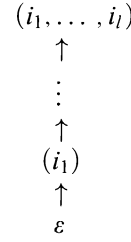


FIGURE 2.2.

LEMMA 2.1.

- (1) If E^1 and E^2 are epistemic structures, then so are $E^1 \cap E^2$ and $E^1 \cup E^2$.
- (2) If E, E^1 and E^2 are epistemic structures satisfying $E = E^1 \cup E^2$ and $E^1 \cap E^2 = \{\varepsilon\}$, then there is a partition (S^1, S^2) of N such that

$$E^k = \{(i_1, \dots, i_m) \in E : i_1 \in S^k\} \cup \{\varepsilon\}$$

for $k = 1, 2$.

PROOF. We prove only (2). Let $S^1 = \{i \in N : (i) \in E^1\}$ and $S^2 = N - S^1$. We denote $\{(i_1, \dots, i_m) \in E : (i_1) \in S^k\} \cup \{\varepsilon\}$ by F^k for $k = 1, 2$. Then $E^k \subseteq F^k$ for $k = 1, 2$ and $F^1 \cup F^2 = E$. Since $F^1 \cap F^2 = \{\varepsilon\}$, we have $F^k = E^k$ for $k = 1, 2$. \dashv

The first assertion implies that the set of all epistemic substructures of an epistemic structure E is a lattice, which has also the minimum element $\{\varepsilon\}$ and maximum element E . Incidentally, it is a Brouwerian lattice, i.e., the dual of a Heyting algebra.

For the purpose of giving a restriction on the set of formulae by E , we use the concept of the *epistemic depth of each formula*. Let A be a formula. We define the epistemic depth $\delta(A)$ of A by induction on the length of A :

- D0. $\delta(p) = \{\varepsilon\}$ for all $p \in PV$;
- D1. $\delta(\neg A) = \delta(A)$;
- D2. $\delta(A \supset B) = \delta(A) \cup \delta(B)$;
- D3. $\delta(\bigwedge \Phi) = \bigcup_{A \in \Phi} \delta(A)$ and $\delta(\bigvee \Phi) = \bigcup_{A \in \Phi} \delta(A)$;
- D4. $\delta(B_i(A)) = \{(i, i_1, \dots, i_m) : (i_1, \dots, i_m) \in \delta(A)\}$,

where (i, ε) is stipulated to be (i) . For example, $\delta(p \wedge B_1(p)) = \{\varepsilon, (1)\}$ and $\delta(B_2B_2B_1(p)) = \{(2, 2, 1)\}$, where $p \in PV$. For a set Γ of formulae, let $\delta(\Gamma) = \bigcup_{A \in \Gamma} \delta(A)$. Note that $\delta(A)$ may not satisfy condition (1), but that the smallest epistemic structure including $\delta(A)$ is uniquely determined, which we denote by $\delta^*(A)$. For example, $\delta^*(B_2B_2B_1(p)) = \{\varepsilon, (2), (2, 2), (2, 2, 1)\}$.

Let E be an epistemic structure. Then we define

$$(2) \quad \mathcal{P}_E = \{A \in \mathcal{P} : \delta(A) \subseteq E\}.$$

Thus \mathcal{P}_E is the set of formulae whose epistemic depths are included in E . If A is nonepistemic, then $\delta(A) = \{\varepsilon\} \subseteq E$ by D0–D3, which implies that \mathcal{P}_E contains all nonepistemic formulae. If $E = \{\varepsilon\}$, then $\mathcal{P}_{\{\varepsilon\}}$ is the set of all nonepistemic formulae. When $E = N^{<\omega}$, $\mathcal{P}_{N^{<\omega}}$ is the entire set \mathcal{P} . For $E^{(2)} = \{\varepsilon, (1), (2), (2, 1)\}$ given above, $\mathcal{P}_{E^{(2)}}$ is the set of formulae describing 2's thinking about the game and 2's thinking about 1's thinking about the game.

We denote the $\{e' : e * e' \in E\}$ by $E(e)$. This is the subtree of E starting at e . and is also an epistemic structure. We denote $\mathcal{P}_{E(e)}$ by $\mathcal{P}_E(e)$, and attach it to each epistemic status $e = (i_1, \dots, i_m)$, which will play a role in defining our logics as well as semantics. The set $\mathcal{P}_E(e)$ is also described as $\{A : B_e(A) \in \mathcal{P}_E\}$. We say that a formula in $\mathcal{P}_E(e)$ is *e-admissible in E*.

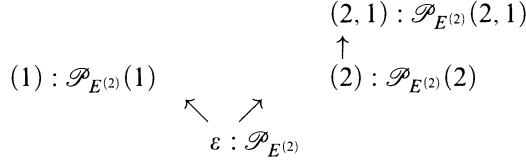


FIGURE 2.3.

In Figure 2.3, $\mathcal{P}_{E(2)}(1)$ and $\mathcal{P}_{E(2)}(2, 1)$ coincide with the set of nonepistemic formulae $\mathcal{P}_{\{\varepsilon\}}$, since at these epistemic statuses, the players do not consider any players’ thinking. At (2), the attached set $\mathcal{P}_{E(2)}(2)$ is $\mathcal{P}_{\{\varepsilon, (1)\}} = \{A : \delta(A) \subseteq \{\varepsilon, (1)\}\}$, that is, player 2 thinks about player 1’s thinking as well as nonepistemic matters. The entire $\mathcal{P}_{E(2)}$ is attached to ε .

Finally, we give descriptions of the games of Section 1 in our formalized language. The idea is that each payoff function is expressed as a set of preferences. Specifically, we assume

Strategy symbols: $s_{11}, s_{12}, s_{21}, s_{22}$;

Preference symbols: $P_1(\cdot; \cdot), P_2(\cdot; \cdot)$.

We write $S_1 = \{s_{11}, s_{12}\}$, $S_2 = \{s_{21}, s_{22}\}$ and $S = S_1 \times S_2$. Let $AF := \{P_i(s; s') : s, s' \in S \text{ and } i = 1, 2\}$, which has $2^4 \times 2 = 32$ atomic formulae. The expression $P_i(s; s')$ is intended to mean that player i (weakly) prefers $s = (s_1, s_2)$ to $s' = (s'_1, s'_2)$. We adopt this set AF as PV and then we have the set of formulae \mathcal{P} and \mathcal{P}_E based on AF . In this language, we describe a given payoff (real-valued) function g_i on S by the following set of preferences:

$$(3) \quad \{P_i(s; s') : g_i(s) \geq g_i(s')\} \cup \{\neg P_i(s; s') : g_i(s') > g_i(s)\}.$$

We denote this set by the symbol \hat{g}_i . That is, payoff function g_i is expressed by the set of preferences.⁴

Then $B_i(\hat{g}_i) := \{B_i(A) : A \in \hat{g}_i\}$ describes “player i believes that g_i is his payoff function”. The set of formulae $B_j B_i(\hat{g}_i) := \{B_j B_i(A) : A \in \hat{g}_i\}$ describes “player j believes that i believes that i ’s payoff function is g_i ”. We denote $\hat{g}_1 \cup \hat{g}_2$ by \hat{g} . Then $B_i(\hat{g})$ describes “player i believes that the game is $g = (g_1, g_2)$ ”.

We introduce two specific formulae for game theoretical considerations. The following formula expresses “ s_1 is a dominant strategy”:

$$(4) \quad \bigwedge \{P_1(s_1, t_2; t_1, t_2) : (t_1, t_2) \in S\},$$

⁴In the terminology of economics, the set, \hat{g}_i , captures only the ordinal property of g_i . This suffices for the purposes of this paper.

which we denote by $\text{Dom}_1(s_1)$. It means that whatever t_2 player 2 chooses, strategy s_1 maximizes 1's payoff. We define $\text{Dom}_2(s_2)$ in the parallel manner. In the nonformalized game theory, it is considered that $\text{Dom}_1(s_{12})$ and $\text{Dom}_2(s_{22})$ hold in Table 1.1, but that neither $\text{Dom}_2(s_{21})$ nor $\text{Dom}_2(s_{22})$ holds in Table 1.2.

To describe the decision criterion of player 2 for the game of Table 1.2, the other formula to be used is

$$(5) \quad \bigwedge \{ P_2(s_1, s_2; s_1, t_2) : t_2 \in S_2 \},$$

which is denoted by $\text{Best}_2(s_2 \mid s_1)$. This describes “ s_2 is a best response to s_1 ” or “ s_2 is a payoff maximizing strategy given the prediction that s_1 would be played by 1”. In Table 1.2, it is considered that $\text{Best}_2(s_{22} \mid s_{12})$ holds. This could be a criterion of decision making, for which player 2 needs the prediction that player 1 would choose s_{12} .

2.2. Epistemic world semantics. Let E be an epistemic structure. In this section, we introduce an E -frame, which is a modification of a Kripke frame. The general idea of an E -frame is that the set of possible worlds is partitioned into subsets indexed by epistemic statuses in E . Each possible world of an epistemic status $e = (i_1, \dots, i_m)$ is imagined by player i_m in the mind of i_{m-1}, \dots , in the mind of i_1 .

DEFINITION 2.2. An $n + 2$ tuple $\mathcal{F} = (W; w_e; R_1, \dots, R_n)$ of a nonempty set W , a distinguished element w_e in W , and n binary relations R_1, \dots, R_n over W is said to be an E -frame if and only if the following F1 and F2 hold:

F1. W is the disjoint union of a family $\{ W_e : e \in E \}$ of non-empty sets indexed by the elements of E with $W_e = \{ w_e \}$. When $w \in W_e$, we write $\lambda_{\mathcal{F}}(w) = e$, which is called the epistemic status of w .

F2. Each R_i satisfies the following conditions:

F2.1. for all $w, w' \in W$, if $w R_i w'$, then $\lambda_{\mathcal{F}}(w') = \lambda_{\mathcal{F}}(w) * i$;

F2.2. R_i is E -serial, i.e., for each $w \in W$, if $\lambda_{\mathcal{F}}(w) * i \in E$, then $w R_i w'$ for some $w' \in W$.

First, $W_e = W_{(i_1, \dots, i_m)}$ is the set of epistemic worlds of epistemic status $e = (i_1, \dots, i_m)$. Each $w \in W_e$ is a possible epistemic state of the mind of player i_m in a possible state of the mind of i_{m-1} of \dots of player i_1 . Condition F2.1 means that if two epistemic worlds are connected by R_i , then their epistemic statuses should be legitimate in the sense of $\lambda_{\mathcal{F}}(w') = \lambda_{\mathcal{F}}(w) * i$. This is equivalent to

F2.1*. for all $w, w' \in W$, if $w R_i w'$ and $w \in W_e$, then $w' \in W_{e*i}$.

Thus, $\{ W_e : e \in E \}$ and R_1, \dots, R_n reflect the epistemic structure E . Condition F2.2 is specific to our choice of the KD-type semantics. If we choose a different type, these should be modified, which was discussed in Kaneko-Suzuki [9]. We take the unique distinguished element w_e , which is the investigator's epistemic world. The uniqueness of w_e would not be needed for the completeness theorem, but is assumed for convenience.

Each $w_e \in W_e$ ($e = (i_1, \dots, i_m) \in E$) describes a possible state of the mind of player i_m in the scope of $B_e(\cdot)$. Hence the formulae considered in w_e are ones in $\mathcal{P}_E(e) = \{ A : B_e(A) \in \mathcal{P}_E \}$. We will associate the set of e -admissible formulae $\mathcal{P}_E(e)$ with each $w \in W_e$, and will give truth valuations to formulae in $\mathcal{P}_E(e)$ for any $e \in E$. See Figure 2.3.

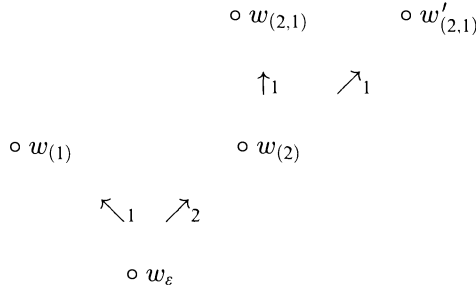


FIGURE 2.4.

EXAMPLE 2.3. For $E^{(2)} = \{\varepsilon, (1), (2), (2, 1)\}$, Figure 2.4 gives an example of an $E^{(2)}$ -frame. This frame is denoted by $\mathcal{F}^{(2)}$. World $w_{(1)}$ represents the state of the mind of player 1 considered by the investigator, but $w_{(2,1)}$ and $w'_{(2,1)}$ represent the possible states of the mind of player 1 imagined by player 2. These are unrelated to $w_{(1)}$. In this frame, the associated sets of formulae are as follows: $\mathcal{P}_{E^{(2)}}$ with w_ε , $\mathcal{P}_{E^{(2)}}(1) = \mathcal{P}_{\{\varepsilon\}}$ with $w_{(1)}$, $\mathcal{P}_{E^{(2)}}(2) = \{A : \delta(A) \subseteq \{\varepsilon, (1)\}\}$ with $w_{(2)}$, and $\mathcal{P}_{E^{(2)}}(2, 1) = \mathcal{P}_{\{\varepsilon\}}$ with $w_{(2,1)}$ and $w'_{(2,1)}$. This is not a standard KD^n Kripke frame in that the arrows named 1 and 2 are missing at $w_{(1)}$, $w_{(2,1)}$ and $w'_{(2,1)}$.

The following is another example of an E -frame, which will be used in Section 4.

EXAMPLE 2.4 (Cane Frame). Let $\ell = (i_1, \dots, i_l) \in N^{<\omega}$ and $\{\ell\}^* = \{\varepsilon, (i_1), (i_1, i_2), \dots, (i_1, \dots, i_l)\}$. Then the following $\mathcal{F}(\ell) = (W; \varepsilon; R_1^\ell, \dots, R_n^\ell)$ is an $\{\ell\}^*$ -frame:

$$(4-1) \quad W = \{\ell\}^* \text{ and } W_e = \{e\} \text{ for all } e \in \{\ell\}^*;$$

$$(4-2) \quad R_i^\ell = \{((i_1, \dots, i_k), (i_1, \dots, i_{k+1})) : i_{k+1} = i\} \text{ for } i \in N.$$

Then $\lambda_{\mathcal{F}(\ell)}(e) = e$. Frame $\mathcal{F}(\ell)$ is depicted in Figure 2.5. Here $\mathcal{P}_{\{\ell\}^*}(i_1, \dots, i_k) = \{A : \delta(\mathbf{B}_{(i_1, \dots, i_k)}(A)) \subseteq \{\ell\}^*\}$ is associated with each $(i_1, \dots, i_k) \in \{\ell\}^*$. We call this $\mathcal{F}(\ell)$ the cane frame generated by ℓ .

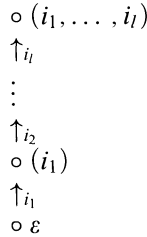


FIGURE 2.5.

Let $\mathcal{F} = (W; w_\varepsilon; R_1, \dots, R_n)$ be an E -frame. An assignment σ in \mathcal{F} is a mapping from $W \times PV$ to $\{\top, \perp\}$. For any assignment σ in \mathcal{F} , we define $(\mathcal{F}, \sigma, w_e) \models A$ or $(\mathcal{F}, \sigma, w_e) \not\models A$ for $A \in \mathcal{P}_E(e)$, $e \in E$ and $w_e \in W_e$ by the following induction on the length of a formula:

V0. for any $p \in PV$, $(\mathcal{F}, \sigma, w_e) \models p$ if and only if $\sigma(w_e, p) = \top$;

and for any $C, D \in \mathcal{P}_E(e)$ and a nonempty finite subset Φ of $\mathcal{P}_E(e)$,

- V1. $(\mathcal{F}, \sigma, w_e) \models \neg C$ if and only if $(\mathcal{F}, \sigma, w_e) \not\models C$;
- V2. $(\mathcal{F}, \sigma, w_e) \models C \supset D$ if and only if $(\mathcal{F}, \sigma, w_e) \not\models C$ or $(\mathcal{F}, \sigma, w_e) \models D$;
- V3. $(\mathcal{F}, \sigma, w_e) \models \bigwedge \Phi$ if and only if $(\mathcal{F}, \sigma, w_e) \models A$ for all $A \in \Phi$;
- V4. $(\mathcal{F}, \sigma, w_e) \models \bigvee \Phi$ if and only if $(\mathcal{F}, \sigma, w_e) \models A$ for some $A \in \Phi$;

and for any $B_i(A) \in \mathcal{P}_E(e)$,

- V5. $(\mathcal{F}, \sigma, w_e) \models B_i(A)$ if and only if $(\mathcal{F}, \sigma, w'_{e*i}) \models A$ for all $w'_{e*i} \in W$ with $w_e R_i w'_{e*i}$.

Since $w_e R_i w'_{e*i}$ implies $\lambda_{\mathcal{F}}(w'_{e*i}) = \lambda_{\mathcal{F}}(w_e) * i = e * i$ by F2.1, the formula A in V5 belongs to $\mathcal{P}_E(\lambda_{\mathcal{F}}(w'_{e*i}))$ for any w'_{e*i} with $w_e R_i w'_{e*i}$. Hence the latter part of V5 is well-defined.

We say that a pair (\mathcal{F}, σ) of an E -frame \mathcal{F} and an assignment σ in \mathcal{F} is an E -model. Let Γ be a set of formulae in \mathcal{P}_E . We say that (\mathcal{F}, σ) is an E -model of Γ if and only if $(\mathcal{F}, \sigma, w_e) \models A$ for all $A \in \Gamma$.

We mention a few basic results on an E -model. Let $(\mathcal{F}, \sigma) = ((W; w_e; R_1, \dots, R_n), \sigma)$ be an E -model. Then, for any $B_e(A) \in \mathcal{P}_E$,

- (6) if $(\mathcal{F}, \sigma, w_e) \models A$ for all $w_e \in W_e$, then $(\mathcal{F}, \sigma, w_e) \models B_e(A)$,

which follows immediately from the definition of the valuation relation.

The other important notion is a restriction of a model. Let E and E' be epistemic structures with $E' \subseteq E$, and let $(\mathcal{F}, \sigma) = ((W; w_e; R_1, \dots, R_n), \sigma)$ be an E -model. Then we say that $(\mathcal{F}', \sigma') = ((W'; w_e; R'_1, \dots, R'_n), \sigma')$ is the *restriction of (\mathcal{F}, σ) to E'* if and only if $W' = \bigcup_{e \in E'} W_e$ and $R'_1, \dots, R'_n, \sigma'$ are the restrictions of R_1, \dots, R_n, σ to W' , respectively. Then the following lemma is an immediate consequence.

LEMMA 2.5. *Let E, E' and E'' be epistemic structures with $E'' \subseteq E' \subseteq E$, and (\mathcal{F}, σ) an E -model.*

- (1) *If (\mathcal{F}', σ') is the restriction of (\mathcal{F}, σ) to E' and if $(\mathcal{F}'', \sigma'')$ is the restriction of (\mathcal{F}', σ') to E'' , then $(\mathcal{F}'', \sigma'')$ is the restriction of (\mathcal{F}, σ) to E'' .*
- (2) *If (\mathcal{F}', σ') is the restriction of (\mathcal{F}, σ) to E' and if $(\mathcal{F}'', \sigma'')$ is the restriction of (\mathcal{F}, σ) to E'' , then $(\mathcal{F}'', \sigma'')$ is the restriction of (\mathcal{F}', σ') to E'' .*

The following lemma, proved in Kaneko-Suzuki [9], will be used later.

LEMMA 2.6. *Let E and E' be epistemic structures with $E' \subseteq E$. Let (\mathcal{F}, σ) be an E -model and (\mathcal{F}', σ') the restriction of (\mathcal{F}, σ) to E' . Then, for any $e' \in E'$, $A \in \mathcal{P}_{E'}(e')$ and $w_{e'} \in W_{e'}$,*

- (7) $(\mathcal{F}, \sigma, w_{e'}) \models A$ if and only if $(\mathcal{F}', \sigma', w_{e'}) \models A$.

2.3. Formal system GL_E and its provability. Let E be an epistemic structure. We introduce the formal system GL_E relative to E . It is the general idea of the logic GL_E that each imaginary player i_k in the epistemic status $e = (i_1, \dots, i_m)$ is given the logical ability described by classical logic. To describe the thought of the imaginary player in the scope of $e = (i_1, \dots, i_m)$, we introduce square brackets $[\]$ and new expressions $B_e[A] = B_{i_1} \cdots B_{i_m}[A]$ for $e = (i_1, \dots, i_m) \in E$ and $A \in \mathcal{P}_E(e)$. Here we distinguish $B_e[A]$ as different from $B_e(A)$, i.e., the latter is a formula in \mathcal{P} , but the former is used to define a formal proof. By making use of $B_e[\dots]$, we express a logical calculus made by i_m in the scope of $i_{m-1} \dots$ of i_1 . We call an expression

$B_e[A]$ a *thought formula*, and say that it is *admissible in E* if and only if $e \in E$ and $A \in \mathcal{P}_E(e)$.⁵

The following are axioms and inference rules: all thought formulae occurring in them are admissible in E :

- e -L1. $B_e[A \supset (B \supset A)]$;
- e -L2. $B_e[(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))]$;
- e -L3. $B_e[(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)]$;
- e -L4. $B_e[\bigwedge \Phi \supset A]$, where $A \in \Phi$;
- e -L5. $B_e[A \supset \bigvee \Phi]$, where $A \in \Phi$;

$$\frac{B_e[A \supset B] \quad B_e[A]}{B_e[B]} \quad e\text{-MP},$$

$$\frac{\{B_e[A \supset B] : B \in \Phi\}}{B_e[A \supset \bigwedge \Phi]} \quad e\text{-}\bigwedge\text{-Rule},$$

$$\frac{\{B_e[A \supset B] : A \in \Phi\}}{B_e[\bigvee \Phi \supset B]} \quad e\text{-}\bigvee\text{-Rule},$$

and for $i = 1, \dots, n$,

- e -MP $_i$. $B_e[B_i(A \supset C) \supset (B_i(A) \supset B_i(C))]$;
- e -D $_i$. $B_e[\neg B_i(\neg A \wedge A)]$;
- (e -Necessitation). $\frac{B_{e^*i}[A]}{B_e[B_i(A)]}$.

A *proof* of $B_e[A]$ in GL_E is a finite tree having the following properties: (1) a thought formula is associated with each node, and an instance of the logical axioms is associated with each leaf; (2) adjoint nodes with the associated thought formulae form an instance of the inference rules; and (3) $B_e[A]$ is associated with the root of the tree. We say that $B_e[A]$ is *provable in GL_E* if and only if there is a proof of $B_e[A]$, in which case we write $\vdash_E B_e[A]$. When $\vdash_E B_e[A]$, we say that A is *provable in GL_E* , denoted simply by $\vdash_E A$.

When e is ε , the above axioms as well as inference rules, except e -Necessitation, become the standard ones for the standard multi-modal KD^n logic restricted to \mathcal{P}_E . When $E = \{\varepsilon\}$, the epistemic axioms and necessitation are not admissible. Hence $GL_{\{\varepsilon\}}$ is *classical* propositional logic.

In the game g^1 of Table 1.1, $B_1(\bigwedge \hat{g}_1^1) \supset B_1(\text{Dom}_1(s_{12}))$ is provable in the logic $GL_{E^{(1)}}$, where $E^{(1)} = \{\varepsilon, (1), (2)\}$. Its proof is given in Diagram 2.1: the first thought formulae are instances of (1)-L4, the first inference is (1)- \bigwedge -Rule, the second inference is (1)-Necessitation, the other initial thought formula is ε -MP $_1$, and the last inference is ε -MP.

Since we abbreviate the outer $B_e[\dots]$, $B_1(\bigwedge \hat{g}_1^1) \supset B_1(\text{Dom}_1(s_{12}))$ is provable in $GL_{E^{(1)}}$. In fact, this proof is admissible even in $GL_{\{\varepsilon, (1)\}}$.

⁵By an epistemic structure E , we give restrictions on the admissible formulae, and by the same E , we restrict the admissible proofs. The latter is reflected by the outer $B_e[\dots]$. If these are separated, we may give more precise arguments on the complexity of interpersonal inferences (proofs). This is discussed in Kaneko-Suzuki [10] and [11].

$$\frac{\frac{\{B_1[\bigwedge \hat{g}_1^1 \supset P_1(s_{12}, t_2; t_1, t_2)] : (t_1, t_2) \in S\}}{B_1[\bigwedge \hat{g}_1^1 \supset \text{Dom}_1(s_{12})]}}{B_e[B_1(\bigwedge \hat{g}_1^1 \supset \text{Dom}_1(s_{12}))]} \quad \frac{B_e[B_1(\bigwedge \hat{g}_1^1 \supset \text{Dom}_1(s_{12})) \supset (B_1(\bigwedge \hat{g}_1^1) \supset B_1(\text{Dom}_1(s_{12})))]}{B_e[B_1(\bigwedge \hat{g}_1^1) \supset B_1(\text{Dom}_1(s_{12}))]}]$$

DIAGRAM 2.1.

For game theoretical considerations, nonlogical axioms are important. Let Γ be a subset of \mathcal{P}_E and A in \mathcal{P}_E . We write $\Gamma \vdash_E A$ if and only if $\vdash_E \bigwedge \Phi \supset A$ for some nonempty finite subset Φ of Γ . We say that Γ is *inconsistent* in GL_E if and only if $\Gamma \vdash_E \neg A \wedge A$ for some $A \in \mathcal{P}_E$, and that Γ is *consistent* in GL_E if and only if it is not inconsistent in GL_E . The following fact will be used: $\Gamma \cup \{A\}$ is consistent in GL_E if and only if $\Gamma \not\vdash_E \neg A$.

For a finite nonempty set $B_i(\Phi) \subseteq \mathcal{P}_E$, it holds that $\vdash_E B_i(\bigwedge \Phi) \equiv \bigwedge B_i(\Phi)$. Thus, $B_i(\bigwedge \Phi) \vdash_E A$ is equivalent to $B_i(\Phi) \vdash_E A$. Using this fact, the formula proved in Diagram 2.1 is written as

$$(8) \quad B_1(\hat{g}_1^1) \vdash_{E^{(1)}} B_1(\text{Dom}_1(s_{12})).$$

The parallel assertion holds for player 2 in the game g^1 . However, this is not the case in the game g^2 of Table 1.2. As mentioned in Section 1, one possible way of 2's decision making is to predict 1's decision making and then to choose a strategy to maximize his own payoff. The prediction part is written as $B_2 B_1(\hat{g}_1^2) \vdash_{E^{(2)}} B_2 B_1(\text{Dom}_1(s_{12}))$, where $E^{(2)} = \{e, (1), (2), (2, 1)\}$. Player 2's choice under this prediction is written as $B_2(\hat{g}_2^2) \vdash_{E^{(2)}} B_2(\text{Best}_2(s_{22} \mid s_{12}))$. In sum,

$$(9) \quad B_2(\hat{g}_2^2), B_2 B_1(\hat{g}_1^2) \vdash_{E^{(2)}} B_2 B_1(\text{Dom}_1(s_{12})) \wedge B_2(\text{Best}_2(s_{22} \mid s_{12})).$$

The right-hand side of (9) is equivalent to $B_2(B_1(\text{Dom}_1(s_{12})) \wedge \text{Best}_2(s_{22} \mid s_{12}))$. We will discuss the decision making of player 2 in the game g^2 throughout the paper in order to illustrate our meta-theorems.

Before going further, we should mention the main result of Kaneko-Suzuki [9].

THEOREM 2.7 (Strong Completeness). *Let E be an epistemic structure. Let Γ be a subset of \mathcal{P}_E and A a formula in \mathcal{P}_E . Then*

- (1) $\Gamma \vdash_E A$ if and only if $(\mathcal{F}, \sigma, w_e) \vDash A$ for all E -models (\mathcal{F}, σ) of Γ ;
- (2) there is an E -model of Γ if and only if Γ is consistent in GL_E .

The following two theorems are relevant for the purpose of the present paper (see Kaneko-Suzuki [9] for their proofs).

THEOREM 2.8 (Conservativity I). *Let E and E' be epistemic structures with $E' \subseteq E$. Let Γ be a subset of $\mathcal{P}_{E'}$ and $A \in \mathcal{P}_{E'}$. Then $\Gamma \vdash_{E'} A$ if and only if $\Gamma \vdash_E A$.*

This theorem states that it suffices to use an epistemic structure large enough for Γ and A . Then we can always evaluate necessary epistemic structures in a more precise manner. Keeping this remark in mind, we can avoid making too precise choices of epistemic structures in stating subsequent theorems.

The relationship of the epistemic logic GL_E to the standard multi-modal KD^n is as follows. For KD^n , we adopt the entire \mathcal{P} , then delete all the outer $B_e[\dots]$

from e -L1– e -L5, e -MP_{*i*}, e -D_{*i*}, e -MP, e - \wedge -Rule, e - \vee -Rule, and finally modify e -Necessitation to

$$\frac{A}{B_i(A)} \text{ (Necessitation).}$$

Then we have the following theorem.

THEOREM 2.9 (Conservativity II). *Let Γ be a subset of \mathcal{P} and $A \in \mathcal{P}$. Let E be any epistemic structure with $\delta(\Gamma \cup \{A\}) \subseteq E$. Then $\Gamma \vdash_E A$ if and only if A is provable from Γ in KD^n .*

2.4. A theorem on decision making in the game g^2 . In this section, we mention one theorem on decision making in the game g^2 of Table 1.2. We will prove this theorem, step-by-step, as we develop various meta-theorems in subsequent sections. In the following, $\hat{\Gamma}$ is intended to be the set of beliefs owned by player 1 in the mind of player 2.⁶

THEOREM 2.10. *Consider the game $g^2 = (g_1^2, g_2^2)$ of Table 1.2. Let $\hat{\Gamma}$ be a subset of \mathcal{P} , and \hat{E} an epistemic structure with $\{\varepsilon, (1), (2), (2, 1)\} \subseteq \hat{E}$ and $\delta(B_2B_1(\hat{\Gamma})) \subseteq \hat{E}$. Let $\hat{E}(2) = \{e : (2) * e \in \hat{E}\}$.*

(1) *The following two statements are equivalent:*

(a) $B_1(\hat{\Gamma}) \vdash_{\hat{E}(2)} B_1(\text{Dom}_1(s_{11}))$ or $B_1(\hat{\Gamma}) \vdash_{\hat{E}(2)} B_1(\text{Dom}_1(s_{12}))$;

(b) $\hat{g}^2, B_1(\hat{g}^2), B_2(\hat{g}^2), B_2B_1(\hat{\Gamma}) \vdash_{\hat{E}} (\bigvee_{s_1} I_1(s_1)) \wedge (\bigvee_{s_2} I_2(s_2))$.

Here $I_1(s_1)$ and $I_2(s_2)$ represent the formulae $B_1(\text{Dom}_1(s_1))$ and

$$\bigvee_{s_1} B_2(B_1(\text{Dom}_1(s_1)) \wedge \text{Best}_2(s_2 \mid s_1)),$$

respectively, and $\hat{g}^2 = \hat{g}_1^2 \cup \hat{g}_2^2$.

(2) *Let $\hat{E}(2, 1) = \{e : (2, 1) * e \in \hat{E}\}$. Then the consistency of $\hat{\Gamma}$ in $\text{GL}_{\hat{E}(2, 1)}$ is equivalent to the consistency of the left-hand side of (b) in $\text{GL}_{\hat{E}}$.*

(3) *The assertion (a) of (1) is equivalent to that $\hat{\Gamma} \vdash_{\hat{E}(2, 1)} \text{Dom}_1(s_{11})$ or $\hat{\Gamma} \vdash_{\hat{E}(2, 1)} \text{Dom}_1(s_{12})$.*

The main game theoretical statement is (b) of (1), which states that each player $i = 1, 2$ has a decision with his decision criterion $I_i(s_i)$. The left-hand side of (b) means that both players truly believe (know) their own payoff functions and that player 2 has a belief on player 1's belief, $B_2B_1(\hat{\Gamma})$. The last component, $\hat{\Gamma}$, is given arbitrarily. The right-hand side of (b) means that both players have their decisions under the presumption that the decision making criteria are given as $I_1(s_1)$ and $I_2(s_1)$ for players 1 and 2.

Claim (1) states that a necessary and sufficient condition for these players to have decisions is (a), i.e., 1's belief (in 2's mind) contains the information that s_{11} or s_{12} is a dominant strategy. According to (8) and (9), \hat{g}_1^2 is an example of $\hat{\Gamma}$ in (a) and (b). In this case, player 2 believes that 1 believes truly 1's payoff function. Another example of $\hat{\Gamma}$ is $\{\text{Dom}_1(s_{12})\}$ (and $\{\text{Dom}_1(s_{11})\}$ as well). Then, player 2 does not derive $B_1(\text{Dom}_1(s_{12}))$ from other beliefs; instead, 2 believes that 1 believes

⁶We state Theorem 2.10 only on the game of Table 1.2. We can extend the theorem in many directions. One possibility is to confine ourselves to a similar game theoretical concept (dominance solvability or iterative elimination of dominated strategies, cf. Moulin [13]), and the other is to adopt Nash equilibrium. For more game theoretical considerations, see Kaneko [5].

dogmatically that s_{12} is a dominant strategy. When $\hat{\Gamma} = \{\text{Dom}_1(s_{11})\}$, 2 believes that 1 has a dogmatic belief, which is false relative to 2's belief.

Claim (2) states that the consistency of the left-hand side of (b) in $\text{GL}_{\hat{E}}$ is equivalent to that of $\hat{\Gamma}$ in $\text{GL}_{\hat{E}(2.1)}$. This will be proved in Section 4. Claim (3) states that we can delete the outer $B_1(\cdot)$ from (a), which is proved in Section 4.

Since $\hat{\Gamma}$ is arbitrarily given in Theorem 2.10, it may not be a good strategy to prove directly, e.g., the derivation of (a) from (b) of (1). In this paper, we choose the way of proving the derivation of (a) from (b) using various meta-theorems given subsequently. The following directs the reader to where the main steps are shown:

- (b) \implies (a): Sections 3.2, 3.3, and 4;
- (2) the former implies the latter: Section 4.
- (3): Section 4.

Particularly, (b) \implies (a) needs a few steps.

Next, consider (a) \implies (b). Suppose (a). It follows from (a) that $B_2B_1(\hat{\Gamma}) \vdash_{\hat{E}} B_2B_1(\text{Dom}_1(s_1))$ for $s_1 = s_{11}$ or s_{12} . Also, since $B_2(\hat{g}_2^2) \vdash_{\hat{E}} B_2(\text{Best}_2(s_{21} \mid s_{11}))$ and $B_2(\hat{g}_2^2) \vdash_{\hat{E}} B_2(\text{Best}_2(s_{22} \mid s_{12}))$, we have $B_2(\hat{g}_2^2), B_2B_1(\hat{\Gamma}) \vdash_{\hat{E}} B_2(B_1(\text{Dom}_1(s_{11})) \wedge \text{Best}_2(s_{21} \mid s_{11}))$ or $B_2(\hat{g}_2^2), B_2B_1(\hat{\Gamma}) \vdash_{\hat{E}} B_2(B_1(\text{Dom}_1(s_{12})) \wedge \text{Best}_2(s_{22} \mid s_{12}))$. This implies $B_2(\hat{g}_2^2), B_2B_1(\hat{\Gamma}) \vdash_{\hat{E}} \bigvee_{s_2} I_2(s_2)$.

Since $B_1(\hat{g}_1^2) \vdash_{\hat{E}} B_1(\text{Dom}_1(s_{12}))$, we have $B_1(\hat{g}_1^2) \vdash_{\hat{E}} \bigvee_{s_1} I_1(s_1)$. This and the conclusion of the previous paragraph imply (b).

§3. Bouquets and cuttings. In this section, we consider the method of bouquet-making which ties two or more models at their roots. The bouquet behaves almost as the direct sum of two models. In Section 3.1, we give a general definition of bouquet-making. In Section 3.2, we consider bouquet-making with no-merging and prove an epistemic separation theorem, which is a refinement of a result given in Kaneko-Nagashima [7]. In Section 3.3, we consider bouquet-making with merging, and obtain the disjunction property theorem of [7].

3.1. Definition of a bouquet. Throughout the following discussions up to Section 5, we assume that two models have only the root in common, that is, for E - and E' -models $(\mathcal{F}, \sigma) = ((W; w_e; R_1, \dots, R_n), \sigma)$ and $(\mathcal{F}', \sigma') = ((W'; w'_e; R'_1, \dots, R'_n), \sigma')$,

$$(10) \quad W \cap W' = \{w_e\} = \{w'_e\}.$$

This assumption can be made by suitable relabelling of the worlds in W and W' . In this section, we assume also that their assignments coincide at their root $w_e = w'_e$:

$$(11) \quad \sigma(w_e, \cdot) = \sigma'(w_e, \cdot).$$

We say that $(\mathcal{F}^*, \sigma^*) = ((W^*; w_e^*; R_1^*, \dots, R_n^*), \sigma^*)$ is the *bouquet made by tying* (\mathcal{F}, σ) and (\mathcal{F}', σ') at their roots, or, simply, $(\mathcal{F}^*, \sigma^*)$ is the *bouquet of* (\mathcal{F}, σ) and (\mathcal{F}', σ') , if and only if the components of $(\mathcal{F}^*, \sigma^*)$ are defined as follows:⁷

⁷This has some similarity to an amalgamation of models (cf. Hughes-Cresswell [4], p. 98), which is used to obtain the rule of disjunction. See Section 3.3 of this paper.

BQ1. $W^* = W \cup W'$; and $\{W_e^*\}_{e \in E \cup E'}$ is defined by

$$W_e^* = \begin{cases} W_e \cup W'_e & \text{if } e \in E \cap E', \\ W_e & \text{if } e \in E - E', \\ W'_e & \text{if } e \in E' - E; \end{cases}$$

BQ2. $w_e^* = w_e = w'_e$;

BQ3. $R_i^* = R_i \cup R'_i$ for $i = 1, \dots, n$;

BQ4. $\sigma^*(w, \cdot) = \begin{cases} \sigma(w, \cdot) & \text{if } w \in W, \\ \sigma'(w, \cdot) & \text{if } w \in W'. \end{cases}$

First, let us see when $(\mathcal{F}^*, \sigma^*)$ is an $E \cup E'$ -model. Since $W \cap W' = \{w_\varepsilon\}$, $W^* = W \cup W'$ is partitioned into $\{W_e^*\}_{e \in E \cup E'}$. For the same reason, neither R_i nor R'_i goes across $W - \{w_\varepsilon\}$ and $W' - \{w_\varepsilon\}$. The bouquet, \mathcal{F}^* , may not satisfy $E \cup E'$ -seriality, which is shown by the following example. As far as $(\mathcal{F}^*, \sigma^*)$ satisfies $E \cup E'$ -seriality, $(\mathcal{F}^*, \sigma^*)$ is an $E \cup E'$ -model.

EXAMPLE 3.1. Consider the frames \mathcal{F} and \mathcal{F}' described by Figures 3.1 and 3.2. They are isomorphic to the cane frames $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\ell_2)$ with $\ell_1 = (1)$ and $\ell_2 = (1, 2)$, respectively, but it is assumed that the sets of worlds have ε in common so that (10) holds. Let σ^1 and σ^2 be any assignments in \mathcal{F} and \mathcal{F}' . The bouquet $(\mathcal{F}^*, \sigma^*)$ of $(\mathcal{F}(\ell_1), \sigma^1)$ and $(\mathcal{F}(\ell_2), \sigma^2)$ is described as Figure 3.3.

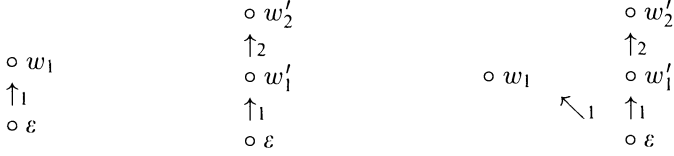


FIGURE 3.1: \mathcal{F} .

FIGURE 3.2: \mathcal{F}' .

FIGURE 3.3: \mathcal{F}^* .

This \mathcal{F}^* does not satisfy $E \cup E'$ -seriality in that w_1 has no successor with respect to R_2^* . Hence $(\mathcal{F}^*, \sigma^*)$ is not an $E \cup E'$ -model.

The following lemma gives a condition for E -seriality.

LEMMA 3.2. Let (\mathcal{F}, σ) and (\mathcal{F}', σ') be E - and E' -models. Then the bouquet $(\mathcal{F}^*, \sigma^*)$ of (\mathcal{F}, σ) and (\mathcal{F}', σ') is an $E \cup E'$ -model if and only if $E \cap E' = \{\varepsilon\}$ or $E = E'$.

PROOF. (If): It suffices to show that \mathcal{F}^* satisfies $E \cup E'$ -seriality. If $E \cap E' = \{\varepsilon\}$ or $E = E'$, then E -seriality or E' -seriality is not affected by bouquet-making.

(Only-If): Suppose neither $E \cap E' = \{\varepsilon\}$ nor $E = E'$. Then there is some $(i_1, \dots, i_m) \in E \cap E'$ with $m \geq 1$ and $(i_1, \dots, i_m, i_{m+1}) \in (E - E') \cup (E' - E)$ for some i_{m+1} . Suppose $(i_1, \dots, i_m, i_{m+1}) \in E - E'$. Consider $w' \in W'_{(i_1, \dots, i_m)}$ in (\mathcal{F}', σ') . Then since $w' \in W' - W$, this w' has no w'' such that $w' R_{i_{m+1}} w''$. Hence $(\mathcal{F}^*, \sigma^*)$ does not satisfy $E \cup E'$ -seriality. The symmetric argument is applied to the case where $(i_1, \dots, i_m, i_{m+1}) \in E' - E$. \dashv

We will consider these two cases: $E \cap E' = \{\varepsilon\}$ and $E = E'$ in Sections 3.2 and 3.3.

The following lemma states that the bouquet is regarded as a direct sum of two models except at the root.

LEMMA 3.3. Let $(\mathcal{F}, \sigma) = ((W; w_\varepsilon; R_1, \dots, R_n), \sigma)$ and $(\mathcal{F}', \sigma') = ((W'; w'_\varepsilon; R'_1, \dots, R'_n), \sigma')$ be E - and E' -models. Assume that the bouquet $(\mathcal{F}^*, \sigma^*)$ of (\mathcal{F}, σ) and (\mathcal{F}', σ') is an $E \cup E'$ -model. Then,

- (1) for any $w \in W - \{w_\varepsilon\}$ and $A \in \mathcal{P}_E(\lambda_{\mathcal{F}}(w))$, $(\mathcal{F}^*, \sigma^*, w) \models A$ if and only if $(\mathcal{F}, \sigma, w) \models A$;
- (2) for any $w' \in W' - \{w'_\varepsilon\}$ and $A \in \mathcal{P}_{E'}(\lambda_{\mathcal{F}'}(w'))$, $(\mathcal{F}^*, \sigma^*, w') \models A$ if and only if $(\mathcal{F}', \sigma', w') \models A$.

PROOF. Consider only (1). Let $w \in W - \{w_\varepsilon\}$. By BQ3, for any $i \in N$ and $u \in W$, $w R_i^* u$ if and only if $w R_i u$, and by BQ4, σ^* coincides with σ over these accessible worlds. Hence $(\mathcal{F}^*, \sigma^*, w) \models A$ if and only if $(\mathcal{F}, \sigma, w) \models A$. \dashv

Let $\{(\mathcal{F}^k, \sigma^k) : k = 1, \dots, m\}$ be E^k -models satisfying (10) and (11) for $k = 1, \dots, m$. When $E^k = E^{k'}$ or $E^k \cap E^{k'} = \{\varepsilon\}$ for any $k \neq k'$, bouquet-making satisfies associativity, i.e., the order of bouquet-making does not matter. Hence the bouquet of multiple models $\{(\mathcal{F}^k, \sigma^k) : k = 1, \dots, m\}$ is well defined.

3.2. Bouquets with no-merging and the epistemic separation theorem. In this section, we consider bouquet-making under the first condition of Lemma 3.2:

$$(12) \quad E \cap E' = \{\varepsilon\}.$$

We call (12) the *no-merging* condition. This means that two models are epistemically separated.

Bouquet making may be regarded as the reverse of taking restrictions of a model, which was defined in Section 2.2.

LEMMA 3.4. Let E, E^1, \dots, E^m be epistemic structures with $E = \bigcup_k E^k$ and $E^k \cap E^{k'} = \{\varepsilon\}$ for $k \neq k'$. Let (\mathcal{F}, σ) an E -model, and $(\mathcal{F}^k, \sigma^k)$ the restriction of (\mathcal{F}, σ) to E^k for $k = 1, \dots, m$. Then the bouquet of $\{(\mathcal{F}^k, \sigma^k) : k = 1, \dots, m\}$ coincides with (\mathcal{F}, σ) .

PROOF. Let $(\mathcal{F}^k, \sigma^k) = ((W^k; w_\varepsilon^k; R_1^k, \dots, R_n^k), \sigma^k)$ for $k = 1, \dots, m$. Since $E = \bigcup_k E^k$ and $E^k \cap E^{k'} = \{\varepsilon\}$ for $k \neq k'$, and since each $(\mathcal{F}^k, \sigma^k)$ is the restriction of (\mathcal{F}, σ) to E^k , it holds that $W = \bigcup_k W^k$, $W^k \cap W^{k'} = \{w_\varepsilon\}$ for $k \neq k'$, and $\sigma^k(w_\varepsilon, \cdot) = \sigma(w_\varepsilon, \cdot)$ for $k = 1, \dots, m$. Hence the bouquet $(\mathcal{F}^*, \sigma^*)$ of $\{(\mathcal{F}^k, \sigma^k) : k = 1, \dots, m\}$ is well defined. Then $W^* = \bigcup_k W^k = W$ and $\{W_e^*\}_{e \in \bigcup_k E^k} = \{W_e\}_{e \in E}$ by BQ1. By BQ4, $\sigma^*(w, \cdot) = \sigma(w, \cdot)$ for all $w \in W = W^*$. Also, $R_i^* = \bigcup_k R_i^k = R_i$. \dashv

We are interested in the restrictions of an E -model (\mathcal{F}, σ) to epistemic substructures of specific forms. For a nonempty $S \subseteq N \cup \{0\}$, we denote the set $\{(i_1, \dots, i_m) \in E : i_l \in S\} \cup \{\varepsilon\}$ by E^S . We call the restriction of (\mathcal{F}, σ) to E^S the *cutting* of (\mathcal{F}, σ) at S . The role of the additional 0 will be clear after Lemma 3.5.

LEMMA 3.5. Let S^1, \dots, S^m be mutually disjoint nonempty subsets of $N \cup \{0\}$. Let (\mathcal{F}, σ) be an E -model, and $(\mathcal{F}^k, \sigma^k)$ the cutting of (\mathcal{F}, σ) at S^k for $k = 1, \dots, m$. Then the bouquet of $\{(\mathcal{F}^k, \sigma^k) : k = 1, \dots, m\}$ is the cutting of (\mathcal{F}, σ) at $S := \bigcup_k S^k$.

PROOF. Consider the cutting (\mathcal{F}', σ') of (\mathcal{F}, σ) at $S = \bigcup_k S^k$. By Lemma 2.5.(2), each $(\mathcal{F}^k, \sigma^k)$ is also the cutting of (\mathcal{F}', σ') at S^k . Here $E^{S^k} \subseteq E^S \subseteq E$

and $E^{S^k} \cap E^{S^{k'}} = \{\varepsilon\}$ if $k \neq k'$. Hence, by Lemma 3.4, the bouquet of $\{(\mathcal{F}^k, \sigma^k) : k = 1, \dots, m\}$ is (\mathcal{F}', σ') . \dashv

Now, we need some syntactical notion. Let S be a nonempty subset of $N \cup \{0\}$. We say that A is an S -formula if and only if (1) $i_1 \in S$ for all $(i_1, \dots, i_m) \in \delta(A)$ and (2) $0 \notin S$ implies $\varepsilon \notin \delta(A)$. For example, $p \wedge \mathbf{B}_1(p)$ is a $\{0, 1\}$ -formula, but not a $\{1\}$ -formula, where $p \in PV$. Also, $I_2(s_2) = \bigvee_{t_1} \mathbf{B}_2(\mathbf{B}_1(\text{Dom}_1(t_1) \wedge \text{Best}_2(s_2 \mid t_1)))$ of Theorem 2.10 is a $\{2\}$ -formula, and $(\bigvee_{s_1} I_1(s_1)) \wedge (\bigvee_{s_2} I_2(s_2))$ is a $\{1, 2\}$ -formula.

The next lemma states that for any S -formula A , the truth value of A in $(\mathcal{F}, \sigma, w_\varepsilon)$ is determined within the cutting at S .

LEMMA 3.6. *Let (\mathcal{F}, σ) be an E -model, $(\mathcal{F}^S, \sigma^S)$ the cutting of (\mathcal{F}, σ) at $S \subseteq N \cup \{0\}$, and A an S -formula in \mathcal{P}_E . Then $(\mathcal{F}, \sigma, w_\varepsilon) \models A$ if and only if $(\mathcal{F}^S, \sigma^S, w_\varepsilon) \models A$.*

PROOF. Since A is an S -formula, we have $i_1 \in S$ for all $(i_1, \dots, i_m) \in \delta(A)$. Hence $\delta(A) \subseteq E^S$, i.e., $A \in \mathcal{P}_{E^S}$, where $E^S = \{(i_1, \dots, i_m) \in E : i_1 \in S\} \cup \{\varepsilon\}$. Hence the assertion follows from Lemma 2.6. \dashv

The following lemma states that when S does not contain 0, the truth valuation of an S -formula is determined by an assignment over $W - \{w_\varepsilon\}$.

LEMMA 3.7. *Suppose $S \subseteq N$. Let \mathcal{F} be an E -frame and σ, σ' two assignments in \mathcal{F} with $\sigma(w, \cdot) = \sigma'(w, \cdot)$ for all $w \neq w_\varepsilon$. Then for any S -formula A in \mathcal{P}_E , $(\mathcal{F}, \sigma, w_\varepsilon) \models A$ if and only if $(\mathcal{F}, \sigma', w_\varepsilon) \models A$.*

PROOF. We say that a subformula C of A is a *direct* subformula of A if and only if C does not occur in the scope of any $\mathbf{B}_j(\cdot)$. Consider a minimal direct subformula C of A . Since $0 \notin S$, C is written as $\mathbf{B}_j(D)$ for some $j \in S$ and some D . We have the assertion of the lemma for this C , since $\sigma(w, \cdot) = \sigma'(w, \cdot)$ for all $w \neq w_\varepsilon$. Then, by induction, we have the assertion for any S -formula. \dashv

Now, we obtain a model-theoretic result by bouquet-tying.

THEOREM 3.8 (Bouquet-Tying). *Let E be an epistemic structure, and S^1, \dots, S^m mutually disjoint nonempty subsets of $N \cup \{0\}$. Let Γ^k be a set of S^k -formulae in \mathcal{P}_E , and $E^k = \{(i_1, \dots, i_m) \in E : i_1 \in S^k\} \cup \{\varepsilon\}$ for $k = 1, \dots, m$. Then Γ^k has an E^k -model for each $k = 1, \dots, m$ if and only if $\bigcup_k \Gamma^k$ has an E -model.*

PROOF. The *if* part can be proved by considering the cutting of an E -model of $\bigcup_k \Gamma^k$ at each S^k . Here we prove the *only-if* part. Let $(\mathcal{F}^k, \sigma^k)$ be an E^k -model of Γ^k for $k = 1, \dots, m$. Note that 0 belongs to at most one S^k . We consider only the case where $0 \in S^{k_0}$. By Lemma 3.7, we can modify the values of σ^k , $k \neq k_0$, into τ^k only at the root w_ε so that the modified τ^k coincides with σ^{k_0} at w_ε . Then each (\mathcal{F}^k, τ^k) is an E^k -model of Γ^k for each $k \neq k_0$ by Lemma 3.7. We write $(\mathcal{F}^{k_0}, \sigma^{k_0})$ also as $(\mathcal{F}^{k_0}, \tau^{k_0})$. Then we make the bouquet (\mathcal{F}, τ) by tying $\{(\mathcal{F}^k, \tau^k) : k = 1, \dots, m\}$. It follows from Lemma 3.4 that each (\mathcal{F}^k, τ^k) is the cutting of (\mathcal{F}, σ) at S^k . By Lemma 3.6, for any S^k -formula A , $(\mathcal{F}, \sigma, w_\varepsilon) \models A$ if and only if $(\mathcal{F}^k, \tau^k, w_\varepsilon) \models A$. Hence (\mathcal{F}, σ) is a $\bigcup_k E^k$ -model of $\bigcup_k \Gamma^k$. By Theorem 2.7, $\bigcup_k \Gamma^k$ is consistent in $\text{GL}_{\bigcup_k E^k}$. By Theorem 2.8, $\bigcup_k \Gamma^k$ is consistent in GL_E . Hence, again by Theorem 2.7, $\bigcup_k \Gamma^k$ has an E -model in GL_E . \dashv

The following is the syntactical counterpart of the above theorem.

THEOREM 3.9 (Epistemic Separation). *Let E be an epistemic structure, S^1, \dots, S^m mutually disjoint nonempty subsets of $N \cup \{0\}$, and $E^k = \{(i_1, \dots, i_m) \in E :$*

$i_1 \in S^k \} \cup \{\varepsilon\}$ for $k = 1, \dots, m$. Let Γ^k be a set of S^k -formulae in \mathcal{P}_E and A^k an S^k -formula in \mathcal{P}_E for $k = 1, \dots, m$.

- (1) $\bigcup_k \Gamma^k \vdash_E \bigvee_k A^k$ if and only if $\Gamma^k \vdash_{E^k} A^k$ for some $k = 1, \dots, m$.
- (2) Suppose that each Γ^k is consistent in GL_E . Then $\bigcup_k \Gamma^k \vdash_E \bigwedge_k A^k$ if and only if $\Gamma^k \vdash_{E^k} A^k$ for all $k = 1, \dots, m$.

PROOF. (1): The *only-if* part is essential. Suppose that $\Gamma^k \not\vdash_{E^k} A^k$ for all $k = 1, \dots, m$. Then it follows from Theorem 2.7 that for each $k = 1, \dots, m$, there is an E^k -model $(\mathcal{F}^k, \sigma^k)$ of $\Gamma^k \cup \{\neg A^k\}$. By Theorem 3.8, we have an E -model (\mathcal{F}, σ) of $\bigcup_k (\Gamma^k \cup \{\neg A^k\})$. Hence $\bigcup_k \Gamma^k \vdash_E \bigvee_k A^k$ does not hold by Theorem 2.7.

(2): Again, the *only-if* part is essential. Suppose $\bigcup_k \Gamma^k \vdash_E \bigwedge_k A^k$. Then $\bigcup_k \Gamma^k \vdash_E A^k$ for all $k = 1, \dots, m$. Take an arbitrary k from $1, \dots, m$. Let \perp^t be a contradictory S^t -formula for $t \neq k$. Then

$$\bigcup_k \Gamma^k \vdash_E \perp^1 \vee \dots \vee \perp^{k-1} \vee A^k \vee \perp^{k+1} \vee \dots \vee \perp^m.$$

By (1) of this theorem, we have $\Gamma^k \vdash_{E^k} A^k$ or $\Gamma^t \vdash_{E^t} \perp^t$ for some $t \neq k$. Since each Γ^t is consistent in GL_E , it is also consistent in GL_{E^t} by Theorem 2.8. Hence $\Gamma^t \not\vdash_{E^t} \perp^t$ for all $t \neq k$. Thus $\Gamma^k \vdash_{E^k} A^k$. \dashv

Let us apply Theorem 3.9 to (b) of Theorem 2.10.(1). First, let \top be $\neg p \vee p$ (p is an atomic formula). Then (b) is equivalent to

$$\hat{g}^2, \mathbf{B}_1(\hat{g}^2), \mathbf{B}_2(\hat{g}^2), \mathbf{B}_2\mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}} \top \wedge \left(\bigvee_{s_1} I_1(s_1) \right) \wedge \left(\bigvee_{s_2} I_2(s_2) \right).$$

Putting $S^0 = \{0\}$, $S^1 = \{1\}$ and $S^2 = \{2\}$, we have $\hat{E}^0 = \{\varepsilon\}$, $\hat{E}^1 = \{(i_1, \dots, i_m) \in \hat{E} : i_1 = 1\} \cup \{\varepsilon\}$ and $\hat{E}^2 = \{(i_1, \dots, i_m) \in \hat{E} : i_1 = 2\} \cup \{\varepsilon\}$. By Theorem 3.9.(2), we have

$$(13) \quad \hat{g}^2 \vdash_{\{\varepsilon\}} \top;$$

$$(14) \quad \mathbf{B}_1(\hat{g}^2) \vdash_{\hat{E}^1} \bigvee_{s_1} I_1(s_1);$$

$$(15) \quad \mathbf{B}_2(\hat{g}^2), \mathbf{B}_2\mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}^2} \bigvee_{s_2} I_2(s_2).$$

Statement (13) is a trivial one, and (14) is derived from $\mathbf{B}_1(\hat{g}^2) \vdash_{\hat{E}^1} \mathbf{B}_1(\text{Dom}_1(\mathbf{s}_{12}))$, which follows from $\hat{g}_1^2 \vdash_{\{\varepsilon\}} \text{Dom}_1(\mathbf{s}_{12})$. On the other hand, to evaluate (15), we need more meta-theorems. In Section 3.3, we give another meta-theorem, which enables us to evaluate (15).

3.3. Bouquets with merging and the epistemic disjunction property. In this section, we extract some consequences from bouquet-making with $E = E'$. Recall that Lemma 3.2 guarantees that when $(\mathcal{F}^1, \sigma^1), \dots, (\mathcal{F}^m, \sigma^m)$ are E -models with identical E and each pair of them satisfies (10) and (11), the bouquet of $(\mathcal{F}^1, \sigma^1), \dots, (\mathcal{F}^m, \sigma^m)$ is also an E -model.

THEOREM 3.10. *Let $(\mathcal{F}^1, \sigma^1), \dots, (\mathcal{F}^m, \sigma^m)$ be E -models so that each pair of them satisfies (10) and (11). Let $(\mathcal{F}^*, \sigma^*)$ be the bouquet of them. Assume $(i) \in E$. Then for any formula $\mathbf{B}_i(A) \in \mathcal{P}_E$,*

- (1) $(\mathcal{F}^*, \sigma^*, w_\varepsilon^*) \models \mathbf{B}_i(A)$ if and only if $(\mathcal{F}^k, \sigma^k, w_\varepsilon^k) \models \mathbf{B}_i(A)$ for all $k = 1, \dots, m$;

(2) $(\mathcal{F}^*, \sigma^*, w_\varepsilon^*) \models \neg B_i(A)$ if and only if $(\mathcal{F}^k, \sigma^k, w_\varepsilon) \models \neg B_i(A)$ for some $k = 1, \dots, m$.

PROOF. (2) is the dual of (1). Consider (1). Let $\mathcal{F}^k = (W^k; w_\varepsilon; R_1^k, \dots, R_n^k)$ for $k = 1, \dots, m$. Suppose $(\mathcal{F}^*, \sigma^*, w_\varepsilon^*) \models B_i(A)$. Then $(\mathcal{F}^*, \sigma^*, w^*) \models A$ for all w^* with $w_\varepsilon^* R_i^* w^*$. By Lemma 3.3, this is equivalent to that for all $k = 1, \dots, m$, $(\mathcal{F}^k, \sigma^k, w) \models A$ for all w with $w_\varepsilon R_i^k w$, which is equivalent to $(\mathcal{F}^k, \sigma^k, w_\varepsilon) \models B_i(A)$ for $k = 1, \dots, m$. We obtain the proof of the converse by tracing this argument back. \dashv

We can give one application of this theorem.

THEOREM 3.11 (Epistemic Disjunction Property). *Let E be an epistemic structure, $B_i(\Gamma)$ a set of formulae in \mathcal{P}_E , and $B_i(\Phi)$ a nonempty finite set of formulae in \mathcal{P}_E . Then $B_i(\Gamma) \vdash_E \bigvee B_i(\Phi)$ if and only if $B_i(\Gamma) \vdash_E B_i(A)$ for some $A \in \Phi$.*

PROOF. The *only-if* part is essential. Suppose that $B_i(\Gamma) \not\vdash_E B_i(A)$ for any $A \in \Phi$. For each $A_k \in \Phi := \{A_1, \dots, A_m\}$, there is an E -model $(\mathcal{F}^k, \sigma^k)$ of $B_i(\Gamma) \cup \{\neg B_i(A_k)\}$. We can assume that each pair of these models satisfies conditions (10) and (11). Consider the bouquet $(\mathcal{F}^*, \sigma^*)$ of $\{(\mathcal{F}^k, \sigma^k) : k = 1, \dots, m\}$. By Theorem 3.10.(1), $(\mathcal{F}^*, \sigma^*, w_\varepsilon) \models B_i(A)$ for any $A \in \Gamma$, i.e., it is an E -model of $B_i(\Gamma)$. Also, $(\mathcal{F}^*, \sigma^*, w_\varepsilon) \models \neg B_i(A_k)$ for all $k = 1, \dots, m$ by Theorem 3.10.(2). Thus $(\mathcal{F}^*, \sigma^*)$ is a model of $B_i(\Gamma) \cup \{\neg \bigvee B_i(\Phi)\}$. Hence $B_i(\Gamma) \not\vdash_E \bigvee B_i(\Phi)$ by Theorem 2.7. \dashv

Theorem 3.11 was first proved in Kaneko-Nagashima [7] for their game logic GL_m using the cut-elimination theorem for it. Without the assumption set $B_i(\Gamma)$, the assertion is often called simply the *disjunction property* or the *rule of disjunction*, which many logical systems are known to satisfy. For a model theoretic proof of the rule of disjunction for a uni-modal logic, see Hughes-Cresswell [4], p. 99.

Let us apply Theorem 3.11 to (15). We have, from (15), that

$$B_2(\hat{g}^2), B_2B_1(\hat{\Gamma}) \vdash_{\hat{E}^2} B_2(B_1(\text{Dom}_1(s_1)) \wedge \text{Best}_2(s_2 \mid s_1))$$

for some $s_1 \in S_1$ and $s_2 \in S_2$. This implies that for some $s_1 \in S_1$,

$$(16) \quad B_2(\hat{g}^2), B_2B_1(\hat{\Gamma}) \vdash_{\hat{E}^2} B_2B_1(\text{Dom}_1(s_1)).$$

To study (16), we need another meta-theorem given in the next section.

§4. Cane extensions and epistemic reductions. In this section, we consider a method of extending a model (\mathcal{F}, σ) by connecting a cane frame of Example 2.4 with the root of (\mathcal{F}, σ) . This method enables us to study properties of formulae of the form $B_\ell(A) = B_{i_1} \dots B_{i_l}(A)$.

Let (\mathcal{F}, σ) be an E -model, and let $\mathcal{F}(\ell) = (\{\ell\}^*; \varepsilon; R_1^\ell, \dots, R_n^\ell)$ be the cane frame generated by $\ell = (i_1, \dots, i_l)$. Recall that $(\ell * E)^*$ is the epistemic structure generated by $\ell * E = \{\ell * e : e \in E\}$.

We say that an $(\ell * E)^*$ -model $(\mathcal{F}^*, \sigma^*) = ((W^*; w_\varepsilon^*; R_1^*, \dots, R_n^*), \sigma^*)$ is a *cane extension* of (\mathcal{F}, σ) with ℓ if and only if \mathcal{F}^* is defined by CE1–CE3 and σ^* satisfies CE4:

CE1. $W^* = \{\ell\}^* \cup \{(\ell, w) : w \in W\}$, where (ℓ, w_ε) and ℓ are regarded as identical; and $\{W_e^*\}_{e \in (\ell * E)^*}$ is defined as follows:

$$W_e^* = \begin{cases} \{e\} & \text{if } e \in \{\ell\}^*, \\ \{(\ell, w) : w \in W_{e'}\} & \text{if } e = \ell * e' \text{ and } e' \in E. \end{cases}$$

CE2. $w_\varepsilon^* = \varepsilon$;

CE3. $R_i^* = R_i^\ell \cup \{((\ell, w), (\ell, w')) : (w, w') \in R_i\}$ for $i = 1, \dots, n$;

CE4. $\sigma^*((\ell, w), p) = \sigma(w, p)$ for all $p \in PV$ and $w \in W$.⁸

The frame \mathcal{F}^* is uniquely determined by CE1–CE3, but the assignment σ^* is only required to coincide with σ over $\{(\ell, w) : w \in W\}$. Hence a cane extension is not uniquely determined. Since ℓ and (ℓ, w_ε) are regarded as identical, the connecting part of (\mathcal{F}, σ) and $\mathcal{F}(\ell)$ consists only of $\ell = (\ell, w_\varepsilon)$, i.e., $W_\ell^* = \{\ell\} = \{(\ell, w_\varepsilon)\}$. Figure 4.1 illustrates a cane extension with $\ell = (i_1, i_2)$.

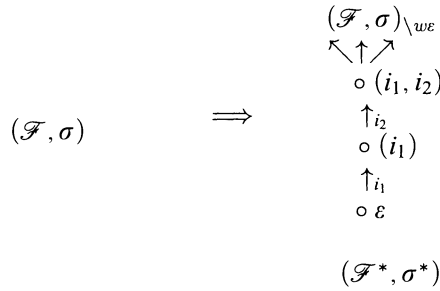


FIGURE 4.1.

Here $(\mathcal{F}, \sigma) \setminus_{w_\varepsilon}$ means that w_ε is removed from (\mathcal{F}, σ) .

LEMMA 4.1. *Let (\mathcal{F}, σ) be an E -model, and $(\mathcal{F}^*, \sigma^*)$ a cane extension of (\mathcal{F}, σ) with $\ell = (i_1, \dots, i_l)$. For any $A \in \mathcal{P}_E$, $(\mathcal{F}, \sigma, w_\varepsilon) \models A$ is equivalent to $(\mathcal{F}^*, \sigma^*, \ell) \models A$.*

PROOF. First, it follows from CE1–CE3 that for any $w, w' \in W$ and $i \in N$, $w R_i w'$ if and only if $(\ell, w) R_i^* (\ell, w')$. Also, $\mathcal{P}_{(\ell * E)^*}(\lambda_{\mathcal{F}^*}(\ell, w)) = \mathcal{P}_E(\lambda_{\mathcal{F}}(w))$ and $\sigma^*((\ell, w), \cdot) = \sigma(w, \cdot)$ for all $w \in W$. Hence the structure of $(\mathcal{F}^*, \sigma^*)$ above ℓ is regarded as equivalent to (\mathcal{F}, σ) . Thus we have the assertion. \dashv

LEMMA 4.2. *Let (\mathcal{F}, σ) be an E -model, and $(\mathcal{F}^*, \sigma^*)$ a cane extension of (\mathcal{F}, σ) with $\ell = (i_1, \dots, i_l)$. Let $A \in \mathcal{P}_E$. Then $(\mathcal{F}, \sigma, w_\varepsilon) \models A$ if and only if $(\mathcal{F}^*, \sigma^*, \varepsilon) \models B_\ell(A)$.*

PROOF. (Only-If): Let $(\mathcal{F}, \sigma, w_\varepsilon) \models A$. Then $(\mathcal{F}^*, \sigma^*, \ell) \models A$ by Lemma 4.1. Since $W_\ell^* = \{\ell\}$, we have $(\mathcal{F}^*, \sigma^*, \varepsilon) \models B_\ell(A)$ by (6).

(If): Let $(\mathcal{F}, \sigma, w_\varepsilon) \models \neg A$. By Lemma 4.1, $(\mathcal{F}^*, \sigma^*, \ell) \models \neg A$. Thus, $(\mathcal{F}^*, \sigma^*, (i_1, \dots, i_{l-1})) \models \neg B_{i_l}(A)$. In the same manner, we have $(\mathcal{F}^*, \sigma^*, (i_1, \dots, i_t)) \models \neg B_{i_{t+1}} \dots B_{i_l}(A)$ for all $t = 0, \dots, l$. Hence $(\mathcal{F}^*, \sigma^*, \varepsilon) \models \neg B_\ell(A)$. \dashv

We have the following reduction theorem.⁹

⁸A similar concept which is known as a *safe extension* of a Kripke model in the literature of modal logic (cf. Chellas [2], p. 98).

⁹When the assumption part Γ is empty, the assertion of the following theorem is found in Chellas [2], p. 99, p. 181.

THEOREM 4.3 (Epistemic Reduction). *Let E be an epistemic structure and $\ell = (i_1, \dots, i_l) \in N^{<\omega}$. Let Γ be a set of formulae with $\Gamma \subseteq \mathcal{P}_E$ and A a formula with $A \in \mathcal{P}_E$. Then $\Gamma \vdash_E A$ if and only if $\mathbf{B}_\ell(\Gamma) \vdash_{(\ell * E)^*} \mathbf{B}_\ell(A)$.*

PROOF. The *if* part is essential. Suppose $\Gamma \not\vdash_E A$. By Theorem 2.7, there is an E -model (\mathcal{F}, σ) for $\Gamma \cup \{\neg A\}$. Let $(\mathcal{F}^*, \sigma^*)$ be a cane extension of (\mathcal{F}, σ) with ℓ . It suffices to show that this is an $(\ell * E)^*$ -model of $\mathbf{B}_\ell(\Gamma) \cup \{\neg \mathbf{B}_\ell(A)\}$. Let C be any formula in Γ . Since $(\mathcal{F}, \sigma, w_\varepsilon) \models C$, we have $(\mathcal{F}^*, \sigma^*, \ell) \models C$. By Lemma 4.2, we have $(\mathcal{F}^*, \sigma^*, \varepsilon) \models \mathbf{B}_\ell(C)$. Also, since $(\mathcal{F}, \sigma, w_\varepsilon) \models \neg A$, we have $(\mathcal{F}^*, \sigma^*, \ell) \models \neg A$. By Lemma 4.2.(2), we have $(\mathcal{F}^*, \sigma^*, \varepsilon) \models \neg \mathbf{B}_\ell(A)$. Hence $(\mathcal{F}^*, \sigma^*)$ is an $(\ell * E)^*$ -model of $\mathbf{B}_\ell(\Gamma) \cup \{\neg \mathbf{B}_\ell(A)\}$, which implies $\mathbf{B}_\ell(\Gamma) \not\vdash_{(\ell * E)^*} \mathbf{B}_\ell(A)$. \dashv

We now complete the proof of (b) \implies (a) of Theorem 2.10.(1). Recall that we have reached (16), i.e., $\mathbf{B}_2(\hat{g}^2)$, $\mathbf{B}_2\mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}^2} \mathbf{B}_2\mathbf{B}_1(\text{Dom}_1(s_1))$. Since $\hat{E}(2) = \{e : (2) * e \in \hat{E}\}$ and $\hat{E}^2 = \{(i_1, \dots, i_m) \in \hat{E} : i_1 = 2\}$, we have $\hat{E}(2) = \{e : (2) * e \in \hat{E}^2\}$. By Theorem 4.3, we have that

$$(17) \quad \hat{g}^2, \mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}(2)} \mathbf{B}_1(\text{Dom}_1(s_1)).$$

Then we apply the Epistemic Separation Theorem (Theorem 3.8) to (17), and then we have

$$(18) \quad \mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}(2)} \mathbf{B}_1(\text{Dom}_1(s_1)).$$

Since this holds for $s_1 = s_{11}$ or s_{12} , we have (a) of Theorem 2.10.(1).

By applying Theorem 4.3 to (a) of Theorem 2.10, we have $\hat{\Gamma} \vdash_{\hat{E}(2.1)} \text{Dom}_1(s_{11})$ or $\hat{\Gamma} \vdash_{\hat{E}(2.1)} \text{Dom}_1(s_{12})$. The converse is straightforward. We have Theorem 2.10.(3).

Now, let us prove Theorem 2.10.(2) that $\hat{\Gamma}$ is consistent in $\text{GL}_{\hat{E}(2.1)}$ if and only if $\hat{g}^1, \mathbf{B}_1(\hat{g}^2)$, $\mathbf{B}_2(\hat{g}^2)$, $\mathbf{B}_2\mathbf{B}_1(\hat{\Gamma})$ is consistent in $\text{GL}_{\hat{E}}$. The *if* part is straightforward. Consider the *only-if* part. Suppose that $\hat{g}^1, \mathbf{B}_1(\hat{g}^2)$, $\mathbf{B}_2(\hat{g}^2)$, $\mathbf{B}_2\mathbf{B}_1(\hat{\Gamma})$ is inconsistent in $\text{GL}_{\hat{E}}$. Then $\hat{g}^1, \mathbf{B}_1(\hat{g}^2)$, $\mathbf{B}_2(\hat{g}^2)$, $\mathbf{B}_2\mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}} \perp \vee \mathbf{B}_1(\perp) \vee \mathbf{B}_2\mathbf{B}_1(\perp)$, where \perp is $\neg p \wedge p$ (p is an atomic formula). Then by Theorem 3.9, we have (i) $\hat{g}^1 \vdash_{\hat{E}} \perp$; (ii) $\mathbf{B}_1(\hat{g}^2) \vdash_{\hat{E}} \mathbf{B}_1(\perp)$; or (iii) $\mathbf{B}_2(\hat{g}^2)$, $\mathbf{B}_2\mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}} \mathbf{B}_2\mathbf{B}_1(\perp)$. The first two are not the case since \hat{g}^1 and $\mathbf{B}_1(\hat{g}^2)$ are consistent. Consider (iii). By Theorem 4.3, we have $\hat{g}^2, \mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}(2)} \mathbf{B}_1(\perp)$. Hence, again by Theorem 3.9, $\mathbf{B}_1(\hat{\Gamma}) \vdash_{\hat{E}(2)} \mathbf{B}_1(\perp)$. By Theorem 4.3 again, we have $\hat{\Gamma} \vdash_{\hat{E}(2.1)} \perp$.

§5. Engrafting and epistemic joint-consistency. In this section, we consider the method of *engrafting*. It connects a model to another model at its epistemic worlds of a given epistemic status. We give the definition of engrafting under a condition corresponding to the no-merging condition (12). Except for the choice of an assignment at the connecting part, a bouquet is a special case of an engrafted model, and so is a cane extension. We obtain certain useful meta-theorems by engrafting.

5.1. Definition of engrafting. Let $(\mathcal{F}, \sigma) = ((W; w_\varepsilon; R_1, \dots, R_n), \sigma)$ and $(\mathcal{F}', \sigma') = ((W'; w'_\varepsilon; R'_1, \dots, R'_n), \sigma')$ be E - and E' -models. Let $\ell = (i_1, \dots, i_l) \in E$. Here, (\mathcal{F}, σ) is the base model, and (\mathcal{F}', σ') is engrafted to (\mathcal{F}, σ) . Specifically, we connect a copy of (\mathcal{F}', σ') to every world $w_\ell \in W_\ell$ in (\mathcal{F}, σ) . Since they are connected to each $w_\ell \in W_\ell$, we should choose either $\sigma(w_\ell, \cdot)$ or $\sigma'(w'_\ell, \cdot)$ for the

well-definedness of the resulting engrafted model. Here, we choose $\sigma(w_\ell, \cdot)$ so that the entire structure of the base model (\mathcal{F}, σ) is preserved.

We impose the following condition on E , E' and $\ell = (i_1, \dots, i_l)$:

$$(19) \quad E \cap (\ell * E') = \{\ell\}.$$

This excludes cases such as Figure 5.1, and is the same as the no-merging condition (12) when $\ell = \varepsilon$. Without (19), engrafting would not define a frame.

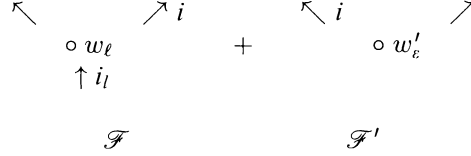


FIGURE 5.1.

We say that $\mathcal{F}^* = (W^*; w_\varepsilon^*; R_1^*, \dots, R_n^*)$ defined by EG1–EG3 is made by engrafting $\mathcal{F}' = (W'; w'_\varepsilon; R'_1, \dots, R'_n)$ into $\mathcal{F} = (W; w_\varepsilon; R_1, \dots, R_n)$ at W_ℓ :

EG1. $E^* = E \cup (\ell * E')$ and $W^* = W \cup (\bigcup_{w_\ell \in W_\ell} \{(w_\ell, w') : w' \in W' - \{w'_\varepsilon\}\})$;

EG2. W^* is partitioned into $\{W_e^* : e \in E^*\}$ which is defined by

$$W_e^* = \begin{cases} W_e & \text{if } e \in E, \\ \bigcup_{w_\ell \in W_\ell} \{(w_\ell, w'_\ell) : w'_\ell \in W'_{e'}\} & \text{if } e = \ell * e' \in \ell * (E' - \{\varepsilon\}); \end{cases}$$

EG3. for any $i \in N$,

$$R_i^* = R_i \cup \{(w_\ell, (w_\ell, w')) : w_\ell \in W_\ell \text{ and } (w'_\varepsilon, w') \in R'_i\} \\ \cup \{((w_\ell, w'), (w_\ell, w'')) : w_\ell \in W_\ell, w' \neq w'_\varepsilon \text{ and } (w', w'') \in R'_i\}.$$

Since $\ell = (i_1, \dots, i_l) \in E$, $E^* = E \cup (\ell * E')$ is an epistemic structure. In EG1, each $w' \in W' - \{w'_\varepsilon\}$ is relabelled as (w_ℓ, w') , and $\{(w_\ell, w') : w' \in W' - \{w'_\varepsilon\}\}$ will be connected to w_ℓ . EG2 defines the partition $\{W_e^* : e \in E^*\}$ of W^* . The first and second cases of EG2 are the base frame and newly engrafted parts. Condition (19) guarantees that W_e^* is the union of $\{(w_\ell, w'_\ell) : w'_\ell \in W'_{e'}\}$ over $w_\ell \in W_\ell$ for $e = \ell * e' \in \ell * (E' - \{\varepsilon\})$. In EG3, the accessibility relation R_i^* is defined to be the union of R_i and the parallel transformations of R'_i to the copies of $W' - \{w'_\varepsilon\}$.

LEMMA 5.1. *The $n + 2$ tuple $\mathcal{F}^* = (W^*; w_\varepsilon^*; R_1^*, \dots, R_n^*)$ defined by EG1–EG3 is an E^* -frame.*

We define an assignment $\sigma^* : W^* \times PV \rightarrow \{\top, \perp\}$ as follows:

EG4. For any $w^* \in W^*$ and $p \in PV$,

$$\sigma^*(w^*, p) = \begin{cases} \sigma(w^*, p) & \text{if } w^* \in W, \\ \sigma'(w', p) & \text{if } w^* = (w_\ell, w') \text{ for } w_\ell \in W_\ell \text{ and } w' \in W' - \{w'_\varepsilon\}. \end{cases}$$

In EG4, the new assignment σ^* inherits σ entirely and σ' in the newly engrafted parts. We say that $(\mathcal{F}^*, \sigma^*)$ is made by engrafting (\mathcal{F}', σ') into (\mathcal{F}, σ) at W_ℓ , or that $(\mathcal{F}^*, \sigma^*)$ is the engrafted model of (\mathcal{F}, σ) and (\mathcal{F}', σ') at W_ℓ .

Let $\sigma(w_\ell, \cdot) = \sigma'(w'_\varepsilon, \cdot)$ for all $w_\ell \in W_\ell$. Then, a cane extension is a special case. When $\ell = \varepsilon$ and (12) holds, a bouquet is a special case of an engrafted model. In

a cane extension, the base is the cane frame $\mathcal{F}(\ell)$. In bouquet-making of models, either model can be regarded as a base model.

The following lemma states that the truth values in (\mathcal{F}, σ) and (\mathcal{F}', σ') are preserved in the engrafted model $(\mathcal{F}^*, \sigma^*)$ except for the formulae $A \in \mathcal{P}_{E'}$ with $\varepsilon \in \delta(A)$. The choice of $\sigma(w_\ell, \cdot)$ for $\sigma^*(w_\ell, \cdot)$ rather than $\sigma'(w'_\varepsilon, \cdot)$ requires the additional condition $\varepsilon \notin \delta(A)$ for Lemma 5.2.(3).

LEMMA 5.2. *Let (\mathcal{F}, σ) and (\mathcal{F}', σ') be E - and E' -models. Suppose (19) for E , E' and ℓ . Let $(\mathcal{F}^*, \sigma^*)$ be made by engrafting (\mathcal{F}', σ') into (\mathcal{F}, σ) at W_ℓ . Then*

- (1) *for any $w \in W$ and $A \in \mathcal{P}_E(\lambda_{\mathcal{F}}(w))$, $(\mathcal{F}, \sigma, w) \models A$ if and only if $(\mathcal{F}^*, \sigma^*, w) \models A$.*
- (2) *for any $w' \in W' - \{w'_\varepsilon\}$, $w_\ell \in W_\ell$ and $A \in \mathcal{P}_{E'}(\lambda_{\mathcal{F}'}(w'))$, $(\mathcal{F}', \sigma', w') \models A$ if and only if $(\mathcal{F}^*, \sigma^*, (w_\ell, w')) \models A$.*
- (3) *for any formula $A \in \mathcal{P}_{E'}$ with $\varepsilon \notin \delta(A)$ and $w'_\varepsilon \in W'_\varepsilon$, $(\mathcal{F}', \sigma', w'_\varepsilon) \models A$ if and only if $(\mathcal{F}^*, \sigma^*, w'_\varepsilon) \models A$.*

PROOF. (1) Observe that by EG4, σ^* coincides with σ over W . Moreover, (19) guarantees that the truth value of $A \in \mathcal{P}_E(\lambda_{\mathcal{F}}(w))$ is not affected by the engrafted part.

(2) Let w_ℓ be any world in W_ℓ . We can prove by induction on the length of a formula that for any $e' \in E' - \{\varepsilon\}$, $w'_{e'} \in W'_{e'}$ and $A \in \mathcal{P}_{E'}(e')$, $(\mathcal{F}', \sigma', w') \models A$ if and only if $(\mathcal{F}^*, \sigma^*, (w_\ell, w')) \models A$.

(3) This follows from (2). \dashv

5.2. Epistemic joint-consistency and the depth lemma. Here, we provide three results: the first two theorems are called the *epistemic joint-consistency theorems*, and the third the *depth lemma*. To illustrate merits of those theorems, we will consider the necessity of the assumption set $B_2B_1(\tilde{\Gamma})$ of Theorem 2.10.(1).(b).

For the following two theorems, we let Γ and Γ' be subsets of \mathcal{P}_E and $\mathcal{P}_{E'}$, respectively, and $\ell = (i_1, \dots, i_l)$.

THEOREM 5.3 (Epistemic Joint-Consistency I). *Assume $\ell \notin E$. Let (\mathcal{F}, σ) and (\mathcal{F}', σ') be E - and E' -models of Γ and Γ' . Then there is an E^o -model of $\Gamma \cup B_\ell(\Gamma')$, where $E^o = E \cup (\ell * E')^*$.*

PROOF. We define $\ell' = (i_1, \dots, i_k)$ and $\ell'' = (i_{k+1}, \dots, i_l)$ by

$$(20) \quad \ell' \in E \quad \text{but} \quad (\ell', i_{k+1}) \notin E.$$

Since $\ell \notin E$, we have $k < l$. Let $(\mathcal{F}'', \sigma'')$ be a cane extension of (\mathcal{F}', σ') with $\ell'' = (i_{k+1}, \dots, i_l)$. The epistemic structure of the cane extension $(\mathcal{F}'', \sigma'')$ is $(\ell'' * E')^*$. Then let $(\mathcal{F}^*, \sigma^*)$ be obtained by engrafting $(\mathcal{F}'', \sigma'')$ into (\mathcal{F}, σ) at $W_{\ell'}$. For this engrafting, (19) also holds by (20), i.e., $E \cap (\ell' * (\ell'' * E')^*) = \{\ell'\}$. The epistemic structure of $(\mathcal{F}^*, \sigma^*)$ is $E \cup (\ell' * (\ell'' * E')^*) = E \cup (\ell * E')^* = E^o$. Hence $(\mathcal{F}^*, \sigma^*)$ is an E^o -model by Lemma 5.1. Then we should show that $(\mathcal{F}^*, \sigma^*, w'_\varepsilon) \models A$ for all $A \in \Gamma \cup B_\ell(\Gamma')$.

By Lemma 5.2.(1), $(\mathcal{F}^*, \sigma^*, w'_\varepsilon) \models A$ for all $A \in \Gamma$.

Let A be any formula in Γ' . Then $(\mathcal{F}', \sigma', w'_\varepsilon) \models A$. Recalling $\ell'' \neq \varepsilon$, we have $(\mathcal{F}'', \sigma'', w''_\varepsilon) \models B_{\ell''}(A)$ by Lemma 4.2. By Lemma 5.2.(3), $(\mathcal{F}^*, \sigma^*, w'_\varepsilon) \models B_{\ell''}(A)$ for all $w''_\varepsilon \in W''_\varepsilon$. Finally, by (6), $(\mathcal{F}^*, \sigma^*, w'_\varepsilon) \models B_{\ell'}B_{\ell''}(A)$, i.e., $(\mathcal{F}^*, \sigma^*, w'_\varepsilon) \models B_\ell(A)$. \dashv

The following is the proof-theoretic counterpart of Theorem 5.3.

THEOREM 5.4 (Epistemic Joint-Consistency II). *Assume $\ell \notin E$. Suppose that Γ and Γ' are consistent in GL_E and $\text{GL}_{E'}$, respectively. Then $\Gamma \cup \text{B}_\ell(\Gamma')$ is also consistent in GL_{E^o} , where $E^o = E \cup (\ell * E')$.*

PROOF. By Theorem 2.7, there is an E -model (\mathcal{F}, σ) of Γ and an E' -model (\mathcal{F}', σ') of Γ' . By Theorem 5.3, there is an E^o -model of $\Gamma \cup \text{B}_\ell(\Gamma')$. Hence, by Theorem 2.7, $\Gamma \cup \text{B}_\ell(\Gamma')$ is also consistent in GL_{E^o} . \dashv

The next theorem is a refinement of the depth lemma proved for S4^n by Kaneko-Nagashima [6], which was used to show the axiomatic indefinability of common knowledge in finitary epistemic logic S4^n without adding an inference rule on the common knowledge operator.

THEOREM 5.5 (Depth Lemma). *Let Γ be a set of formulae, A a formula and ℓ an epistemic status. Let $E = \delta^*(\Gamma \cup \{\text{B}_\ell(A)\})$. If $\Gamma \vdash_E \text{B}_\ell(A)$ and $\ell \notin \delta^*(\Gamma)$, then Γ is inconsistent in GL_E or $\vdash_{\delta^*(A)} A$.*

PROOF. Suppose that Γ is consistent in GL_E and $\not\vdash_{\delta^*(A)} A$. Then Γ is consistent in $\text{GL}_{\delta^*(\Gamma)}$ by Theorem 2.8. By Theorem 2.7, there is an $\delta^*(\Gamma)$ -model (\mathcal{F}, σ) of Γ and a $\delta^*(A)$ -model (\mathcal{F}', σ') of $\neg A$. Since $\ell \notin \delta^*(\Gamma)$, we can apply Theorem 5.3 to (\mathcal{F}, σ) and (\mathcal{F}', σ') with respect to ℓ . Hence we have a $\delta^*(\Gamma) \cup (\ell * \delta^*(A))^*$ -model $(\mathcal{F}^*, \sigma^*)$ of $\Gamma \cup \{\text{B}_{i_1} \dots \text{B}_{i_{l-1}} \text{B}_{i_l}(\neg A)\}$. This is also an $\delta^*(\Gamma) \cup (\ell * \delta^*(A))^*$ -model of $\Gamma \cup \{\neg \text{B}_{i_1} \dots \text{B}_{i_{l-1}} \text{B}_{i_l}(A)\}$. Since $\delta^*(\Gamma) \cup (\ell * \delta^*(A))^* = \delta^*(\Gamma \cup \{\text{B}_\ell(A)\}) = E$, we have $\Gamma \not\vdash_E \text{B}_\ell(A)$ by Theorem 2.7. \dashv

Consider Theorem 2.10.(1).(b), i.e., $\hat{g}^2, \text{B}_1(\hat{g}^2), \text{B}_2(\hat{g}^2), \text{B}_2\text{B}_1(\hat{\Gamma}) \vdash_{\hat{E}} (\bigvee_{s_1} I_1(s_1)) \wedge (\bigvee_{s_2} I_2(s_2))$. Theorem 5.5 can be used to show that the nested occurrences of B_2 and B_1 for additional $\text{B}_2\text{B}_1(\hat{\Gamma})$ is unavoidable to have (b). Indeed, we have obtained already (16) from (b), i.e., $\text{B}_2(\hat{g}^2), \text{B}_2\text{B}_1(\hat{\Gamma}) \vdash_{\hat{E}^2} \text{B}_2\text{B}_1(\text{Dom}_1(s_1))$. Theorem 5.5 states that if $\text{B}_2\text{B}_1(\hat{\Gamma})$ is missing, this would not hold. Thus, we could not have $\bigvee_{s_1} I_2(s_2)$ without $\text{B}_2\text{B}_1(\hat{\Gamma})$.

§6. Concluding remarks. We have developed methods of surgical operations, and obtained various meta-theorems, while applying them to game theoretical decision making. The game theoretical problem we have considered is simple, but problems for general n -person games can be investigated by our methods. Some are found in Kaneko-Suzuki [11].

Here, we give two remarks on the treatment of the Axiom of Positive Introspection (Axiom 4), and on an extension of GL_E given in Kaneko-Suzuki [10] and [11].

(1) The Axiom of Positive Introspection, i.e., $\text{B}_e[\text{B}_i(A) \supset \text{B}_i\text{B}_i(A)]$, was adopted in Kaneko-Suzuki [9]. For this, we need some changes in various definitions for GL_E and its semantics. For example, an epistemic status (i_1, \dots, i_m) is assumed to have no repetitive occurrences, i.e., $i_k \neq i_{k+1}$, and accordingly, concatenation $e * e'$ of epistemic statuses e and e' should be modified. Nevertheless, the essential part of the present paper remains, but we need more complicated proofs.

Although some arguments in game theoretical applications (e.g., see Kaneko [5]) rely upon the Axiom of Positive Introspection, the elimination of the axiom enables us to study positive introspection more precisely. In the Appendix of Kaneko-Suzuki [11], the relationship between the logic GL_E and that including the Axiom of Positive Introspection is considered from the proof-theoretical point of view.

(2) An epistemic structure E is used to restrict the admissible formulae. Kaneko-Suzuki [10] and [11] impose another constraint on GL_E so that an epistemic structure required for proofs is distinguished from E . The required epistemic structure is denoted by F , and the resulting epistemic logic is GL_{EF} , where $F \subseteq E$. In semantics, F is a restriction on a frame. Then, we have the completeness theorem for the systems with those modifications.

In this paper, we did not consider the additional constraint F for our horticulture. Nevertheless, we can incorporate the constraint F into our considerations. The required modification is straightforward for bouquet-making and a cane extension. Engrafting needs slightly more careful considerations.

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