

Abstract. This paper provides a Gentzen style formulation of the game logic framework GL_m ($0 \leq m \leq \omega$), and proves the cut-elimination theorem for GL_m . As its application, we prove the term existence theorem for GL_m used in Part I.

Key words: infinitary predicate KD_4 , common knowledge, Nash equilibrium, undecidability on playability.

1. Introduction

This is a sequel to our development of the game logic framework. In Part I (Kaneko-Nagashima [4]) of this paper, we presented the framework in the Hilbert style formulation and showed some applications of the framework to game theory – the epistemic axiomatization of Nash equilibrium and the undecidability on the playability of a game. To obtain the undecidability results (Theorems 6.2 and 6.3 of Part I), we used the result called the *term existence theorem*. This is a metatheorem stating an evaluation of provability on an existential formula. The Hilbert style formulation is convenient in presentation, but is difficult in managing an evaluation of such provability. In general, it would be better to reformulate the Hilbert style formulation into a Gentzen style sequent calculus for the purpose of evaluating provability. This paper provides a Gentzen style formulation of the game logic framework, and proves the cut-elimination theorem. As its application, we prove the term existence theorem used in Part I.

First, we provide the sequent calculus corresponding to GL_ω , which is an infinitary predicate extension of the sequent calculus formulation of propositional KD_4 along the line of Ohnishi Matsumoto [5] and [6]. Then we present the cut-elimination theorem for sequent calculus GL_ω . We discuss also the infinitary predicate extensions of K_4 , K as well as S_4 . Section 4 provides a sequent calculus formulation of logic GL_m for a finite m . This provides

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also an alternative sequent calculus formulation of GL_ω . The corresponding cut-elimination theorem will be provided, which will be proved in Section 5.

In Section 3, we prove the term existence theorem in sequent calculus GL_ω , using the cut-elimination theorem. As steps to this result, we provide a more basic term existence theorem and another result which we call a *separation theorem*. These two results hold, in fact, in GL_m for any m .

We will find that the separation theorem manifests cognitive relativism in that the epistemic world of each player is separated from the others'. This enables us to prove the results of Section 3. These results fail to hold for the S_4 -type extension, in which knowledge must be always true relative to the thinker and ultimately relative to the investigator.

In our sequent calculus, the Barcan rules, $\wedge K_i(\Phi) \supset K_i(\wedge\Phi)$ for an allowable set Φ and $\forall x K_i(A(x)) \supset K_i(\forall x A(x))$, are formulated as inference rules, $(B-\wedge)$ and $(B-\forall)$, which violate the subformula property. Hence the cut-elimination theorem for GL_m does not give a proof with the full subformula property. Nevertheless, this is not an obstacle in obtaining the term existence theorem. For other applications, unfortunately, those inferences become obstacles. A final remark is that since the Barcan rules are not needed in the finitary propositional fragment of GL_ω , the cut-elimination theorem for the finitary propositional fragment of GL_ω provides a cut-free proof with the full subformula property.

2. Sequent Calculus GL_ω and its Variations

2.1 Sequent Calculus GL_ω and the Cut-Elimination Theorem

We work on the same language \mathcal{P}_ω as in Part I, and prepare an auxiliary symbol \rightarrow . We call the expression $\Gamma \rightarrow \Theta$ a *sequent* iff Γ and Θ are finite sets of formulae. The sets Γ and Θ are called the *antecedent* and *succedent* of the sequent $\Gamma \rightarrow \Theta$. The expression $\Gamma, \Delta \rightarrow \Theta, \Lambda$ is used to denote $\Gamma \cup \Delta \rightarrow \Theta \cup \Lambda$. We omit the set-theoretical bracket $\{A\}$, for example, $\{A\}, \Gamma \rightarrow \Theta, \{B\}$ is denoted as $A, \Gamma \rightarrow \Theta, B$.

Our sequent calculus GL_ω is defined as follows:

Initial Sequents: An initial sequent is of the form $A \rightarrow A$ for any formula A .

Inference Rules: We have three kinds of inference rules: structural, operational and K -inference rules.

Structural Inferences:

$$\frac{\Gamma \rightarrow \Theta}{\Delta, \Gamma \rightarrow \Theta, \Lambda} \text{ (th)}$$

$$\frac{\Gamma \rightarrow \Theta, M \quad M, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda} (M) \text{ (cut),}$$

where M is called the *cut-formula*.

Operational Inferences:

$$\frac{A, \Gamma \rightarrow \Theta}{\wedge \Phi, \Gamma \rightarrow \Theta} (\wedge \rightarrow) \quad (A \in \Phi) \qquad \frac{\{\Gamma \rightarrow \Theta, A : A \in \Phi\}}{\Gamma \rightarrow \Theta, \wedge \Phi} (\rightarrow \wedge)$$

$$\frac{\{A, \Gamma \rightarrow \Theta : A \in \Phi\}}{\vee \Phi, \Gamma \rightarrow \Theta} (\vee \rightarrow) \qquad \frac{\Gamma \rightarrow \Theta, A}{\Gamma \rightarrow \Theta, \vee \Phi} (\rightarrow \vee) \quad (A \in \Phi)$$

$$\frac{\Gamma \rightarrow \Theta, A \quad B, \Gamma \rightarrow \Theta}{A \supset B, \Gamma \rightarrow \Theta} (\supset \rightarrow) \qquad \frac{A, \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \supset B} (\rightarrow \supset)$$

$$\frac{\Gamma \rightarrow \Theta, A}{\neg A, \Gamma \rightarrow \Theta} (\neg \rightarrow) \qquad \frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A} (\rightarrow \neg)$$

$$\frac{A(t), \Gamma \rightarrow \Theta}{\forall x A(x), \Gamma \rightarrow \Theta} (\forall \rightarrow) \qquad \frac{\Gamma \rightarrow \Theta, A(a)}{\Gamma \rightarrow \Theta, \forall x A(x)} (\rightarrow \forall)$$

$$\frac{A(a), \Gamma \rightarrow \Theta}{\exists x A(x), \Gamma \rightarrow \Theta} (\exists \rightarrow) \qquad \frac{\Gamma \rightarrow \Theta, A(t)}{\Gamma \rightarrow \Theta, \exists x A(x)} (\rightarrow \exists),$$

where Φ is an allowable set, t is a term, and a is a free variable which must not occur in the lower sequents of $(\rightarrow \forall)$ and $(\exists \rightarrow)$.

In an operational inference, the formulae to be changed in the upper sequents are called the *side formulae*, and the formula newly created in the lower sequent is called the *principal formula*. The free variable a in $(\rightarrow \forall)$ and $(\exists \rightarrow)$ is called an *eigenvariable*.

The following two inference rules are specific to our system.

K-Inferences:

$$\frac{\Gamma, K_i(\Delta) \rightarrow \Theta}{K_i(\Gamma, \Delta) \rightarrow K_i(\Theta)} (K \rightarrow K),$$

where $|\Theta|$, the cardinality of Θ , is at most one. Recall that $K_i(\Gamma)$ denotes the set $\{K_i(A) : A \in \Gamma\}$ and $K_i(\Gamma, \Delta)$ is $K_i(\Gamma \cup \Delta)$.

The last inference rules are as follows:

$$\frac{\{\Gamma \rightarrow \Theta, K_i(A) : A \in \Phi\} \quad K_i(\wedge\Phi), \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta} (B-\wedge),$$

$$\frac{\Gamma \rightarrow \Theta, K_i(A(a)) \quad K_i(\forall xA(x)), \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta} (B-\forall),$$

where Φ is an allowable set and the free variable a must not occur in $K_i(\forall xA(x)), \Gamma \rightarrow \Theta$ of $(B-\forall)$. We call these the *Barcan inferences*. The *side formulae* of $(B-\wedge)$ (and $(B-\forall)$) are $K_i(A)$ ($A \in \Phi$) and $K_i(\wedge\Phi)$ ($K_i(A(a))$ and $K_i(\forall xA(x))$, respectively). We need these to derive the Barcan sequents $\wedge K_i(\Phi) \rightarrow K_i(\wedge\Phi)$ and $\forall xK_i(A(x)) \rightarrow K_i(\forall xA(x))$.

In a similar manner to in Part I, a *proof* is defined to be a countable tree with the following properties: (i) every path from the root is finite; (ii) a sequent is associated with each node, and the sequent associated with each leaf is an initial sequent; and (iii) adjoining nodes together with the associated sequents form an instance of the above inference rules; and (iv) all formulae occurring in the proof belongs to \mathcal{P}_t for some $t < \omega$.¹ A sequent $\Gamma \rightarrow \Theta$ is said to be *provable* in GL_ω , denoted by $\vdash_\omega \Gamma \rightarrow \Theta$, iff there is a proof such that $\Gamma \rightarrow \Theta$ is associated with its root. This $\Gamma \rightarrow \Theta$ is called the *endsequent* of the proof.

Without K -inferences, the above system is an infinitary extension of Gentzen's LK . If we restrict the system to the finitary propositional fragment, then it is the sequent calculus formulation of $KD4$ with n knowledge operators. Ohnishi-Matsumoto [5] and [6] first formulated some modal propositional logics including propositional T , $S4$ and $S5$, in sequential calculi. Ours is an infinitary predicate extension of $KD4$ with the Barcan rules along this line of research. We will discuss the extensions of some other systems later.

We use the same notations GL_ω and \vdash_ω as in Part I. This is due to the following theorem.

THEOREM 2.1. *Let Φ be an allowable set of closed formulae and A a formula. Then $\vdash_\omega \wedge\Phi \rightarrow A$ if and only if $\Phi \vdash_\omega A$ in the sense of Part I.*

The following lemma together with the deduction theorem (Lemma 2.1) of Part I implies Theorem 2.1.

LEMMA 2.2. 1): *If $\vdash_\omega A$ in the sense of Part I, then $\vdash_\omega \rightarrow A$.*

¹Condition (iv) is used only for the equivalence between game logics between the Hilbert and Gentzen styles.

2): If $\vdash_\omega \Gamma \rightarrow \Theta$, then $\vdash_\omega \wedge \Gamma \supset \vee \Theta$ in the sense of Part I, where $\wedge \Gamma$ (and $\vee \Theta$) is interpreted as $\neg A \vee A$ (and $\neg A \wedge A$) when Γ (and Θ , respectively) is empty.

Assertion (1) is proved inductively on the structure of a proof of A in the Hilbert Style. Assertion (2) is proved inductively on the structure of a proof of $\Gamma \rightarrow \Theta$. For example, $(PI_i): K_i(A) \supset K_i K_i(A)$ is proved in the sequent calculus as follows:

$$\frac{K_i(A) \rightarrow K_i(A)}{K_i(A) \rightarrow K_i K_i(A)} (K \rightarrow K),$$

which implies $\vdash_\omega \rightarrow K_i(A) \supset K_i K_i(A)$ by $(\rightarrow \supset)$. Conversely, let us prove that $(K \rightarrow K)$ is allowed in the Hilbert style formulation: Suppose $\vdash_\omega (\wedge \Gamma) \wedge (\wedge K_i(\Delta)) \supset \vee \Theta$. By Lemma 3.3.1) of Part I and (MP_i) , $\vdash_\omega K_i((\wedge \Gamma) \wedge (\wedge K_i(\Delta))) \supset K_i(\vee \Theta)$. Using $\vdash_\omega \wedge K_i(\Sigma) \equiv K_i(\wedge \Sigma)$ for any finite set Σ and also (PI_i) , we have $\vdash_\omega \wedge K_i(\Gamma, \Delta) \supset K_i(\vee \Theta)$. Since $|\Theta| \leq 1$, $\vdash_\omega \wedge K_i(\Gamma, \Delta) \supset \vee K_i(\Theta)$.

In GL_ω , the sequent $\wedge K_i(\Phi) \rightarrow K_i(\wedge \Phi)$ is provable for any allowable set Φ , using $(B-\wedge)$: For any $A \in \Phi$, $\vdash_\omega \wedge K_i(\Phi) \rightarrow K_i(A)$, which implies $\vdash_\omega \wedge K_i(\Phi) \rightarrow K_i(\wedge \Phi), K_i(A)$ by (th). Hence

$$\frac{\{\wedge K_i(\Phi) \rightarrow K_i(\wedge \Phi), K_i(A) : A \in \Phi\} \quad K_i(\wedge \Phi) \rightarrow K_i(\wedge \Phi)}{\wedge K_i(\Phi) \rightarrow K_i(\wedge \Phi)} (B-\wedge).$$

In the case of finite Φ , $\wedge K_i(\Phi) \rightarrow K_i(\wedge \Phi)$ is provable without $(B-\wedge)$.

In the same way, we can show $\vdash_\omega \forall x K_i(A(x)) \rightarrow K_i(\forall x A(x))$.

Note that Lemma 2.2.2) could not hold without assuming the Barcan axioms for the Hilbert style formulation and the Barcan inferences for the Gentzen style formulation.

In Section 4, we will give an alternative formulation of GL_ω and prove the cut-elimination theorem for it in Section 5. From this cut-elimination theorem, we will prove the following in Section 4.

THEOREM 2.3. [Cut-Elimination for GL_ω] If $\vdash_\omega \Gamma \rightarrow \Theta$, there is a cut-free proof of $\Gamma \rightarrow \Theta$ in GL_ω .

In GL_ω , if the Barcan inference $(B-\wedge)$ or $(B-\forall)$ occurs in a proof, some formulae of the form $K_i(\cdot)$ occur in the upper sequent but not in the lower sequent of the inference. Hence the above cut-elimination theorem does not imply the full subformula property that every formula occurring in a

cut-free proof is a subformula of a formula occurring in its endsequent.² Nevertheless, the subformula property holds for the other kinds of formulae. This partial violation of the subformula property is sometimes an obstacle and sometimes not. Fortunately, it does not prevent us from proving the term existence theorem, which is the present primary purpose of Part II.

In logic GL_ω , we give a special attention to a part of a cut-free proof, where the violation of the subformula property is kept minimal. Consider a proof P . In the path from the root to a leaf, the lower sequent of the lowest occurrence of $(K \rightarrow K)$ is called a *boundary*. If the path does not have such an inference, the *boundary* is the initial sequent. The part of P from the endsequent to all boundaries is called the *trunk* of P . In the trunk of a cut-free proof, there is no occurrence of inference $(K \rightarrow K)$. Since the side formulae of inference $(B-\wedge)$ or $(B-\forall)$ are of the form $K_j(B)$ for some j and B , the following holds for a cut-free proof P :

$$(2.1) \quad \begin{array}{l} \text{any formula occurring in the trunk of } P \text{ is} \\ \text{a subformula of some formula occurring in the endsequent} \\ \text{or has the outermost symbol } K_j \text{ for some } j. \end{array}$$

This property will be used in proving the term existence theorem in Section 3.

When we prohibit the use of $(K \rightarrow K)$, $(B-\wedge)$ and $(B-\forall)$, the system is essentially the same as Gentzen's LK with the infinitary modification, which corresponds to the base logic GL_0 defined in Part I. We denote the provability in GL_0 by \vdash_0 . Then it is shown as Proposition 4.1 of Part I that

$$(2.2) \quad \vdash_\omega \Gamma \rightarrow \Theta \text{ implies } \vdash_0 \epsilon\Gamma \rightarrow \epsilon\Theta.$$

Recall that $\epsilon\Gamma$ is obtained from Γ by eliminating all K_j ($j = 1, \dots, n$). Theorem 4.2 of this paper states that cut-elimination holds for GL_0 and implies the full subformula property, since GL_0 does not allow $(B-\wedge)$ and $(B-\forall)$. Hence logic GL_0 is contradiction-free, which together with (2.2) implies that logic GL_ω is also contradiction-free.

The above cut-elimination theorem does not rely on the Barcan inferences. That is, Theorem 2.3 holds when we prohibit one or both of $(B-\wedge)$ and $(B-\forall)$. When we prohibit both $(B-\wedge)$ and $(B-\forall)$, a cut-free proof satisfies the full subformula property. Using this fact, we can prove that the assertion of Lemma 2.4 of Part I, i.e., $C(A) \rightarrow K_i(C(A))$, could not necessarily

²A *subformula* of a given formula A is defined in the standard inductive manner, but note the step for quantification that if $\forall xB(x)$ and $\exists xB(x)$ are subformulae of A , then $B(t)$ is a subformula of A for any term t .

be provable in GL_ω without $(B-\wedge)$ and $(B-\forall)$. This will be discussed more in a separate paper.

When we restrict our attention to the finitary propositional fragment of our logic, the Barcan inferences $(B-\wedge)$ and $(B-\forall)$ become unnecessary, as was already stated. Therefore a cut-free proof in the finitary propositional fragment of GL_ω has the full subformula property.

2.2 Variations of GL_ω

The sequent calculus formulation of $GL_{\omega p}$ is obtained from GL_ω by replacing $(K \rightarrow K)$ by the following $(K \rightarrow K)_p$:

$$\frac{\Gamma \rightarrow \Theta}{K_i(\Gamma) \rightarrow K_i(\Theta)} (K \rightarrow K)_p,$$

where $|\Theta| \leq 1$. Logic $GL_{\omega p}$ is weaker than GL_ω , e.g., $K_i(A) \rightarrow K_i K_i(A)$ is not necessarily provable in $GL_{\omega p}$, but is provable in GL_ω . Cut-elimination as well as the other metatheorems of Section 3 hold for $GL_{\omega p}$. In this sense, $GL_{\omega p}$ has a status similar to GL_ω . However, the epistemic axiomatization of Nash equilibrium needs some modification in $GL_{\omega p}$, i.e., the common knowledge formula should be modified by using $\bigcup_{m < \omega} K_p(m)$ instead of $\bigcup_{m < \omega} K(m)$. Recall that $K_p(m) = \{K_{i_1} \dots K_{i_m} : \text{each } K_{i_t} \text{ is one of } K_1, \dots, K_n\}$ and $K(m) = \{K_{i_1} \dots K_{i_m} : \text{each } K_{i_t} \text{ is one of } K_1, \dots, K_n \text{ with } i_t \neq i_{t+1} \text{ for } t = 1, \dots, m-1\}$. In game theory applications, extensions of KD_4 seem more natural than those of KD or K , which would become clearer only by working more on game theoretical applications.

Among others, the infinitary predicate extension of modal logic S_4 is of special interest. The sequent calculus of the infinitary predicate extension of S_4 is obtained from GL_ω by replacing $(K \rightarrow K)$ by the following two inferences:

$$\frac{\Gamma, \Delta \rightarrow \Theta}{K_i(\Gamma), \Delta \rightarrow \Theta} (K \rightarrow) \qquad \frac{K_i(\Gamma) \rightarrow A}{K_i(\Gamma) \rightarrow K_i(A)} (\rightarrow K).$$

We denote this system by $GL_\omega S_4$. The cut-elimination theorem for $GL_\omega S_4$ is obtained by modifying the proof for GL_m given in Section 5. One important consequence is that a cut-free proof in the finitary propositional fragment of $GL_\omega S_4$ satisfies the full subformula property.

In the full predicate calculus $GL_\omega S_4$, the Barcan inference $(B-\wedge)$ is needed to prove $C(A) \rightarrow K_i(C(A))$. In this case, $(B-\wedge)$ and also $(B-\forall)$ become obstacles in applying the cut-elimination theorem in that (2.1) does not hold. Therefore we do not obtain the results in Section 3 for $GL_\omega S_4$.

3. Applications of the Cut-Elimination Theorem

In this section, we assume the cut-elimination theorem (Theorem 2.3), whose proof will be given in Sections 4 and 5, and will prove several theorems. From them, we will obtain the term existence theorem (Theorem 3.5) used for the undecidability results in Part I.

3.1 Term Existence Theorems and Separation Theorem

In this subsection, we will give some results on GL_ω , all of whose proofs will be given in the next subsection.

The first result is term existence for an individual player.

THEOREM 3.1. [*Term Existence I*] *If $\vdash_\omega K_i(\Gamma) \rightarrow \exists x_1 \dots \exists x_\ell K_i(A(x_1, \dots, x_\ell))$, then $\vdash_\omega K_i(\Gamma) \rightarrow K_i(A(t_1, \dots, t_\ell))$ for some terms t_1, \dots, t_ℓ .*

When the system has closed terms and no free variables occur in Γ and in $\exists x_1 \dots \exists x_\ell K_i(A(x_1, \dots, x_\ell))$, we can assert that t_1, \dots, t_ℓ are closed terms.

It follows from Theorem 3.1 that $K_i(\exists x A(x)) \rightarrow \exists x K_i(A(x))$ is not necessarily provable. Indeed, suppose that the system has a unary predicate $P(\cdot)$ and $\vdash_\omega K_i(\exists x \forall y (P(y) \supset P(x))) \rightarrow \exists x K_i(\forall y (P(y) \supset P(x)))$. Then it follows from Theorem 3.1 that $\vdash_\omega K_i(\exists x \forall y (P(y) \supset P(x))) \rightarrow K_i(\forall y (P(y) \supset P(t)))$ for some term t . By (2.2), we have $\vdash_0 \exists x \forall y (P(y) \supset P(x)) \rightarrow \forall y (P(y) \supset P(t))$. Since $\vdash_0 \rightarrow \exists x \forall y (P(y) \supset P(x))$, we have $\vdash_0 \rightarrow \forall y (P(y) \supset P(t))$. By the cut-elimination theorem for GL_0 , however, this is not provable. Hence $K_i(\exists x \forall y (P(y) \supset P(x))) \rightarrow \exists x K_i(\forall y (P(y) \supset P(x)))$ is not provable.

An assertion parallel to Theorem 3.1 holds for a disjunctive formula:

(3.1) If $\vdash_\omega K_i(\Gamma) \rightarrow \bigvee_{A \in \Phi} K_i(A)$, then $\vdash_\omega K_i(\Gamma) \rightarrow K_i(A)$ for some $A \in \Phi$,

which is also proved in the same way as Theorem 3.1. Using this fact, we find that $K_i(A \vee B) \rightarrow K_i(A) \vee K_i(B)$ is not necessarily provable. Hence the statements of Proposition 3.1 of Part I are, indeed, unparalleled.

A result similar to the above term existence (and disjunctive property) theorem is known for intuitionistic logic, cf., Harrop [2] and [3] (see also van Dalen [7]). From the view point of formal systems, logic GL_ω has the restriction of the succedents of the upper and lower sequents of $(K \rightarrow K)$ to contain at most one formula, while intuitionistic logic has the same restriction on any sequent (cf., Gentzen [1]). In fact, to prove the above theorem, we will modify the Barcan inference so that this restriction holds for all sequents in the trunk of the cut-free proof of $K_i(\Gamma) \rightarrow \exists x_1 \dots \exists x_\ell K_i(A(x_1, \dots, x_\ell))$, which

will be stated in Lemma 3.2. Thus Theorem 3.1 is based on the property of our logic similar to that of intuitionistic logic.

To state Lemma 3.2 and to prove Theorem 3.1, we use the following, slightly different formulations of $(B-\wedge)$ and $(B-\forall)$:

$$\frac{\{\Gamma \rightarrow \Theta, K_i(A) : A \in \Phi\} \quad K_i(\wedge\Phi), \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda} (B-\wedge)^*,$$

$$\frac{\Gamma \rightarrow \Theta, K_i(A(a)) \quad K_i(\forall xA(x)), \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda} (B-\forall)^*,$$

where Φ is an allowable set and the free variable a must not occur in $K_i(\forall xA(x)), \Gamma, \Delta \rightarrow \Theta, \Lambda$ of $(B-\forall)^*$. When we have (th), $(B-\wedge)$ and $(B-\forall)$ are equivalent to $(B-\wedge)^*$ and $(B-\forall)^*$, respectively, in that provability \vdash_ω , as well as the cut-elimination theorem, is not affected by the use of $(B-\wedge)^*$ and $(B-\forall)^*$ instead of $(B-\wedge)$ and $(B-\forall)$. Nevertheless, $(B-\wedge)^*$ and $(B-\forall)^*$ are more convenient in proving Theorem 3.1, and the original ones are more for other purposes.

LEMMA 3.2. *Suppose $\vdash_\omega K_i(\Gamma) \rightarrow \exists x_1 \dots \exists x_\ell K_i(A(x_1, \dots, x_\ell))$. Then there is a cut-free proof P of $K_i(\Gamma) \rightarrow \exists x_1 \dots \exists x_\ell K_i(A(x_1, \dots, x_\ell))$ such that the succedent of each sequent in the trunk of P has at most one formula.*

To prove the term existence theorem stated in Part I, i.e., Theorem 3.5 in the following, we need one more theorem. A formula A is said to be *indecomposable* iff A is atomic or the outermost symbol of A is K_j for some $j = 1, \dots, n$. We say that for $i = 1, \dots, n$, a formula A is a K_i -formula iff the outermost symbol of every maximal indecomposable subformula of A is K_i ; and that A is a K_{-i} -formula iff K_i occurs only in the scope of K_j for some $j \neq i$. These two notions are mutually exclusive. For example, $K_1(K_2(A) \supset B) \supset K_1(B)$ is a K_1 -formula, $K_2(K_1(A) \supset B) \supset K_3(B)$ a K_{-1} -formula, and $K_1(K_2(A) \supset B) \supset K_2(B)$ is neither a K_1 -formula nor a K_{-1} -formula.

THEOREM 3.3. [Separation Theorem] *Let Γ_i, Θ_i be finite sets of K_i -formulae, and Γ_{-i}, Θ_{-i} finite sets of K_{-i} -formulae ($i = 1, \dots, n$). If $\vdash_\omega \Gamma_i, \Gamma_{-i} \rightarrow \Theta_i, \Theta_{-i}$, then $\vdash_\omega \Gamma_i \rightarrow \Theta_i$ or $\vdash_\omega \Gamma_{-i} \rightarrow \Theta_{-i}$.*

This theorem is proved based on the following lemma.

LEMMA 3.4. *Let Γ_i, Θ_i be finite sets of K_i -formulae, and Γ_{-i}, Θ_{-i} finite sets of K_{-i} -formulae ($i = 1, \dots, n$). Let P be a cut-free proof of $\Gamma_i, \Gamma_{-i} \rightarrow \Theta_i, \Theta_{-i}$. Then every formula occurring in the trunk of P is either a K_i -formula or a K_{-i} -formula.*

Using Theorem 3.3 repeatedly, we have a refinement: Let Γ_i, Θ_i be finite sets of K_i -formulae for $i = 0, 1, \dots, n$, where Γ_0 and Θ_0 are finite sets of nonepistemic formulae. Then

$$(3.2) \quad \begin{array}{l} \text{If } \vdash_{\omega} \Gamma_0, \Gamma_1, \dots, \Gamma_n \rightarrow \Theta_0, \Theta_1, \dots, \Theta_n, \\ \text{then } \vdash_{\omega} \Gamma_i \rightarrow \Theta_i \text{ for some } i = 0, 1, \dots, n, \end{array}$$

Now we can state the term existence theorem used in Part I.

THEOREM 3.5. [*Term Existence II*] *Let Γ be a finite set of nonepistemic formulae, and A a nonepistemic formula. If $\vdash_{\omega} C(\Gamma) \rightarrow \exists x_1 \dots \exists x_{\ell} C(A(x_1, \dots, x_{\ell}))$, then $\vdash_{\omega} C(\Gamma) \rightarrow C(A(t_1, \dots, t_{\ell}))$ for some terms t_1, \dots, t_{ℓ} .*

Theorems 3.1 and 3.3 can be obtained for GL_m and GL_{mp} ($m \leq \omega$) as well as for their finitary fragments. However, they fail to hold for the S_4 -type extensions. For example, $\vdash_{\omega S_4} K_1(\exists x K_1(P(x))) \rightarrow \exists x K_1(P(x))$, but Theorem 3.1 does not hold for this sequent in $GL_{\omega S_4}$, where $P(\cdot)$ is a unary predicate. Also, $\vdash_{\omega S_4} K_1(P(a)), K_2(\neg P(a)) \rightarrow \cdot$, but Theorem 3.3 does not hold for this sequent in $GL_{\omega S_4}$. Nevertheless, it still remains open whether Theorem 3.5 holds in $GL_{\omega S_4}$.

Theorem 3.3 manifests cognitive relativism in GL_{ω} in that the epistemic world of each player (even in the mind of another player) is separated from the others'. In contrast, the S_4 -type extension $GL_{\omega S_4}$ does not permit this separation as was mentioned above, but assumes that knowledge must be true relative to the thinker and ultimately relative to the investigator. In GL_{ω} , cognitive relativism manifested as the separation of epistemic worlds enables us to obtain our results.

3.2 Proofs of the Results of Section 3.1

Recall that we use $(B-\wedge)^*$ and $(B-\vee)^*$ instead of $(B-\wedge)$ and $(B-\vee)$ in the proofs of Lemma 3.2 and Theorem 3.1.

PROOF OF LEMMA 3.2.

Let P be a cut-free proof of $K_i(\Gamma) \rightarrow \exists x_1 \dots \exists x_{\ell} K_i(A(x_1, \dots, x_{\ell}))$ in GL_{ω} . Consider the trunk of P . By the form of the endsequent and (2.1), the trunk has only four types of inferences, (th), $(\rightarrow \exists)$, $(B-\wedge)^*$ and $(B-\vee)^*$, and each boundary is either an initial sequent or the lower sequent of inference ($K \rightarrow K$).

We prove that for any sequent $\Lambda \rightarrow \Theta$ in the trunk of P , there is a sequent $\Lambda \rightarrow \Theta^*$ with its cut-free proof P' such that

- (i): Θ^* is a subset of Θ ;

- (ii): the succedent of any sequent in the trunk of P' has at most one formula.

Of course, Θ^* must have at most one formula. If this is done, we have a cut-free proof of $K_i(\Gamma) \rightarrow \exists x_1 \dots \exists x_\ell K_i(A(x_1, \dots, x_\ell))$ with the property (ii). We prove this assertion by induction on the tree structure of the trunk of P from each boundary.

First, consider a boundary of the trunk of P . Then $\Lambda \rightarrow \Theta$ is an initial sequent or the lower sequent of inference ($K \rightarrow K$). Thus Θ has at most one formula. Therefore the subproof of $\Lambda \rightarrow \Theta$ in P is a cut-free proof with the properties (i) and (ii).

Next, consider a sequent $\Delta \rightarrow \Psi$ in the trunk of P which is not a boundary. The induction hypothesis is now that for any sequent $\Lambda \rightarrow \Theta$ immediately above $\Delta \rightarrow \Psi$, there is a cut-free proof of $\Lambda \rightarrow \Theta^*$ with the properties (i) and (ii). We consider the three possible cases, (th), $(\rightarrow \exists)$, $(B-\wedge)^*$ and $(B-\vee)^*$. That is, $\Lambda \rightarrow \Theta$ and $\Delta \rightarrow \Psi$ are upper and lower sequents of one of these inferences.

Consider (th). The upper sequent $\Lambda \rightarrow \Theta$ of (th) satisfies $\Lambda \subseteq \Delta$ and $\Theta \subseteq \Psi$. By the induction hypothesis, there is a cut-free proof of $\Lambda \rightarrow \Theta^*$ with (i) and (ii). Define Ψ^* to be Θ^* . Adding (th) to the proof of $\Lambda \rightarrow \Theta^*$, we have a cut-free proof of $\Delta \rightarrow \Psi^*$ with (i) and (ii).

Consider $(\rightarrow \exists)$. This $(\rightarrow \exists)$ is represented as

$$\frac{\Lambda \rightarrow \Theta', \exists y_1 \dots \exists y_k K_i(A(t, y_1, \dots, y_k))}{\Lambda \rightarrow \Theta', \exists y \exists y_1 \dots \exists y_k K_i(A(y, y_1, \dots, y_k))} ,$$

where $\Delta \rightarrow \Psi$ is $\Lambda \rightarrow \Theta', \exists y \exists y_1 \dots \exists y_k K_i(A(y, y_1, \dots, y_k))$. The induction hypothesis states that there is a cut-free proof P' of $\Lambda \rightarrow \Theta^*$ with the properties (i), i.e., $\Theta^* \subseteq \Theta' \cup \{\exists y_1 \dots \exists y_k K_i(A(t, y_1, \dots, y_k))\}$, and (ii). If Θ^* does not contain $\exists y_1 \dots \exists y_k K_i(A(t, y_1, \dots, y_k))$, we conclude that the proof P' of $\Lambda \rightarrow \Theta^*$ is the desired one. If Θ^* consists of $\exists y_1 \dots \exists y_k K_i(A(t, y_1, \dots, y_k))$, we add the following step to the proof P' , which is the desired one:

$$\frac{\Lambda \rightarrow \exists y_1 \dots \exists y_k K_i(A(t, y_1, \dots, y_k))}{\Lambda \rightarrow \exists y \exists y_1 \dots \exists y_k K_i(A(y, y_1, \dots, y_k))} (\rightarrow \exists).$$

Consider $(B-\wedge)^*$:

$$\frac{\{\Sigma \rightarrow \Xi, K_j(B) : B \in \Phi\} \quad K_j(\wedge \Phi), \Pi \rightarrow \Upsilon}{\Sigma, \Pi \rightarrow \Xi, \Upsilon} ,$$

where $\Delta \rightarrow \Psi$ is $\Sigma, \Pi \rightarrow \Xi, \Upsilon$. By the induction hypothesis, for each $B \in \Phi$ we have a cut-free proof P_B of $\Sigma \rightarrow \Xi_B$ with the properties (i), i.e., $\Xi_B \subseteq \Xi \cup$

$\{K_j(B)\}$, and (ii), and also we have a cut-free proof P' of $K_j(\wedge\Phi), \Pi \rightarrow \Upsilon^*$ with (i), i.e., $\Upsilon^* \subseteq \Upsilon$, and (ii).

If Ξ_B does not contain $K_j(B)$ for some $B \in \Phi$, then it is a subset of Ξ . Hence we obtain a proof of $\Sigma, \Pi \rightarrow \Xi_B$ by adding a (th) to the proof P_B , which has the properties (i) and (ii). Now consider the case where Ξ_B consists of $K_j(B)$ for any $B \in \Phi$. Then we have a proof of $\Sigma, \Pi \rightarrow \Upsilon^*$ combining proofs P_B ($B \in \Phi$) and P' in the following way:

$$\frac{\{\Sigma \rightarrow K_j(B) : B \in \Phi\} \quad K_j(\wedge\Phi), \Pi \rightarrow \Upsilon^*}{\Sigma, \Pi \rightarrow \Upsilon^*} (B-\wedge)^*.$$

This cut-free proof of $\Sigma, \Pi \rightarrow \Upsilon^*$ satisfies (i) and (ii).

Finally, consider $(B-\vee)^*$:

$$\frac{\Sigma \rightarrow \Xi, K_j(B(a)) \quad K_j(\forall xA(x)), \Pi \rightarrow \Upsilon}{\Sigma, \Pi \rightarrow \Xi, \Upsilon},$$

where $\Delta \rightarrow \Psi$ is $\Sigma, \Pi \rightarrow \Xi, \Upsilon$. By the induction hypothesis, we have a cut-free proof P' of $\Sigma \rightarrow \Xi^*$ with the properties (i), i.e., $\Xi^* \subseteq \Xi \cup \{K_j(B(a))\}$, and (ii), and also we have a cut-free proof P'' of $K_j(\forall xA(x)), \Pi \rightarrow \Upsilon^*$ with (i), i.e., $\Upsilon^* \subseteq \Upsilon$, and (ii).

If Ξ^* does not contain $K_j(B(a))$, then it is a subset of Ξ . Hence we obtain a proof of $\Sigma, \Pi \rightarrow \Xi^*$ by adding a (th) to the proof P' , which has the properties (i) and (ii). Now consider the case where Ξ^* consists of $K_j(B(a))$. Then we have a proof of $\Sigma, \Pi \rightarrow \Upsilon^*$ combining proofs P' and P'' in the following way:

$$\frac{\Sigma \rightarrow K_j(B(a)) \quad K_j(\forall xA(x)), \Pi \rightarrow \Upsilon^*}{\Sigma, \Pi \rightarrow \Upsilon^*} (B-\vee)^*.$$

This cut-free proof of $\Sigma, \Pi \rightarrow \Upsilon^*$ satisfies (i) and (ii). ■

PROOF OF THEOREM 3.1. By Lemma 3.2, there is a cut-free proof P of $K_i(\Gamma) \rightarrow \exists x_1 \dots \exists x_\ell K_i(A(x_1, \dots, x_\ell))$ such that the succedent of any sequent in the trunk of P has at most one formula. By induction on the structure of the trunk of P , we prove that for any sequent $\Lambda \rightarrow \Theta$ in the trunk,

- (*) : if Θ is represented as $\{\exists x_k \dots \exists x_\ell K_i(A(t_1, \dots, t_{k-1}, x_k, \dots, x_\ell))\}$ with $k \leq \ell$, then $\vdash_\omega \Lambda \rightarrow K_i(A(t_1, \dots, t_\ell))$ for some terms t_k, \dots, t_ℓ .

From this, we have the conclusion that $\vdash_\omega K_i(\Gamma) \rightarrow K_i(A(t_1, \dots, t_\ell))$ for some terms t_1, \dots, t_ℓ . As will be argued in Subsection 5.1, we can assume that for any proof, some free variables do not occur in it.

For a boundary $\Lambda \rightarrow \Theta$, the premise of $(*)$ does not hold, since it is an initial sequent of the form $K_j(\cdot) \rightarrow K_j(\cdot)$ or the lower sequent of $(K \rightarrow K)$.

Consider a sequent $\Lambda \rightarrow \Theta$ which is not a boundary in the trunk. Now the induction hypothesis is that for any sequent immediately above $\Lambda \rightarrow \Theta$, $(*)$ holds. We assume that Θ is represented as $\{\exists x_k \dots \exists x_\ell K_i(A(t_1, \dots, t_{k-1}, x_k, \dots, x_\ell))\}$. We will consider the following four cases, (th) , $(\rightarrow \exists)$, $(B-\wedge)^*$ and $(B-\vee)^*$, depending on the last inference which is applied to get $\Lambda \rightarrow \Theta$.

Consider (th) . Then its upper sequent has the form $\Lambda' \rightarrow \Theta'$ with $\Lambda' \subseteq \Lambda$ and $\Theta' \subseteq \Theta$. If Θ' consists of $\exists x_k \dots \exists x_\ell K_i(A(t_1, \dots, t_{k-1}, x_k, \dots, x_\ell))$, then $\vdash_\omega \Lambda' \rightarrow K_i(A(t_1, \dots, t_\ell))$ for some t_k, \dots, t_ℓ by the induction hypothesis, which together with (th) implies $\vdash_\omega \Lambda \rightarrow K_i(A(t_1, \dots, t_\ell))$. If Θ' is empty, then $\vdash_\omega \Lambda \rightarrow K_i(A(t_1, \dots, t_\ell))$ for any terms t_1, \dots, t_ℓ by (th) .

Consider $(\rightarrow \exists)$. Then its upper sequent is $\Lambda \rightarrow \exists x_{k+1} \dots \exists x_\ell K_i(A(t_1, \dots, t_k, x_{k+1}, \dots, x_\ell))$ for some terms t_1, \dots, t_k . If $k = \ell$, the upper sequent is $\Lambda \rightarrow K_i(A(t_1, \dots, t_\ell))$, which is already the assertion for the lower sequent. If $k < \ell$, then $\vdash_\omega \Lambda \rightarrow K_i(A(t_1, \dots, t_\ell))$ for some terms t_{k+1}, \dots, t_ℓ by the induction hypothesis, which is also the assertion.

Consider $(B-\wedge)^*$:

$$\frac{\{\Sigma \rightarrow K_j(B) : B \in \Phi\} \quad K_j(\wedge \Phi), \Delta \rightarrow \exists x_k \dots \exists x_\ell K_i(A(t_1, \dots, t_{k-1}, x_k, \dots, x_\ell))}{\Sigma, \Delta \rightarrow \exists x_k \dots \exists x_\ell K_i(A(t_1, \dots, t_{k-1}, x_k, \dots, x_\ell))}.$$

By the induction hypothesis, $\vdash_\omega K_j(\wedge \Phi), \Delta \rightarrow K_i(A(t_1, \dots, t_\ell))$ for some terms t_k, \dots, t_ℓ . Then we have a proof of $\Sigma, \Delta \rightarrow K_i(A(t_1, \dots, t_\ell))$ by adding the following:

$$\frac{\{\Sigma \rightarrow K_j(B) : B \in \Phi\} \quad K_j(\wedge \Phi), \Delta \rightarrow K_i(A(t_1, \dots, t_\ell))}{\Sigma, \Delta \rightarrow K_i(A(t_1, \dots, t_\ell))} (B-\wedge)^*.$$

Finally, consider $(B-\vee)^*$:

$$\frac{\Sigma \rightarrow K_j(B(a)) \quad K_j(\forall x A(x)), \Delta \rightarrow \exists x_k \dots \exists x_\ell K_i(A(t_1, \dots, t_{k-1}, x_k, \dots, x_\ell))}{\Sigma, \Delta \rightarrow \exists x_k \dots \exists x_\ell K_i(A(t_1, \dots, t_{k-1}, x_k, \dots, x_\ell))},$$

where a must not occur in $\Sigma, K_j(\forall x A(x)), \Delta, \exists x_k \dots \exists x_\ell K_i(A(t_1, \dots, t_{k-1}, x_k, \dots, x_\ell))$. By the induction hypothesis, $\vdash_\omega K_j(\forall x A(x)), \Delta \rightarrow K_i(A(t_1, \dots, t_\ell))$ for some terms t_k, \dots, t_ℓ . Also, there is a proof P' of $\Sigma \rightarrow K_j(B(a))$. Then we obtain another proof P'' of $\Sigma \rightarrow K_j(B(b))$ by substituting b for all occurrences of a in P' , where b is a new free variable not occurring in P' and in the right upper sequent. Then we have a proof of $\Sigma, \Delta \rightarrow K_i(A(t_1, \dots, t_\ell))$ by adding the following:

$$\frac{\Sigma \rightarrow K_j(B(b)) \quad K_j(\forall x A(x)), \Delta \rightarrow K_i(A(t_1, \dots, t_\ell))}{\Sigma, \Delta \rightarrow K_i(A(t_1, \dots, t_\ell))} (B-\vee)^*.$$

■

In the following, we use the original $(B-\wedge)$ and $(B-\forall)$.

PROOF OF LEMMA 3.4. As was noted in (2.1), the violation of the subformula property is caused by in the occurrences of $(B-\wedge)$ in the trunk of P . The side formulae of $(B-\wedge)$ and $(B-\forall)$ are of the form $K_j(\cdot)$ for some $j = 1, \dots, n$; they are K_i -formulae if $j = i$ or K_{-i} -formulae if $j \neq i$. For any formula of other form, the subformula property holds in the trunk. Hence every formula occurring in the trunk must be a K_i -formula or a K_{-i} -formula, since the endsequent consists of K_i -formulae and K_{-i} -formulae. ■

PROOF OF THEOREM 3.3. Suppose $\vdash_\omega \Gamma_i, \Gamma_{-i} \rightarrow \Theta_i, \Theta_{-i}$, where Γ_i, Θ_i are finite sets of K_i -formulae and Γ_{-i}, Θ_{-i} are finite sets of K_{-i} -formulae. Then there is a cut-free proof P of $\Gamma_i, \Gamma_{-i} \rightarrow \Theta_i, \Theta_{-i}$ by the cut-elimination theorem. Consider the trunk of P . Lemma 3.4 states that any sequent in the trunk is represented as $\Delta_i, \Delta_{-i} \rightarrow \Lambda_i, \Lambda_{-i}$, where Δ_i, Λ_i and $\Delta_{-i}, \Lambda_{-i}$ consist of K_i -formulae and K_{-i} -formulae, respectively. We prove the following by induction on the structure of the trunk from boundaries:

$$(*): \vdash_\omega \Delta_i \rightarrow \Lambda_i \text{ or } \vdash_\omega \Delta_{-i} \rightarrow \Lambda_{-i}.$$

We call $\Delta_i \rightarrow \Lambda_i$ and $\Delta_{-i} \rightarrow \Lambda_{-i}$ the K_i -part and K_{-i} -part.

A boundary is an initial sequent or the lower sequent of $(K \rightarrow K)$. In the first case, it is represented as $A \rightarrow A$. Then A is either a K_i -formula or K_{-i} -formula by Lemma 3.4. In the second case, it is of the form $K_j(\Delta) \rightarrow K_j(\Lambda)$, and hence the formulae are K_i -formulae if j is i , or are K_{-i} -formulae if j is not i . Thus the assertion $(*)$ holds.

Consider a sequent $\Delta \rightarrow \Lambda$ in the trunk which is not a boundary. Now the induction hypothesis is that every sequent immediately above $\Delta \rightarrow \Lambda$ satisfies $(*)$. There are three cases we have to consider: $\Delta \rightarrow \Lambda$ is the lower sequent of (th), some operational inference, and $(B-\wedge)$ or $(B-\forall)$.

(th): In this case, it follows from Lemma 3.4 that its upper and lower sequents are described as $\Delta'_i, \Delta'_{-i} \rightarrow \Lambda'_i, \Lambda'_{-i}$ and $\Delta_i, \Delta_{-i} \rightarrow \Lambda_i, \Lambda_{-i}$, where $\Delta'_i \subseteq \Delta_i, \Delta'_{-i} \subseteq \Delta_{-i}, \Lambda'_i \subseteq \Lambda_i$ and $\Lambda'_{-i} \subseteq \Lambda_{-i}$. By the induction hypothesis, $\vdash_\omega \Delta'_i \rightarrow \Lambda'_i$ or $\vdash_\omega \Delta'_{-i} \rightarrow \Lambda'_{-i}$. Thus, by (th), $\vdash_\omega \Delta_i \rightarrow \Lambda_i$ or $\vdash_\omega \Delta_{-i} \rightarrow \Lambda_{-i}$.

(Operational Inferences): By Lemma 3.4, there are only two cases: (a) the side formulae are K_i -formulae; and (b) they are K_{-i} -formulae.

Consider case (a). If the K_{-i} -part of some upper sequent is provable, the K_{-i} -part of the lower sequent is provable, since the K_{-i} -parts of the upper and lower sequents are the same. If the K_{-i} -part is not provable for any upper sequent, then the K_i -part of every upper sequent is provable by the

induction hypothesis, which implies that the K_i -part of the lower sequent is also provable.

Case (b) is parallel to case (a). Indeed, if the K_{-i} -part of some upper sequent is provable, the K_i -part of the lower sequent is also provable, and if the K_{-i} -part of every upper sequent is provable, the inference can be directly applied to the K_{-i} -parts of the upper sequents of the inference and the K_{-i} -part of the lower sequent is provable.

(**B- \wedge**): Suppose that $\Delta \rightarrow \Lambda$ is the lower sequent of ($B-\wedge$). Then ($B-\wedge$) has the following form:

$$\frac{\{\Delta_i, \Delta_{-i} \rightarrow \Lambda_i, \Lambda_{-i}, K_j(A) : A \in \Phi\} \quad K_j(\wedge\Phi), \Delta_i, \Delta_{-i} \rightarrow \Lambda_i, \Lambda_{-i}}{\Delta_i, \Delta_{-i} \rightarrow \Lambda_i, \Lambda_{-i}}.$$

Let $j = i$. Then if the K_{-i} -part of one of the upper sequents is provable, then $\vdash_\omega \Delta_{-i} \rightarrow \Lambda_{-i}$. If the K_i -parts of all upper sequents are provable, then $\vdash_\omega \Delta_i \rightarrow \Lambda_i$, since

$$\frac{\{\Delta_i \rightarrow \Lambda_i, K_i(A) : A \in \Phi\} \quad K_i(\wedge\Phi), \Delta_i \rightarrow \Lambda_i}{\Delta_i \rightarrow \Lambda_i} (B-\wedge).$$

When $j \neq i$, the proof is parallel. Indeed, if the K_i -part of one of the upper sequents is provable, then $\vdash_\omega \Delta_i \rightarrow \Lambda_i$, and if the K_{-i} -parts of all upper sequents are provable, then $\vdash_\omega \Delta_{-i} \rightarrow \Lambda_{-i}$ by ($B-\wedge$).

An argument similar to the above is applied to the case of ($B-\forall$). ■

PROOF OF THEOREM 3.5 By Lemma 2.2, $\vdash_\omega C(\Gamma) \rightarrow \exists x_1 \dots \exists x_\ell C(A(x_1, \dots, x_\ell))$ is equivalent to $\vdash_\omega \wedge C(\Gamma) \rightarrow \exists x_1 \dots \exists x_\ell C(A(x_1, \dots, x_\ell))$. Since $\vdash_\omega \Gamma, K_1(C(\Gamma)), \dots, K_n(C(\Gamma)) \rightarrow \wedge C(\Gamma)$, we have $\vdash_\omega \Gamma, K_1(C(\Gamma)), \dots, K_n(C(\Gamma)) \rightarrow \exists x_1 \dots \exists x_\ell C(A(x_1, \dots, x_\ell))$. This implies $\vdash_\omega \Gamma, K_1(C(\Gamma)), \dots, K_n(C(\Gamma)) \rightarrow \exists x_1 \dots \exists x_\ell K_1(A(x_1, \dots, x_\ell))$. Hence it follows from (3.2) that $\vdash_\omega \Gamma \rightarrow \quad, \vdash_\omega K_j(C(\Gamma)) \rightarrow \quad$ for $j \neq 1$ or $\vdash_\omega K_1(C(\Gamma)) \rightarrow \exists x_1 \dots \exists x_\ell K_1(A(x_1, \dots, x_\ell))$. In the first two cases, we have $\vdash_0 \Gamma \rightarrow \quad$ by (2.2), which implies $\vdash_\omega C(\Gamma) \rightarrow \quad$. Thus $\vdash_\omega C(\Gamma) \rightarrow C(A(t_1, \dots, t_\ell))$ for any terms t_1, \dots, t_ℓ . In the third case, it follows from Theorem 3.1 that $\vdash_\omega K_1(C(\Gamma)) \rightarrow K_1(A(t_1, \dots, t_\ell))$ for some terms t_1, \dots, t_ℓ . This implies $\vdash_0 \Gamma \rightarrow A(t_1, \dots, t_\ell)$ by (2.2). Hence $\vdash_\omega K(\Gamma) \rightarrow K(A(t_1, \dots, t_\ell))$ for any $K \in \bigcup_{t < \omega} \mathbf{K}(t)$. It follows from this that $\vdash_\omega C(\Gamma) \rightarrow K(A(t_1, \dots, t_\ell))$ for any $K \in \bigcup_{t < \omega} \mathbf{K}(t)$. Thus $\vdash_\omega C(\Gamma) \rightarrow C(A(t_1, \dots, t_\ell))$. ■

4. Sequent Calculus GL_m for m ($0 \leq m \leq \omega$)

In logic GL_m for a finite m , the logical and introspective abilities of players are known up to the depth m in the sense of $K = K_{i_1} \dots K_{i_m}$. In GL_ω ,

those abilities are known up to any depth, since $(K \rightarrow K)$ can be applied indefinitely many times in GL_ω . Hence, for GL_m , we need a sequent calculus different from GL_ω in Section 2. In this section, we present the sequent calculus formulation of GL_m . In fact, the new formulation works also for $m = \omega$, and cut-elimination holds for all m ($0 \leq m \leq \omega$). In this sense, the new formulation is a generalization of sequent calculus GL_ω . We also present some other systems.

4.1 Sequent Calculus GL_m and Cut-Elimination

In sequent calculus GL_m , each sequent has the form $K[\Gamma \rightarrow \Theta]$, where Γ and Θ are finite sets of formulae and the outer K is an element of $\bigcup_{t < 1+m} \mathbf{K}(t)$. When K is null, $K[\Gamma \rightarrow \Theta]$ is regarded as $\Gamma \rightarrow \Theta$. When m is finite, a sequent with an outer K of at most depth m is allowed, and when m is ω , a sequent with an outer K of any depth is allowed. When K is represented as $K_{i_1}K_{i_2}\dots K_{i_m}$, an infinitary extension of Gentzen's LK is given to player i_m in the mind of player i_{m-1} in the mind of player i_{m-2} ... of player i_1 . Sequents with different outer K and K' are connected by two inference rules corresponding to $(K \rightarrow K)$.

Specifically, sequent calculus GL_m ($0 \leq m \leq \omega$) is formulated as follows:

Initial Sequents: An initial sequent is of the form $K[A \rightarrow A]$, where $K \in \bigcup_{t < 1+m} \mathbf{K}(t)$ and A is a formula.

Inference Rules: The structural and operational inference rules are the same as those in Section 2, except the outer $K \in \bigcup_{t < 1+m} \mathbf{K}(t)$ associated with each sequent in inferences, for example, (cut) and $(\wedge \rightarrow)$ are given as

$$\frac{K[\Gamma \rightarrow \Theta, M] \quad K[M, \Delta \rightarrow \Lambda]}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (M) \text{ (cut)}$$

$$\frac{K[A, \Gamma \rightarrow \Theta]}{K[\wedge \Phi, \Gamma \rightarrow \Theta]} (\wedge \rightarrow) (A \in \Phi).$$

The inference $(K \rightarrow K)$ is modified as follows: for any $KK_i \in \bigcup_{t < 1+m} \mathbf{K}(t)$,

$$\frac{KK_i[\Gamma, K_i(\Delta) \rightarrow \Theta]}{K[K_i(\Gamma, \Delta) \rightarrow K_i(\Theta)]} (K \rightarrow K)_C \quad \frac{KK_i[\Gamma, K_i(\Delta) \rightarrow \Theta]}{KK_i[K_i(\Gamma, \Delta) \rightarrow K_i(\Theta)]} (K \rightarrow K)_U,$$

where $|\Theta| \leq 1$. When $m = 0$, these rules are not allowed, and when $m = 1$, the outer K is the null symbol.

The Barcan inferences take the following form: for $K \in \bigcup_{t < m} \mathbf{K}(t)$ or $K = K'K_i \in \bigcup_{t < 1+m} \mathbf{K}(t)$,

$$\frac{\{K[\Gamma \rightarrow \Theta, K_i(A)] : A \in \Phi\} \quad K[K_i(\wedge \Phi), \Gamma \rightarrow \Theta]}{K[\Gamma \rightarrow \Theta]} (B-\wedge),$$

$$\frac{K[\Gamma \rightarrow \Theta, K_i(A(a))] \quad K[K_i(\forall xA(x)), \Gamma \rightarrow \Theta]}{K[\Gamma \rightarrow \Theta]} \quad (B-\forall),$$

where Φ is an allowable set and the free variable a must not occur in $K[K_i(\forall xA(x)), \Gamma \rightarrow \Theta]$ of $(B-\forall)$. When $m = 0$, these inferences are not allowed, and when $m = 1$, the outer K of $(B-\wedge)$ (and $(B-\forall)$) is the null symbol or is the same as the outermost K_i of the side formulae of $(B-\wedge)$ (and $(B-\forall)$, respectively).

The two inferences $(K \rightarrow K)_C$ and $(K \rightarrow K)_U$ are needed to describe the idea that each player has the inference ability described by $(K \rightarrow K)_C$ as well as he knows the ability. In the cases of $(B-\wedge)$ and $(B-\forall)$, he can use and also knows the Barcan rules. This knowledge is described in the case where the innermost symbol of K coincides with the outermost symbol of the side formulae in $(B-\wedge)$ and $(B-\forall)$.

When $m = 0$, no K -inference is allowed and the outer K is null. Thus GL_0 is simply an infinitary extension of Gentzen's LK .

We use the same notations GL_m and \vdash_m as in Part I. This is due to the following theorem.

THEOREM 4.1. *Let Φ be an allowable set of closed formulae and A a formula. Then, for any m ($0 \leq m \leq \omega$), $\vdash_m \wedge \Phi \rightarrow A$ if and only if $\Phi \vdash_m A$ in the sense of Part I.*

The proof of Theorem 4.1 is routine, so we omit the proof. On the other hand, the next theorem will be given in Section 5.

THEOREM 4.2. *[Cut-Elimination for GL_m] For any m with $0 \leq m \leq \omega$, if $K[\Gamma \rightarrow \Theta]$ is provable in GL_m , there is a cut-free proof of $K[\Gamma \rightarrow \Theta]$.*

In logic GL_m for $m \geq 1$, when $(B-\wedge)$ and $(B-\forall)$ occur in a cut-free proof, the cut-free proof does not satisfy the full subformula property as in GL_ω of Section 2. In GL_0 , however, since $(B-\wedge)$ and $(B-\forall)$ are not allowed, the cut-elimination theorem ensures the full subformula property for a cut-free proof.

The relationship between the sequent calculus GL_ω in this section and that in Section 2 is given by the following proposition.

PROPOSITION 4.3. *A sequent $\Gamma \rightarrow \Theta$ is provable (without cut respectively) in the present GL_ω if and only if it is provable (without cut) in the GL_ω of Section 2.*

PROOF. Suppose that P is a proof of $\Gamma \rightarrow \Theta$ in GL_ω in the sense of this section. In the proof P , changing every sequent $K[\Delta \rightarrow \Lambda]$ to $\Delta \rightarrow \Lambda$

by removing the outer K , we obtain a proof of $\Gamma \rightarrow \Theta$ in GL_ω in the sense of Section 2.

Conversely, suppose that P is a proof of $\Gamma \rightarrow \Theta$ in GL_ω in the sense of Section 2. We associate an outer K with each sequent by induction from the endsequent as follows. We associate the null symbol with the endsequent. Consider an inference (η) in P , and assume that K is associated with the lower sequent of (η) . If (η) is not $(K \rightarrow K)$, we associate the same K with every upper sequent of (η) . Suppose that (η) is $(K \rightarrow K)$ and introduces K_i . Then we associate K with the upper sequent of (η) if the innermost symbol of K is K_i , and associate KK_i with it otherwise. Then we have a proof P' of $\Gamma \rightarrow \Theta$ in the present GL_ω .

Note that the proof obtained in each of the above paragraphs is cut-free if the original proof is cut-free. \blacksquare

The proof of Proposition 4.3 associates a proof in the present formulation of GL_ω with one in GL_ω in the sense of Section 2. The associated proofs have the same structures of inference rules. Therefore, the cut-elimination theorem for GL_ω in the present formulation provides a cut-free proof in GL_ω in the sense of Section 2, and *vice versa*. In Section 5, we prove the cut-elimination theorem (Theorem 4.2) for GL_m ($0 \leq m \leq \omega$), which implies Theorem 2.3.

REMARK 4.4. In logic GL_m ($0 \leq m \leq \omega$), the trunk of a proof can be defined in the same way as in Section 2 and then (2.1) holds for it. Consequently, the proofs of Theorems 3.1 and 3.3 for GL_ω in Section 3 work almost directly in GL_m for any finite m . Hence we have these two theorems for GL_m for any finite m . However, Theorem 3.5 is specific to the case $m = \omega$.

4.2 Logics GL_{mp} ($1 \leq m \leq \omega$)

First, recall that logic GL_{mp} which is defined by the axiom set $\Delta_{mp} = \{K(A) : A \in \bigcup_{i=1}^n \Delta_{ip} \text{ and } K \in \bigcup_{t < m} K_p(t)\}$. Recall that Δ_{ip} is the set of axioms describing the logical ability of player i , except (PI_i) , which was given in Subsection 2.2 of Part I. The sequent calculus formulation of logic GL_{mp} is obtained from GL_m with the following modifications: An outer K of each sequent in initial sequents, structural and operational inferences is taken from $\bigcup_{t < 1+m} K_p(t)$ instead of $\bigcup_{t < 1+m} K(t)$. We replace $(K \rightarrow K)_C$, $(K \rightarrow K)_U$, $(B-\wedge)$ and $(B-\vee)$ by the following: for any $K \in \bigcup_{t < m} K_p(t)$,

$$\frac{KK_i[\Gamma \rightarrow \Theta]}{K[K_i(\Gamma) \rightarrow K_i(\Theta)]} (K \rightarrow K)_p, \text{ where } |\Theta| \leq 1,$$

$$\frac{\{K[\Gamma \rightarrow \Theta, K_i(A)] : A \in \Phi\} \quad K[K_i(\wedge\Phi), \Gamma \rightarrow \Theta]}{K[\Gamma \rightarrow \Theta]} (B-\wedge)_p$$

$$\frac{K[\Gamma \rightarrow \Theta, K_i(A(a))] \quad K[K_i(\forall xA(x)), \Gamma \rightarrow \Theta]}{K[\Gamma \rightarrow \Theta]} (B-\forall)_p,$$

where the free variable a must not occur in $K[K_i(\forall xA(x)), \Gamma \rightarrow \Theta]$ of $(B-\forall)_p$. If the additional condition $|\Theta| \leq 1$ for $(K \rightarrow K)_p$ is replaced by $|\Theta| = 1$, then the system is denoted by GL_{mK} , which corresponds to the logic defined by Δ_0 without (PI_i) and (\perp_i) .

Cut-elimination is obtained for GL_{mp} as well as for GL_{mK} from the proof given in Section 5 with the desired modifications.

When $m = \omega$, again, we do not need outer K and KK_i in the inference rules as in GL_ω . Logics $GL_{\omega p}$ and $GL_{\omega K}$ are the infinitary predicate extensions of modal logics KD and K .

Proposition 2.2 (Faithful Representation) of Part I is a special case of the following proposition.

PROPOSITION 4.5. [*Faithful Representation*] For a finite m , $\vdash_{(m+1)p} K_i(\Gamma) \rightarrow K_i(\Theta)$ if and only if $\vdash_{mp} \Gamma \rightarrow \Theta$.

PROOF. It is enough to show the *only-if* part. Let P be a proof of $K_i(\Gamma) \rightarrow K_i(\Theta)$ in $GL_{(m+1)p}$. We can define boundaries and the trunk of P in the same way as in GL_ω . In the trunk, by the form of the endsequent, every formula has the form $K_j(B)$ for some j and B , and only inferences (th), $(B-\wedge)_p$ and $(B-\forall)_p$ may occur. We prove by induction on the tree structure of the trunk from boundaries to the root that $\Delta^\# \rightarrow \Lambda^\#$ is provable in GL_{mp} for any sequent $\Delta \rightarrow \Lambda$ in the trunk of P , where $\Delta^\#, \Lambda^\#$ are obtained from Δ, Λ by eliminating the outermost K_j ($j = 1, \dots, n$) of each formula in Δ, Λ .

A boundary $\Delta \rightarrow \Lambda$ is either an initial sequent or the lower sequent of $(K \rightarrow K)_p$. If $\Delta \rightarrow \Lambda$ is an initial sequent, it has the form $K_j(A) \rightarrow K_j(A)$. Then $A \rightarrow A$ is provable in GL_{mp} . Next suppose that a boundary is the lower sequent of $(K \rightarrow K)_p$, which is expressed as

$$\frac{K_j[\Sigma \rightarrow \Xi]}{K_j(\Sigma) \rightarrow K_j(\Xi)} .$$

All sequents above this inference $(K \rightarrow K)_p$ have the outermost symbol K_j , i.e., $K_j K'[\cdot \rightarrow \cdot]$, where K' may be the null symbol. We eliminate this outermost K_j of each sequent in the proof of $K_j[\Sigma \rightarrow \Xi]$ and obtain a proof of $\Sigma \rightarrow \Xi$ in GL_{mp} .

Now consider inferences (th), $(B-\wedge)_p$ and $(B-\forall)_p$. The induction hypothesis is that for every upper sequent $\Delta \rightarrow \Lambda$ of (th) or $(B-\wedge)_p$, $\Delta^\# \rightarrow \Lambda^\#$ is provable in GL_{mp} . We consider only $(B-\wedge)_p$.

Consider $(B-\wedge)_p$:

$$\frac{\{\Sigma \rightarrow \Psi, K_j(A) : A \in \Phi\} \quad K_j(\wedge\Phi), \Sigma \rightarrow \Psi}{\Sigma \rightarrow \Psi} .$$

By the induction hypothesis, the sequent $\Sigma^\# \rightarrow \Psi^\#, A$ is provable for any $A \in \Phi$ and $\wedge\Phi, \Sigma^\# \rightarrow \Psi^\#$ is also provable in GL_{mp} . Hence $\Sigma^\# \rightarrow \Psi^\#$ is provable in GL_{mp} since

$$\frac{\frac{\{\Sigma^\# \rightarrow \Psi^\#, A : A \in \Phi\}}{\Sigma^\# \rightarrow \Psi^\#, \wedge\Phi} (\rightarrow \wedge) \quad \wedge\Phi, \Sigma^\# \rightarrow \Psi^\#}{\Sigma^\# \rightarrow \Psi^\#} (cut).$$

■

5. Proof of the Cut-Elimination Theorem for GL_m

5.1 Preliminaries

Our proof of the cut-elimination theorem GL_m ($0 \leq m \leq \omega$) is based on the original proof of Gentzen [1]. There are several differences between Gentzen's LK and our GL_m . We have additional inference rules $(K \rightarrow K)_C$, $(K \rightarrow K)_U$, $(B-\wedge)$, $(B-\vee)$, and also our system is infinitary. We have to give careful attentions to these differences.

As in Gentzen [1], we focus our attention to a proof P having a (cut) only at the last inference:

$$(5.1) \quad \frac{K[\Gamma \rightarrow \Theta, M] \quad K[M, \Delta \rightarrow \Lambda]}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (M)(cut).$$

Then we show that

(*): for any proof with a (cut) only at the last inference, there is a cut-free proof with the same endsequent.

If this is done, we can eliminate every (cut) from an arbitrary proof by induction on the tree structure of a proof from initial sequents.

To prove the assertion (*), we use triple induction. For this purpose, we define the “grade”, the “left rank” and “right rank” of the (cut).

We assign to each formula A an ordinal number, $\text{gr}(A)$, called the *grade* of formula A . The grade $\text{gr}(A)$ is defined by induction on the structure of a formula as follows:

(1): $\text{gr}(A) = 0$ for every atomic formula A ;

- (2): $\text{gr}(\neg A) = \text{gr}(A) + 1$;
- (3): $\text{gr}(A \supset B) = \max(\text{gr}(A), \text{gr}(B)) + 1$;
- (4): $\text{gr}(\forall x A(x)) = \text{gr}(A) + 1$;
- (5): $\text{gr}(\exists x A(x)) = \text{gr}(A) + 1$;
- (6): $\text{gr}(K_i(A)) = \text{gr}(A) + 2$ for $i = 1, \dots, n$;
- (7): $\text{gr}(\wedge \Phi) = \sup\{\text{gr}(A) : A \in \Phi\} + 1$;
- (8): $\text{gr}(\vee \Phi) = \sup\{\text{gr}(A) : A \in \Phi\} + 1$.

Here $\alpha + \beta$ is the standard sum of two ordinal numbers α, β .³ The *grade* of the (cut) of (5.1) with outer $K = K_{i_1} \dots K_{i_\ell}$, denoted by γ , is defined by

$$(5.2) \quad \gamma = \begin{cases} \text{gr}(M) + \ell & \text{if the outermost symbol of } M \text{ is } K_{i_\ell} \\ \text{gr}(M) + \ell + 1 & \text{otherwise.} \end{cases}$$

Thus, the grade of the (cut) is the sum of the grade of the cut-formula and the depth of the outer K if the outermost symbol of M coincides with the innermost symbol of K . We count the depth of K as $\ell + 1$ if they do not coincide. The second case is applied if $\ell = 0$.

We also associate other two ordinal numbers, called the left and right ranks, with the cut-formula M in (5.1). The left rank is defined as follows. Let P be a proof of the form (5.1) with the cut-formula M . We will define $\rho_\ell(\eta)$ inductively for each sequent η in P in the following. For an initial sequent η , we define

$$(5.3) \quad \rho_\ell(\eta) = \begin{cases} 1 & \text{if } \eta \text{ has the form } K'[M \rightarrow M] \text{ for some outer } K'; \\ 0 & \text{otherwise.} \end{cases}$$

Now let η be the lower sequent of some occurrence (J) of an inference in P , and suppose that the left rank $\rho_\ell(\xi)$ of M at each upper sequent ξ of (J) is already defined. Then

$$(5.4) \quad \rho_\ell(\eta) = \begin{cases} \sup\{\rho_\ell(\xi) : \xi \text{ is an upper sequent of } (J)\} + 1 & \text{if the succedent of } \eta \text{ contains } M; \\ 0 & \text{otherwise.} \end{cases}$$

The *left rank* ρ_ℓ of the (cut) of the proof of (5.1) is the left rank of M at the left upper sequent $K[\Gamma \rightarrow \Theta, M]$ of (5.1). The *right rank* ρ_r of the proof of the (cut) of (5.1) is defined in the dual manner.

³ Any formula in the space \mathcal{P}_ω , which was defined in Part I and we are now working on, has a grade smaller than ω^2 . More precisely, it can be verified that $\text{gr}(A) < \omega(t + 1)$ for any $A \in \mathcal{P}_t$ and t ($0 \leq t < \omega$), which implies $\text{gr}(A) < \omega^2$ for any $A \in \mathcal{P}_\omega$.

To prove the assertion (*), we carry out three inductions:

Induction Step 1 (Subsection 5.2.1): Under the induction hypothesis that (*) holds for any proof where the grade of the (cut) is less than γ , we prove (*) for any proof where the grade is γ , the left rank is one and the right rank is also one.

Induction Step 2 (Subsection 5.2.2): Under the induction hypothesis that (*) holds for any proof where the grade is γ , the left rank is one and the right rank is less than ρ_r , we prove (*) for any proof where the grade is γ , the left rank is one and the right rank is ρ_r .

Induction Step 3 (Subsection 5.2.3): Under the induction hypothesis that (*) holds for any proof where the grade is γ , the left rank is less than ρ_ℓ and the right rank is ρ_r , we prove (*) for any proof where the grade is γ , the left rank is ρ_ℓ and the right rank is ρ_r .

Before going to the main body of the proof, we mention certain lemmata about the substitution of free variables.

LEMMA 5.1. *Let $\vdash_m K[\Gamma \rightarrow \Theta]$. Then there is a proof P of $K[\Gamma \rightarrow \Theta]$ in GL_m such that there remain an infinite number of free variables not occurring in P .*

PROOF. Let P' be a proof of $K[\Gamma \rightarrow \Theta]$. Since Γ, Θ are finite sets and each formula has at most finite number of free variables as noted in Part I, they contain only a finite number of free variables. Denote the set of free variables not occurring in $K[\Gamma \rightarrow \Theta]$ by $\{b_0, b_1, \dots\}$. We substitute the free variable b_{2t} for each free variable a_t in P' but not in $K[\Gamma \rightarrow \Theta]$ ($t = 0, 1, \dots$). The new tree is denoted by P . Then we can prove, by induction on the proof P' , that P is a proof of $K[\Gamma \rightarrow \Theta]$. There remain an infinite number of free variables not occurring in P . ■

From this lemma, we can always assume that there remain an infinite number of free variables not occurring in a proof.

LEMMA 5.2. *Let P be a proof of $K[\Gamma(a) \rightarrow \Theta(a)]$ in GL_m .*

- (1): *Let b be a free variable not occurring in P . The tree P' obtained from P by substituting b for every occurrence of a is a proof of $K[\Gamma(b) \rightarrow \Theta(b)]$.*
- (2): *Let t be a term. Then there is a proof P' of $K[\Gamma(t) \rightarrow \Theta(t)]$ which is obtained from P by substituting free variables not occurring in P for some finite number of free variables in P and by substituting t for every occurrence of a .*

In (1) and (2), P' is cut-free whenever P is cut-free.

PROOF. (1): By induction on the proof P .
 (2): This assertion can be proved in the same way as (1) under the *assumption* that every eigenvariable occurring in P is neither the free variable a nor is included in t . When P does not satisfy the assumption, we change the proof P into another one with the same endsequent so that it satisfies this assumption as follows. Let b be a free variable occurring as an eigenvariable in P , and suppose that b is a itself or is included in the term t . We substitute a new free variable not occurring in P for every occurrence of b above the inference whose eigenvariable is b . This new tree P^* is also a proof of $K[\Gamma(a) \rightarrow \Theta(a)]$ and does not include b as an eigenvariable. Since the number of such free variables is at most finite, repeating this process finite times, we obtain a proof P^{**} of $K[\Gamma(a) \rightarrow \Theta(a)]$ which satisfies the assumption. Then we obtain a proof P' by substituting the term t for all occurrences of the free variable a . This is a proof of $K[\Gamma(t) \rightarrow \Theta(t)]$ desired. ■

5.2 Reductions

In the following, we consider a proof P whose last inference is of the form (5.1).

1) Suppose that M belongs to at least one of $\Gamma, \Delta, \Theta, \Lambda$.

1.1) When $M \in \Gamma$, we change the last part of the proof into

$$\frac{K[M, \Delta \rightarrow \Lambda]}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} \text{ (th).}$$

The upper sequent of this inference is the right upper sequent of (5.1). Thus we simply eliminate the (cut).

1.2) When $M \in \Theta, \Delta$ or Λ , we can eliminate the (cut) in a similar way.

From 1.1) and 1.2), we can assume $M \notin \Gamma \cup \Theta \cup \Delta \cup \Lambda$. This assumption is made throughout the remaining part of Subsection 5.2. Then neither of the upper sequents of (5.1) is an initial sequent, i.e., each is the lower sequent of some inference. That is, the last part has the following form:

$$\frac{\frac{\overset{\circ}{K}[\Gamma \rightarrow \Theta, M]}{(I_1)} \quad \frac{\overset{\circ}{K}[M, \Delta \rightarrow \Lambda]}{(I_2)}}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (M) \text{ (cut).}$$

We consider every case according to inferences (I_1) and (I_2) for the upper sequents of the (cut).

5.2.1. Induction Step 1

Consider a proof of the form (5.1) with grade γ and $\rho_\ell = \rho_r = 1$. We show by induction that we can find a cut-free proof with the same endsequent. The induction hypothesis is:

(5.5) for any proof of the form (5.1) with the grade smaller than γ , we can find a cut-free proof with the same endsequent.

2) Suppose that at least one of (I_1) and (I_2) is (th).

2.1) When (I_1) is (th), the last part of the proof is expressed as

$$\frac{\frac{K[\Gamma' \rightarrow \Theta']}{K[\Gamma \rightarrow \Theta, M]} \text{ (th)}}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (M)(\text{cut}),$$

where $\Gamma' \subseteq \Gamma, \Theta' \subseteq \Theta$ and $M \notin \Theta$. Then we can eliminate the (cut) as follows:

$$\frac{K[\Gamma' \rightarrow \Theta']}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} \text{ (th)}.$$

2.2) When (I_2) is (th), we can eliminate the (cut) in the dual manner.

Therefore we assume in the remaining of Subsection 5.2.1 that neither (I_1) nor (I_2) is (th). Hence M is the principal formula of (I_1) as well as of (I_2) . Also, neither (I_1) nor (I_2) is $(B-\wedge)$ and also neither is $(B-\vee)$, since $\rho_\ell = \rho_r = 1$. Thus we have the following cases.

- (a): When M is of the form $\neg A$, (I_1) and (I_2) are $(\rightarrow \neg)$ and $(\neg \rightarrow)$.
- (b): When M is of the form $A \supset B$, (I_1) and (I_2) are $(\rightarrow \supset)$ and $(\supset \rightarrow)$.
- (c): When M is of the form $\forall x A(x)$, (I_1) and (I_2) are $(\rightarrow \forall)$ and $(\forall \rightarrow)$, respectively; and similarly when M is of the form $\exists x A(x)$, (I_1) and (I_2) are $(\rightarrow \exists)$ and $(\exists \rightarrow)$, respectively.
- (d): When M is of the form $\wedge \Phi$, (I_1) and (I_2) are $(\rightarrow \wedge)$ and $(\wedge \rightarrow)$. When M is of the form $\vee \Phi$, (I_1) and (I_2) are $(\rightarrow \vee)$ and $(\vee \rightarrow)$.
- (e): When M is of the form $K_i(A)$, there are two cases based on the innermost symbol of the outer K of the lower sequents of (I_1) and (I_2) . If the innermost symbol is not K_i , both (I_1) and (I_2) are $(K \rightarrow K)_C$, and otherwise, both are $(K \rightarrow K)_U$.

3) Suppose that (I_1) and (I_2) are operational inferences whose principal formulae are M . In these cases, the reductions are similar to the original ones in Gentzen [1]. Nevertheless, since the evaluations of grades are specific here, we give full reduction steps in the cases of \wedge and \forall .

3.1) When the outermost symbol of M is \wedge , the last part of the proof is

$$\frac{\frac{\{K[\Gamma \rightarrow \Theta, B] : B \in \Phi\}}{K[\Gamma \rightarrow \Theta, \wedge\Phi]} (\rightarrow \wedge) \quad \frac{K[A, \Delta \rightarrow \Lambda]}{K[\wedge\Phi, \Delta \rightarrow \Lambda]} (\wedge \rightarrow)}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (\wedge\Phi) (\text{cut}),$$

where $A \in \Phi$ and Φ is an allowable set. The grade of this (cut) is $\gamma = \text{gr}(\wedge\Phi) + \ell + 1$. This last part is reduced into

$$\frac{K[\Gamma \rightarrow \Theta, A] \quad K[A, \Delta \rightarrow \Lambda]}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (A) (\text{cut}).$$

If the outermost symbol of A is the innermost symbol K_{j_ℓ} of K , the grade of the new (cut) is $\text{gr}(A) + \ell$, and otherwise, it is $\text{gr}(A) + \ell + 1$. In either case, the grade of this (cut) is smaller than $\gamma = \text{gr}(\wedge\Phi) + \ell + 1$. Hence we can eliminate this (cut) by the induction hypothesis.

3.2) When the outermost logical connective of M is \forall , the last part is:

$$\frac{\frac{K[\Gamma \rightarrow \Theta, A(a)]}{K[\Gamma \rightarrow \Theta, \forall xA(x)]} (\rightarrow \forall) \quad \frac{K[A(t), \Delta \rightarrow \Lambda]}{K[\forall xA(x), \Delta \rightarrow \Lambda]} (\forall \rightarrow)}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (\forall xA(x)) (\text{cut}).$$

Let P' be the subproof of $K[\Gamma \rightarrow \Theta, A(a)]$ in P . Lemma 5.2.2 ensures that there is a proof P'' of $K[\Gamma \rightarrow \Theta, A(t)]$ which is obtained from P' by substituting new free variables for some free variables in P' and substituting t for a . Then we can reduce the last part into

$$\frac{K[\Gamma \rightarrow \Theta, A(t)] \quad K[A(t), \Delta \rightarrow \Lambda]}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (A(t)) (\text{cut}).$$

Since the grade of the new (cut) is $\text{gr}(A(t)) + \ell + 1$ or $\text{gr}(A(t)) + \ell$, it is smaller than the grade of the original (cut), $\text{gr}(\forall xA(x)) + \ell + 1$. Hence we can find a cut-free proof of $K[\Gamma, \Delta \rightarrow \Theta, \Lambda]$ by the induction hypothesis.

Note that Lemma 5.2.2 is used in the case of \exists , too.

4) Suppose that (I_1) and (I_2) are K -inferences. Recall that neither (I_1) nor (I_2) is $(B-\wedge)$ and also neither is $(B-\forall)$. We have to consider the following two cases: both (I_1) and (I_2) are $(K \rightarrow K)_C$ or both are $(K \rightarrow K)_U$.

4.1) When M is of the form $K_i(A)$ for some A and K_i is not the innermost symbol of K , the last part of the proof is:

$$\frac{\frac{KK_i[\Gamma, K_i(\Xi) \rightarrow A]}{K[K_i(\Gamma, \Xi) \rightarrow K_i(A)]} (K \rightarrow K)_C \quad \frac{KK_i[A, \Delta, K_i(\Pi) \rightarrow \Lambda]}{K[K_i(A), K_i(\Delta, \Pi) \rightarrow K_i(\Lambda)]} (K \rightarrow K)_C}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \rightarrow K_i(\Lambda)]} (\text{cut}).$$

The grade of this (cut) is $\gamma = \text{gr}(A) + \ell + 3$ by the definition of grades. The last part is reduced into

$$\frac{KK_i[\Gamma, K_i(\Xi) \rightarrow A] \quad KK_i[A, \Delta, K_i(\Pi) \rightarrow \Lambda]}{\frac{KK_i[\Gamma, \Delta, K_i(\Xi, \Pi) \rightarrow \Lambda]}{K[K_i(\Gamma, \Delta, \Xi, \Pi) \rightarrow K_i(\Lambda)]} (K \rightarrow K)_C} (\text{cut})$$

The grade of this new (cut) is $\text{gr}(A) + \ell + 2$ or $\text{gr}(A) + \ell + 1$, which is smaller than $\gamma = \text{gr}(A) + \ell + 3$. Thus we can eliminate this (cut) by the induction hypothesis.

4.2) Suppose that M is of the form $K_i(A)$ for some A and K_i is the innermost symbol of K . The last part of the proof is different from that in 4.1) only in that the outer KK_i should be simply K in the uppermost sequents in the last part and accordingly, two $(K \rightarrow K)_C$'s should be replaced by $(K \rightarrow K)_U$. Here the grade of the (cut) is $\gamma = \text{gr}(A) + \ell + 2$. This last part is reduced into

$$\frac{\frac{K[\Gamma, K_i(\Xi) \rightarrow A] \quad K[A, \Delta, K_i(\Pi) \rightarrow \Lambda]}{K[\Gamma, \Delta, K_i(\Xi, \Pi) \rightarrow \Lambda]} (K \rightarrow K)_U}{K[K_i(\Gamma, \Delta, \Xi, \Pi) \rightarrow K_i(\Lambda)]} (\text{cut})$$

The grade of this new (cut) is $\text{gr}(A) + \ell$ or $\text{gr}(A) + \ell + 1$ by (5.2), and is smaller γ . Hence we can eliminate this (cut) by the induction hypothesis.

5.2.2. Induction Step 2

Consider a proof of the form (5.1) where the grade is γ , the left rank ρ_ℓ is 1 and the right rank is $\rho_r > 1$. We prove by induction that there is a cut-free proof with the same endsequent. The induction hypothesis is:

$$(5.6) \quad \begin{array}{l} \text{for any proof of the form (5.1) with the grade } \gamma, \\ \text{the left rank equal to 1 and the right rank lower than } \rho_r, \\ \text{we can find a cut-free proof with the same endsequent.} \end{array}$$

5) When (I_2) is (th), we can change the proof into one with a lower right rank. Then we can eliminate the (cut) by the induction hypothesis.

When (I_1) is (th), the (cut) is eliminated in the same manner as in 2.1) since $\rho_\ell = 1$. Hence we can assume in the remaining of Subsection 5.2.2 that (I_1) is not (th).

In the following reduction steps except for 7.3), the outer K and the cut-formula M remain unchanged. Hence the grade of the (cut) remains unchanged.

6) Suppose that (I_2) is an operational inference.

6.1) When (I_2) is $(\supset \rightarrow)$, the last part of the proof is

$$\frac{K[\Gamma \rightarrow \Theta, M] \quad \frac{K[M, \Delta' \rightarrow \Lambda, A] \quad K[B, M, \Delta' \rightarrow \Lambda]}{K[A \supset B, M, \Delta' \rightarrow \Lambda]} (\supset \rightarrow)}{K[\Gamma, [A \supset B], \Delta' \rightarrow \Theta, \Lambda]} (\text{cut}),$$

where $[A \supset B]$ is $A \supset B$ and Δ is $\Delta' \cup \{A \supset B\}$ if M is not $A \supset B$, and $[A \supset B]$ is empty and Δ is Δ' if M is $A \supset B$.

6.1.1) When M is not $A \supset B$, the last part is reduced into

$$\frac{\frac{K[\Gamma \rightarrow \Theta, M] \quad K[M, \Delta' \rightarrow \Lambda, A]}{K[\Gamma, \Delta' \rightarrow \Theta, \Lambda, A]} (\text{cut}) \quad \frac{K[\Gamma \rightarrow \Theta, M] \quad K[M, B, \Delta' \rightarrow \Lambda]}{K[B, \Gamma, \Delta' \rightarrow \Theta, \Lambda]} (\text{cut})}{K[A \supset B, \Gamma, \Delta' \rightarrow \Theta, \Lambda]} (\supset \rightarrow).$$

Since these (cut)'s have lower right ranks than ρ_r , we can eliminate them by the induction hypothesis. Note that even when M is $A \supset B$, this reduction is legitimate.

6.1.2) When M is $A \supset B$, we have a cut-free proof of $K[A \supset B, \Gamma, \Delta \rightarrow \Theta, \Lambda]$ by 6.1.1) and continue

$$\frac{K[\Gamma \rightarrow \Theta, M] \quad K[A \supset B, \Gamma, \Delta \rightarrow \Theta, \Lambda]}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (\text{cut}).$$

Since the right and left ranks of this (cut) are 1, we can eliminate this (cut) by the induction hypothesis.

6.2) When (I_2) is $(\vee \rightarrow)$, the last part of the proof is

$$\frac{K[\Gamma \rightarrow \Theta, M] \quad \frac{\{K[M, A, \Delta' \rightarrow \Lambda] : A \in \Phi\}}{K[M, \vee \Phi, \Delta' \rightarrow \Lambda]} (\vee \rightarrow)}{K[\Gamma, \vee \Phi, \Delta' \rightarrow \Theta, \Lambda]} (\text{cut}),$$

where Φ is an allowable set, and Δ is $\Delta' \cup \{\vee \Phi\}$. The right rank of M at each upper sequent of the $(\vee \rightarrow)$ is lower than the right rank ρ_r of the (cut) by (5.4). We combine the subproof rooted at each upper sequent of the $(\vee \rightarrow)$

with the subproof rooted at the right upper sequent of the (cut) as follows: for each $A \in \Phi$,

$$\frac{K[\Gamma \rightarrow \Theta, M] \quad K[M, A, \Delta' \rightarrow \Lambda]}{K[\Gamma, A, \Delta' \rightarrow \Theta, \Lambda]} \text{ (cut)}.$$

Since the right rank of each of these (cut)'s is lower than the right rank ρ_r of the original (cut), we can find a cut-free proof of $K[\Gamma, A, \Delta' \rightarrow \Theta, \Lambda]$ for each $A \in \Phi$ by the induction hypothesis. Then we continue

$$\frac{\{K[\Gamma, A, \Delta' \rightarrow \Theta, \Lambda] : A \in \Phi\}}{K[\Gamma, \vee \Phi, \Delta' \rightarrow \Theta, \Lambda]} (\vee \rightarrow).$$

6.3) When (I_2) is one of the other operational inferences, we can reduce the proof into one with a lower right rank (see Gentzen [1]). In the cases of $(\rightarrow \vee)$ and $(\exists \rightarrow)$, we need Lemma 5.2.1.

7) Suppose that (I_2) is a K -inference.

7.1) When (I_2) is $(B-\wedge)$, the last part of the proof is

$$\frac{\frac{K[\Gamma \rightarrow \Theta, M] \quad \frac{\{K[M, \Delta \rightarrow \Lambda, K_i(A)] : A \in \Phi\} \quad K[K_i(\wedge \Phi), M, \Delta \rightarrow \Lambda]}{K[M, \Delta \rightarrow \Lambda]} (B-\wedge)}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} \text{ (cut)}}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]}$$

where Φ is an allowable set. This is reduced into

$$\frac{\left\{ \frac{K[\Gamma \rightarrow \Theta, M] \quad K[M, \Delta \rightarrow \Lambda, K_i(A)]}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda, K_i(A)]} \right\}_{A \in \Phi} \quad \frac{K[\Gamma \rightarrow \Theta, M] \quad K[M, K_i(\wedge \Phi), \Delta \rightarrow \Lambda]}{K[K_i(\wedge \Phi), \Gamma, \Delta \rightarrow \Theta, \Lambda]}}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} (B-\wedge),$$

where the above inferences are (cut)'s with the cut-formula M . Since these (cut)'s have lower right ranks than ρ_r by (5.4), we can eliminate the (cut)'s by the induction hypothesis.

7.2) When (I_2) is $(B-\vee)$, the reduction is parallel to that in 7.1), except the use of Lemma 5.2.1.

7.3) When (I_2) is $(K \rightarrow K)_C$, the last part is, by the assumption $\rho_\ell = 1$,

$$\frac{\frac{K K_i[\Gamma, K_i(\Xi) \rightarrow A]}{K[K_i(\Gamma, \Xi) \rightarrow K_i(A)]} (K \rightarrow K)_C \quad \frac{K K_i[\Delta, K_i(A), K_i(\Pi) \rightarrow \Lambda]}{K[K_i(A), K_i(\Delta, \Pi) \rightarrow K_i(\Lambda)]} (K \rightarrow K)_C}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \rightarrow K_i(\Lambda)]} \text{ (cut)}.$$

The grade of this (cut) is $\gamma = \text{gr}(A) + \ell + 3$. We reduce the last part into

$$\frac{\frac{K K_i[\Gamma, K_i(\Xi) \rightarrow A]}{K K_i[K_i(\Gamma, \Xi) \rightarrow K_i(A)]} (K \rightarrow K)_U \quad K K_i[K_i(A), \Delta, K_i(\Pi) \rightarrow \Lambda]}{\frac{K K_i[\Delta, K_i(\Gamma, \Xi, \Pi) \rightarrow \Lambda]}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \rightarrow K_i(\Lambda)]} (K \rightarrow K)_C} \text{ (cut)}$$

The grade of this new (cut) is $\text{gr}(A) + \ell + 3$, which is the same as γ . Since this new (cut) has a lower right rank than ρ_r , we can eliminate the (cut) by the induction hypothesis.

7.4) When (I_2) is $(K \rightarrow K)_U$, the last part of the proof is the same as that in 7.3), except that the outer KK_i is K in the uppermost sequents in the last part and both (I_1) and (I_2) are $(K \rightarrow K)_U$. In this case, the innermost symbol of K is K_i . The grade of this (cut) is $\gamma = \text{gr}(A) + \ell + 2$. The last part is reduced into

$$\frac{\frac{K[\Gamma, K_i(\Xi) \rightarrow A]}{K[K_i(\Gamma, \Xi) \rightarrow K_i(A)]} (K \rightarrow K)_U \quad K[K_i(A), \Delta, K_i(\Pi) \rightarrow \Lambda]}{\frac{K[\Delta, K_i(\Gamma, \Xi, \Pi) \rightarrow \Lambda]}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \rightarrow K_i(\Lambda)]} (K \rightarrow K)_U} \text{ (cut)}$$

The grade of this (cut) is the same as $\gamma = \text{gr}(A) + \ell + 2$, and this new (cut) has a lower right rank than ρ_r . By the induction hypothesis, we can eliminate the (cut).

5.2.3. Induction Step 3

Now we consider a proof of the form (5.1) where the grade is γ , the left rank is $\rho_\ell > 1$ and the right rank is ρ_r . In this case, the succedent of at least one upper sequent of the inference (I_1) has M . The induction hypothesis is:

(5.7) for any proof of the form (5.1) with the grade γ ,
the left rank lower than ρ_ℓ and the right rank equal to ρ_r ,
we can find a cut-free proof with the same endsequent.

8) When (I_1) is (th), it is easy to reduce the proof into one with a lower left rank. By the induction hypothesis, we can eliminate the (cut).

In the following reduction steps, the outer K and the cut-formula M remain unchanged. Hence the grade of the (cut) remains the same, too.

9) Suppose that (I_1) is an operational inference.

9.1) When (I_1) is $(\supset \rightarrow)$, the last part of the proof is

$$\frac{\frac{K[\Gamma' \rightarrow \Theta, M, A] \quad K[B, \Gamma' \rightarrow \Theta, M]}{K[A \supset B, \Gamma' \rightarrow \Theta, M]} (\supset \rightarrow) \quad K[M, \Delta \rightarrow \Lambda]}{K[A \supset B, \Gamma', \Delta \rightarrow \Theta, \Lambda]} \text{ (cut)},$$

where Γ is $\Gamma' \cup \{A \supset B\}$ and $A \supset B \notin \Gamma'$. We have to consider the following three cases: (a) neither A nor B is M ; (b) A is M ; and (c) B is M . In

either case, we can reduce, in the standard manner, the above last part into a derivation having (cut)'s with the grade γ , lower left ranks and the right rank ρ_r . For example, in case (c), the last part is reduced into

$$\frac{\frac{K[\Gamma' \rightarrow \Theta, M, A] \quad K[M, \Delta \rightarrow \Lambda]}{K[\Gamma', \Delta \rightarrow \Theta, \Lambda, A]} \text{ (cut)} \quad \frac{K[M, \Delta \rightarrow \Lambda]}{K[B, \Gamma', \Delta \rightarrow \Theta, \Lambda]} \text{ (th)}}{K[A \supset B, \Gamma', \Delta \rightarrow \Theta, \Lambda]} (\supset \rightarrow).$$

Hence we can find a cut-free proof of $K[A \supset B, \Gamma', \Delta \rightarrow \Theta, \Lambda]$ by the induction hypothesis.

9.2) We omit the other cases of operational inferences for (I_1) . Note that Lemma 5.2.1 is needed for the cases of $(\rightarrow \forall)$ and $(\exists \rightarrow)$.

10) Suppose that (I_1) is a K -Inference.

10.1) The inference (I_1) can be neither $(K \rightarrow K)_C$ nor $(K \rightarrow K)_U$, since $\rho_\ell > 1$.

10.2) When (I_1) is $(B-\wedge)$, the proof is

$$\frac{\frac{\{K[\Gamma \rightarrow \Theta, M, K_i(A)] : A \in \Phi\} \quad K[K_i(\wedge\Phi), \Gamma \rightarrow \Theta, M]}{K[\Gamma \rightarrow \Theta, M]} \text{ (B-}\wedge)}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} \quad K[M, \Delta \rightarrow \Lambda] \text{ (cut)},$$

where Φ is an allowable set. This is reduced into

$$\frac{\left\{ \frac{K[\Gamma \rightarrow \Theta, K_i(A), M] \quad K[M, \Delta \rightarrow \Lambda]}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda, K_i(A)]} \right\}_{A \in \Phi} \quad \frac{K[K_i(\wedge\Phi), \Gamma \rightarrow \Theta, M] \quad K[M, \Delta \rightarrow \Lambda]}{K[K_i(\wedge\Phi), \Gamma, \Delta \rightarrow \Theta, \Lambda]}}{K[\Gamma, \Delta \rightarrow \Theta, \Lambda]} \text{ (B-}\wedge),$$

where the above inferences are (cut)'s with the cut-formula M . Since each (cut) has a lower left rank than ρ_ℓ , we can eliminate these (cut)'s.

10.3) When (I_1) is $(B-\forall)$, the reduction is similar to 10.2), except the use of Lemma 5.2.1. ■

REMARK 5.3. In each reduction step, the order of $(B-\wedge)$ or $(B-\forall)$ and $(K \rightarrow K)_C$ or $(K \rightarrow K)_U$ remains unchanged. Using this fact, we can prove the following: Consider a proof P of $K[\Gamma \rightarrow \Theta]$ in GL_m . If no Barcan inferences $(B-\wedge)$ and $(B-\forall)$ occur in the trunk of P , then there is a cut-free proof of P' of $K[\Gamma \rightarrow \Theta]$ such that no $(B-\wedge)$ and $(B-\forall)$ occur in the trunk of P' .

References

- [1] GENTZEN, G., 1935, 'Untersuchungen über das logische Schliessen', *Mathematische Zeitschrift* **39**, 176-210, 405-431. English translation, 'Investigations into Logical Deduction', *The Collected Papers of Gerhard Gentzen*, 1969.

- [2] HARROP, R., 1956, 'On Disjunctions and Existential Statements in Intuitionistic Systems of Logic', *Math. Annalen* **132**, 247–361.
- [3] HARROP, R., 1960, 'Concerning Formulas of the Types $A \rightarrow B \vee C$, $A \rightarrow (Ex)B(x)$ in Intuitionistic Formal System', *J. Symbolic Logic* **25**, 247–361.
- [4] KANEKO, M. and T. NAGASHIMA, 1996, 'Game Logic and its Applications I', To appear in *Studia Logica* **57**.
- [5] OHNISHI, M., AND K. MATSUMOTO, 1957, 'Gentzen Method in Modal Calculi. I', *Osaka Math. J.* **9**, 113–130.
- [6] OHNISHI, M., AND K. MATSUMOTO, 1959, 'Gentzen Method in Modal Calculi. II', *Osaka Math. J.* **11**, 115–120.
- [7] VAN DALEN, D., 1986, 'INTUITIONISTIC LOGIC', *Handbook of Philosophical Logic* (III), eds. D. Gabbay and F. Guenther, Reidel, 225–339.

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