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A Map of Common Knowledge Logics*

Abstract. In order to capture the concept of common knowledge, various extensions of multi-modal epistemic logics, such as fixed-point ones and infinitary ones, have been proposed. Although we have now a good list of such proposed extensions, the relationships among them are still unclear. The purpose of this paper is to draw a map showing the relationships among them. In the propositional case, these extensions turn out to be all Kripke complete and can be comparable in a meaningful manner. F. Wolter showed that the predicate extension of the Halpern-Moses fixed-point type common knowledge logic is Kripke incomplete. However, if we go further to an infinitary extension, Kripke completeness would be recovered. Thus there is some gap in the predicate case. In drawing the map, we focus on what is happening around the gap in the predicate case. The map enables us to better understand the common knowledge logics as a whole.

Keywords: epistemic propositional and predicate logics, common knowledge extension, fixed-point approach, infinitary approach, Kripke-completeness, Kripke-incompleteness, embedding theorem.

1. Introduction

Multi-agent epistemic logics have been developed for investigations of interactions of agents such as game theoretical problems. In such situations, common knowledge is important in discussing knowledge (or beliefs) shared among agents. Various extensions of mutli-agent logics have been proposed in order to capture the concept of common knowledge. These extensions are divided into two types: the *fixed-point* approach, e.g., Halpern-Moses [5], and the *infinitary* approach, e.g., Kaneko-Nagashima [10, 11]. Some of them are given as propositional logics and some others as predicate logics. Also, some are considered from the model-theoretic viewpoint and some others from the proof-theoretic viewpoint. Now, we have a good list of extensions of epistemic logics, but their mutual relationships are yet unclear. The purpose of this paper is to draw a map showing the relationships among those extensions.

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	Finitary Base Logic	Finitary	Infinitary
Propositional	$KD4^n$	HM	GL_ω
Predicate	$QKD4^n$	QHM	QGL_ω

Diagram 1.

In the propositional case, these extensions turn out to be all Kripke complete, and are directly comparable in the sense that either two logics are deductively equivalent or if one is an extension of another, it is also a *conservative* extension. Here, conservativeness means that the extension adds no superfluous properties as far as the formulae in the original logic are concerned.

In contrast, Wolter [24] proved in the predicate case that the set of valid formulae in the Kripke semantics is not recursively enumerable in the presence of common knowledge. This implies the *Kripke-incompleteness theorem* that any finitary predicate extension of the Halpern-Moses type common knowledge logic cannot capture the Kripke semantics with common knowledge. In other words, the latter has no finitary proof theory. Nevertheless, it is also known from Tanaka-Ono [22] and Tanaka [19] that this difficulty does not occur in the infinitary approach. Thus, the predicate case has some great difference between the fixed-point and infinitary approaches. In drawing a map of common knowledge logics, we will focus especially on what this difference is.

Diagram 1 gives some extant extensions of multi-agent epistemic logics. Since we discuss a variety of common knowledge extensions, we make a particular choice of basic epistemic axioms on belief operators. We adopt KD4-type axioms, starting with the propositional multi-agent epistemic logic $KD4^n$, where n is the number of agents. The prefix Q means the predicate extension of a propositional logic together with the Barcan axiom. The logic HM is the fixed-point type extension of $KD4^n$ due to Halpern-Moses [5], where the common knowledge of a formula is determined by adding one axiom and one inference rule to $KD4^n$. The *game logic* GL_ω is an infinitary extension of $KD4^n$, and QGL_ω is its predicate extension, due to Kaneko-Nagashima [10] and [11], where the common knowledge of a formula is expressed as an infinitary conjunctive formula.[1]

The Kripke completeness of $KD4^n$ and that of $QKD4^n$ are known as variants of the completeness results given in the modal logic literature (cf.,

[1] We may find some other approach such as Segerberg [16]. Tanaka [20] discussed completeness of such logics in the predicate case from the viewpoints of noncompact logics.

Hughes-Cresswell [7]). It is also known from Halpern-Moses [5] (see also Lismont-Mongin [14], Fagin, *et al.* [1] and Meyer-van der Hoek [15]) that HM is Kripke complete. It follows from Tanaka-Ono [22] that GL_ω is also Kripke complete. These completeness results in the propositional case imply that if one logic is an extension of another, then it is a conservative extension.

On the other hand, it is known in the predicate case that there is a gap between QHM and QGL_ω. As already mentioned, the Kripke completeness of $QKD4^n$ is known, and that of QGL_ω could be expected from Tanaka-Ono [22] and Tanaka [21]. However, we have Wolter's Kripke-incompleteness theorem for QHM.

From the syntactical point of view, the distance between QHM and QGL_ω is large in that QGL_ω allows infinitary conjunctions and disjunctions as well as infinitary proofs. To consider the question of where the gap occurs, we provide two other logics, propositional CX and CY, and their predicate extensions QCX and QCY. These new logics are intermediate between HM, QHM and GL_ω, QGL_ω in that they have the set of finitary formulae in the same way as HM and QHM but admit infinitary proofs similar to GL_ω and QGL_ω.

In the propositional case, both CX and CY are shown to be deductively equivalent to HM. In the predicate case, QCY is Kripke complete, which is shown in Tanaka [19]. This result together with Wolter's incompleteness theorem implies that QCY is deductively different from QHM. The completeness of QCX remains open, though QCX looks more natural as a common knowledge logic than QCY.

The deductive equivalence of HM, CX and CY to GL_ω, as far as relevant formulae are concerned, are shown by proving that HM, CX and CY are faithfully embedded into GL_ω. In the predicate case, only QCY is faithfully embedded into to QGL_ω. The faithful embedding of QCX into QGL_ω remains also open.

In another attempt to avoid the incompleteness of QHM, Sturm-Wolter-Zakharyaschev [18] considered a fragment of QHM rather than an extension of it. They considered the *monodic* (not monadic) fragment of QHM with no function symbols and no equality, and proved its Kripke completeness. This fragment is located between HM and QHM. Thus, the gap we are considering occurs after the monodic fragment of QHM.

The Kripke completeness of the logic QCY is crucial for comparisons of logics in this paper. Tanaka [19] considered the behavior of QCY in details as well as its Kripke completeness in the case of no function symbols. The original motivation for predicate common knowledge logics is to consider theories with interactions between agents such as game theoretical problems (cf., Kaneko-Nagashima [10]). Hence, it is preferable to include function symbols as well as equality in such a first-order theory. A proof of the completeness theorem for QCY with function symbols can be obtained by modifying Tanaka's [19] proof of the completeness of QCY without function symbols.

This paper is organized as follows: In Section 2, we formulate the base logics KD4^n, QKD4^n and the Kripke semantics. We state the soundness-completeness theorems for these logics. In Section 3, we formulate HM and QHM and state the soundness-completeness theorem for HM. We also mention Wolter's [24] incompleteness theorem for QHM. In Section 4, we provide new logics CX, QCX, and CY, QCY. We show that CX and CY are deductively equivalent to HM, which relies upon the completeness result for HM. Then we state the soundness-completeness theorem for QCY shown by Tanaka [19]. In Section 5, we compare the provabilities of these logics for various types of formulae. In Section 6, we show the embedding theorems for CY, QCY into GL_ω, QGL_ω. In Section 7, we will draw a precise map of common knowledge logics from the results in the preceding sections, which will be given as Diagram 2.

2. Logics KD4^n and QKD4^n

In this section, we formulate the propositional KD4^n and predicate QKD4^n so that they are directly comparable. Common knowledge logics will be defined as extensions of these logics. In contrast with the diversity of syntactical systems, the Kripke semantics is uniquely defined and enables us to make comparisons of various syntactical systems. As stated in Section 1, we will make the choice of KD4-type logical axioms throughout this paper.

2.1. Language

We use the following list of symbols:

Free variables: $\mathbf{a}_0$, $\mathbf{a}_1$ ...; *Bound variables*: $\mathbf{x}_0$, $\mathbf{x}_1$, ...;
Logical connectives: $\neg$ (not), $\supset$ (implies), $\wedge$ (and), $\vee$ (or);
Quantifiers: $\forall$ (for all), $\exists$ (exists);
Function symbols: $\mathbf{f}_0$, $\mathbf{f}_1$, ...; *Predicate symbols*: $\mathbf{P}_0$, $\mathbf{P}_1$, ...;
Unary belief operator symbols: B_1, ..., B_n;
Unary common knowledge operator symbol: C;
Parentheses: (,).

The subscripts 1, ..., n of B_1, ..., B_n are the names of *agents*. We denote the set of agents $\{1, 2, \ldots, n\}$ by N. In the following, we consider the case of $n \geq 2$. We denote the set of all free variables by FV. We assume that there are countably infinite numbers of free variables and bound variables. We follow the tradition of using distinct letters for free and bound variables. This distinction eases some steps of our developments, especially, the infinitary approach, though some other steps look more complicated. Each $\mathbf{f}_k$ is assumed to be an *l-ary* function symbol for some $l \geq 0$, and each $\mathbf{P}_k$ is an l-ary predicate symbol for some $l \geq 0$. When $l = 0$, $\mathbf{f}_k$ is a *constant* symbol and $\mathbf{P}_k$ is a *propositional* variable. We denote the list of these function and predicate symbols by $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...]$. We assume that there is at least one 0-ary predicate symbol but there may be no function symbols.

We define *terms* inductively as follows: free variables are terms; and if $\mathbf{f}_k$ is an l-ary function symbol ($l \geq 0$) and if $t_1, \ldots, t_l$ are terms, then $\mathbf{f}_k(t_1, ..., t_l)$ is a term. A constant symbol is a term by the second step. We call $\mathbf{P}_k(t_1, ..., t_l)$ an *atomic formula* iff $\mathbf{P}_k$ is an l-ary predicate symbol and $t_1, \ldots, t_l$ are terms. Then *formulae* are defined in the standard finitary manner. We denote the set of all formulae by $\mathcal{P}$. We denote, by $\mathcal{P}_{-\mathrm{C}}$, the set of formulae in $\mathcal{P}$ that have no occurrences of C, and by $\mathcal{P}_{-\mathrm{B}}$, the set of formulae in $\mathcal{P}$ which have no occurrences of $\mathrm{B}_1, \ldots, \mathrm{B}_n$. The set of formulae containing neither C nor $\mathrm{B}_1, \ldots, \mathrm{B}_n$ is $\mathcal{P}_{-\mathrm{BC}} = \mathcal{P}_{-\mathrm{B}} \cap \mathcal{P}_{-\mathrm{C}}$. We say that a formula A (or a term t) is *closed* iff no free variable occurs in A (respectively, in t).

We define the *propositional fragment* $\mathcal{P}_{-\mathrm{Q}}$ of $\mathcal{P}$ to be the set of formulae generated from the 0-ary predicate symbols without quantifiers. We also denote $\mathcal{P}_{-\mathrm{C}} \cap \mathcal{P}_{-\mathrm{Q}}$ and $\mathcal{P}_{-\mathrm{BC}} \cap \mathcal{P}_{-\mathrm{Q}}$ by $\mathcal{P}_{-\mathrm{CQ}}$ and $\mathcal{P}_{-\mathrm{BCQ}}$. The set $\mathcal{P}_{-\mathrm{CQ}}$ is the propositional fragment containing no common knowledge operator C, and is used to define $\mathrm{KD4}^n$.

Since the concept of a subformula is slightly subtle in our treatment, it would be better to write down its explicit definition. For any $A \in \mathcal{P}$, we define the set $\mathrm{Sub}(A)$ of subformulae of A inductively as follows:

(0) $\mathrm{Sub}(A) = \{A\}$ for any atomic formula A;

(1) $\mathrm{Sub}(\neg A) = \mathrm{Sub}(A) \cup \{\neg A\}$;

(2) $\mathrm{Sub}(A * B) = \mathrm{Sub}(A) \cup \mathrm{Sub}(B) \cup \{A * B\}$, where $*$ is $\supset$, $\wedge$ or $\vee$;

(3) $\mathrm{Sub}(QxA(x)) = \bigcup_{t \text{ is a term}} \mathrm{Sub}(A(t)) \cup \{QxA(x)\}$, where Q is $\forall$ or $\exists$;

(4) $\mathrm{Sub}(\mathrm{B}_i(A)) = \mathrm{Sub}(A) \cup \{\mathrm{B}_i(A)\}$ for $i \in N$;

(5) $\mathrm{Sub}(\mathrm{C}(A)) = \mathrm{Sub}(A) \cup \{\mathrm{C}(A)\}$.

We call B a *subformula* of A iff $B \in \mathrm{Sub}(A)$. The set $\mathcal{P}_{-\mathrm{B}}$ is *subformula-closed*, i.e., if $A \in \mathcal{P}_{-\mathrm{B}}$ and B is a subformula of A, then $B \in \mathcal{P}_{-\mathrm{B}}$. The sets $\mathcal{P}_{-\mathrm{C}}$, $\mathcal{P}_{-\mathrm{BC}}$ and $\mathcal{P}_{-\mathrm{Q}}$ are also subformula-closed.

In the above definition of a subformula, the subtle step is (3) and the others are standard. This is made so as to be coherent to our language of having both free and bound variables as well as to our formal systems, specifically, the axiom schemata L6 and L7 in the next subsection.

2.2. $\mathbf{KD4}^n$ and $\mathbf{QKD4}^n$

We give the following seven axiom schemata and five inference rules: For any formulae A, B, C, and term t,

L1: $A \supset (B \supset A)$;

L2: $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

L3: $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;

L4: $A \wedge B \supset A$ and $A \wedge B \supset B$;

L5: $A \supset A \vee B$ and $B \supset A \vee B$;

L6: $\forall x A(x) \supset A(t)$;

L7: $A(t) \supset \exists x A(x)$,

and

$$\frac{A \supset B \quad A}{B} \text{ (MP)}$$

$$\frac{A \supset B \quad A \supset C}{A \supset B \wedge C} \ (\wedge\text{−Rule}) \qquad \frac{A \supset C \quad B \supset C}{A \vee B \supset C} \ (\vee\text{−Rule})$$

$$\frac{A \supset B(a)}{A \supset \forall x B(x)} \ (\forall\text{−Rule}) \qquad \frac{A(a) \supset B}{\exists x A(x) \supset B} \ (\exists\text{−Rule}),$$

where the free variable a does not occur in the lower formulae of $\forall$-Rule and $\exists$-Rule.

The above logical axioms and inference rules form the classical logic. We designate the set of L1–L5 and MP, $\wedge$-Rule and $\vee$-Rule by PCL, and the set of all axioms and inference rules by QCL. In fact, the choice of a set of formulae is still needed to determine a logic. In the following, the *classical propositional logic* is understood to be PCL within $\mathcal{P}_{-\mathrm{BCQ}}$, and the *classical predicate logic* is QCL within $\mathcal{P}_{-\mathrm{BC}}$.

The following are axiom schemata and inference rule for belief operators B_i for $i = 1, \ldots, n$:

K: $\mathrm{B}_i(A \supset C) \supset (\mathrm{B}_i(A) \supset \mathrm{B}_i(C))$;

D: $\neg \mathrm{B}_i(\neg A \wedge A)$;

4: $\mathrm{B}_i(A) \supset \mathrm{B}_i\,\mathrm{B}_i(A)$;

$\forall$-B: $\forall x\,\mathrm{B}_i(A(x)) \supset \mathrm{B}_i(\forall x A(x))$;

and

Necessitation: $\dfrac{A}{\mathrm{B}_i(A)}$.

In the literature of epistemic logics, 4 and $\forall$-B are called, respectively, the *Positive Introspection* axiom and *Barcan* axiom. Throughout this paper, we assume the Barcan axiom, $\forall$-B, in the predicate case. See Remark 2.6 for the case without the Barcan axiom. The necessitation rule is abbreviated as Nec.

We define the propositional $\mathrm{KD4}^n$ and predicate $\mathrm{QKD4}^n$ as follows:

$\mathrm{KD4}^n$: PCL + (K + D + 4 + Nec) within $\mathcal{P}_{-\mathrm{CQ}}$;

$\mathrm{QKD4}^n$: QCL + (K + D + 4 + $\forall$-B + Nec) within $\mathcal{P}_{-\mathrm{C}}$.

In $\mathrm{KD4}^n$, all formulae in the axioms and inference rules are restricted to $\mathcal{P}_{-\mathrm{CQ}}$. On the other hand, in $\mathrm{QKD4}^n$, those are restricted to $\mathcal{P}_{-\mathrm{C}}$. When we employ $\mathcal{P}_{-\mathrm{CQ}}$, the axioms L6, L7, $\forall$-Rule, $\exists$-Rule and $\forall$-B are automatically excluded, but when we do $\mathcal{P}_{-\mathrm{C}}$, these reappear.

A proof in these logics is defined in the standard manner. In $\mathrm{QKD4}^n$, for example, a *proof* of A in $\mathcal{P}_{-\mathrm{C}}$ is a finite tree satisfying the following properties: (1) a formula in $\mathcal{P}_{-\mathrm{C}}$ is associated with each node of the tree and A is associated with the root; (2) the formula associated with each leaf is an instance of the axiom schemata of $\mathrm{QKD4}^n$; and (3) adjoining nodes together with the associated formulae form an instance of the inference rules of $\mathrm{QKD4}^n$. We remark that a proof is a finite tree here, but that we will allow infinite proofs after Section 3.

We say that A is *provable* in $\mathrm{KD4}^n$ iff there is a proof of A in $\mathrm{KD4}^n$, which is denoted by $\vdash_{\mathrm{KD4}^n} A$. Similarly, we define the provability relation $\vdash_{\mathrm{QKD4}^n}$ for $\mathrm{QKD4}^n$. Since all the instances of the axiom schemata for $\mathrm{KD4}^n$ are allowed as axioms for $\mathrm{QKD4}^n$, it holds that for any A in $\mathcal{P}_{-\mathrm{CQ}}$,

$$\vdash_{\mathrm{KD4}^n} A \text{ implies } \vdash_{\mathrm{QKD4}^n} A. \tag{2.1}$$

In fact, the converse also holds, which will be mentioned as Corollary 2.4.

2.3. Kripke semantics with constant domains

Contrary to the diversity of syntactical systems to be discussed in this paper, it suffices to consider only one semantics. This common semantics together with the (soundness-) completeness result for each system enables us to make direct comparisons of those syntactical systems. The common semantics is the Kripke semantics with constant domains.

Recall that the list of function and predicate symbols is given as $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...]$. Let M be a nonempty set. A *classical interpretation* $[\tilde{f}_0, \tilde{f}_1, ...; \tilde{P}_0, \tilde{P}_1, ...]$ on M consists of *interpretations* $\tilde{f}_k$ and $\tilde{P}_k$ of $\mathbf{f}_k$ and $\mathbf{P}_k$, respectively, given by:

F1: the interpretation $\tilde{f}_k$ of each l-ary $\mathbf{f}_k$ is a function from M^l to M;

F2: the interpretation $\tilde{P}_k$ of each l-ary $\mathbf{P}_k$ is a function from M^l to $\{\top, \bot\}$ (when $l = 0$, $\tilde{P}_k$ is simply either $\top$ or $\bot$).

A *Kripke frame* $\mathcal{F} = (W; R_1, ..., R_n; M)$ is an $(n+2)$-tuple of a *set of possible worlds* W, *accessibility relations* $R_1, ..., R_n$ over W and a *set* M *of individuals.* We assume the following conditions:

K1: W is an arbitrary nonempty set;

K2: R_i is a subset of $W \times W$ for $i = 1, \ldots, n$;

K3: M is an arbitrary nonempty set.

An *interpretation* $\mathcal{I}$ is a function which assigns to each $w \in W$ a classical interpretation $\mathcal{I}(w) = [\tilde{f}_0, \tilde{f}_1, ...; \tilde{P}_0^w, \tilde{P}_1^w, ...]$ on M with the property that only the interpretations $\tilde{P}_k^w$ of predicate symbols $\mathbf{P}_k$ may depend upon w but the interpretation $\tilde{f}_k$ of each function symbol $\mathbf{f}_k$ is constant over W. A *Kripke model* $\mathcal{M}$ is a pair $(\mathcal{F}, \mathcal{I})$ of a Kripke frame $\mathcal{F}$ and an interpretation $\mathcal{I}$. Since we restrict our attentions to KD4-type logics, *we assume throughout this paper* that each accessibility relation R_i is serial and transitive.

We interpret free variables as independent of a possible world. Hence, we have the following simple definition of an assignment: A function $\sigma \colon FV \to M$ is called an *assignment.* One assignment σ is applied to all possible worlds in W.

Let a pair $(\mathcal{I}, \sigma)$ of an interpretation $\mathcal{I}$ and an assignment σ be given. The *valuation* $V(\cdot, (\mathcal{I}, \sigma))$ is the function from the set of terms to M defined inductively by

T1: $V(\mathbf{a}_k, (\mathcal{I}, \sigma)) = \sigma(\mathbf{a}_k)$ for all $\mathbf{a}_k \in FV$;

T2: $V(\mathbf{f}_k(t_1, ..., t_l), (\mathcal{I}, \sigma)) = \tilde{f}_k(V(t_1, (\mathcal{I}, \sigma)), ..., V(t_l, (\mathcal{I}, \sigma)))$.

For any free variable a, we write $\sigma \underset{a}{=} \sigma'$, $\sigma(b) = \sigma'(b)$ for all $b \in FV - \{a\}$. Also, we denote the set of finite sequences $(i_1, ..., i_m)$ in N by N^*.[2] Note that

[2] For a different purpose, it would be more convenient to adopt $N^{**} = \{(i_1, ..., i_m) \in N^* : i_t \neq i_{t+1}$ for $t = 1, ..., m-1\}$ than N^*.

the null sequence ϵ belongs to N^*. We say that $u \in W$ is *reachable from* w in a Kripke frame $\mathcal{F} = (W; R_1, ..., R_n; M)$ iff there is a finite sequence $(w_1, ..., w_m)$ $(m \geq 1)$ in W and $(i_1, ..., i_{m-1}) \in N^*$ such that $w = w_1$, $u = w_m$ and $(w_t, w_{t+1}) \in R_{i_t}$ for $t = 1, ..., m-1$. Note that w is reachable from w itself.

Let $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ be a Kripke model. Then we define the *valuation relation* $(\mathcal{M}, \sigma, w) \models$ inductively as follows:

E0: for any atomic formula $\mathbf{P}_k(t_1, ..., t_l)$,
$(\mathcal{M}, \sigma, w) \models \mathbf{P}_k(t_1, ..., t_l) \iff \tilde{P}_k^w(V(t_1, (\mathcal{I}, \sigma)), ..., V(t_l, (\mathcal{I}, \sigma))) = \top$;

E1: $(\mathcal{M}, \sigma, w) \models \neg A \iff (\mathcal{M}, \sigma, w) \not\models A$;

E2: $(\mathcal{M}, \sigma, w) \models A \supset B \iff (\mathcal{M}, \sigma, w) \not\models A$ or $(\mathcal{M}, \sigma, w) \models B$;

E3: $(\mathcal{M}, \sigma, w) \models A \wedge B \iff (\mathcal{M}, \sigma, w) \models A$ and $(\mathcal{M}, \sigma, w) \models B$;

E4: $(\mathcal{M}, \sigma, w) \models A \vee B \iff (\mathcal{M}, \sigma, w) \models A$ or $(\mathcal{M}, \sigma, w) \models B$;

E5: $(\mathcal{M}, \sigma, w) \models \forall x A(x) \iff (\mathcal{M}, \sigma', w) \models A(a)$ for all $\sigma' \underset{a}{=} \sigma$;

E6: $(\mathcal{M}, \sigma, w) \models \exists x A(x) \iff (\mathcal{M}, \sigma', w) \models A(a)$ for some $\sigma' \underset{a}{=} \sigma$;

E7: $(\mathcal{M}, \sigma, w) \models \mathrm{B}_i(A) \iff (\mathcal{M}, \sigma, v) \models A$ for any v with $(w, v) \in R_i$;

E8: $(\mathcal{M}, \sigma, w) \models \mathrm{C}(A) \iff (\mathcal{M}, \sigma, u) \models A$ for all u reachable from w,

where a is the first variable not occurring in $\forall x A(x)$ (or $\exists x A(x)$) in E5 (or E6, respectively). We write $\mathcal{M} \models A$ iff $(\mathcal{M}, \sigma, w) \models A$ for all assignments σ and worlds $w \in W$.

We note that the semantic valuation relation $(\mathcal{M}, \sigma, w) \models A$ for each A is completely determined by the valuations of subformulae of A. This note will be relevant for the consideration of the conservativeness of an extension of a logic, e.g., Corollary 2.4.

It is known that E8 has the following equivalent formulation:

LEMMA 2.1. *In the definition of* $(\mathcal{M}, \sigma, w) \models$, E8 *can be replaced by*

E8*: $(\mathcal{M}, \sigma, w) \models \mathrm{C}(A) \iff (\mathcal{M}, \sigma, u) \models \mathrm{B}_e(A)$ *for all* $e \in N^*$,

where $\mathrm{B}_e(A)$ *is an abbreviation of* $\mathrm{B}_{i_1} \mathrm{B}_{i_2} ... \mathrm{B}_{i_m}(A)$ *for* $e = (i_1, ..., i_m) \in N^*$.

Condition E8* describes the intuitive understanding of the common knowledge of A that A is true, each player believes A, each believes that each believes A, and so on. The other related concept, the *common belief* of A, is defined by excluding the first sentence "A is true". That is, A is commonly believed but may not be true. Semantically, it is defined by assuming $m \geq 2$ for any sequent $(w_1, ..., w_m)$ in the definition of reachability. Nevertheless, the common belief of A is defined as the formula $\mathrm{B}_1\,\mathrm{C}(A) \wedge \cdots \wedge \mathrm{B}_n\,\mathrm{C}(A)$ using the common knowledge operator C. See Remark 6.6 for further explanations. We focus on common knowledge rather than common belief throughout the paper.

The Barcan axiom is true in any Kripke model with a constant domain, which is stated as (1) of the following lemma. The converse, $\mathrm{B}_i(\forall x A(x)) \supset \forall x\, \mathrm{B}_i(A(x))$, holds without the assumption of constant domains. The second statement of the lemma is conceptually related to the Barcan axiom in that if we regard C as an infinitary conjunction, the infinitary conjunction is commutative with belief operator B_i. See also Section 6. This is independent of the assumption of a constant domain.

LEMMA 2.2. *Let* $\mathcal{M} = (\mathcal{F}, \mathcal{I}) = ((W; R_1, ..., R_n; M), \mathcal{I})$ *be a Kripke model,* σ *an assignment and* w *a world in* W. *Then*

(1) $(\mathcal{M}, \sigma, w) \vDash \forall x\, \mathrm{B}_i(A(x)) \supset \mathrm{B}_i(\forall x A(x))$;

(2) $(\mathcal{M}, \sigma, w) \vDash \mathrm{C}(A) \supset \mathrm{B}_i\, \mathrm{C}(A)$ *for all* $i \in N$.

The formulae in the first statement are already adopted as an axiom schema for $\mathrm{QKD4}^n$, and those in the second will be adopted for our common knowledge extensions. Although we have $(\mathcal{M}, \sigma, w) \vDash \forall x\, \mathrm{C}(A(x)) \supset \mathrm{C}(\forall x A(x))$, this is a derived property in our common knowledge extensions (except in QS4(C) of Section 3).

The following completeness result has been known in the modal logic literature (cf., Hughes-Cresswell [7]).

THEOREM 2.3. (1) (soundness and completeness for $\mathrm{KD4}^n$). *Let* A *be a formula in* $\mathcal{P}_{-\mathrm{CQ}}$. *Then* $\vdash_{\mathrm{KD4}^n} A$ *if and only if* $\mathcal{M} \vDash A$ *for all Kripke models* $\mathcal{M} = (\mathcal{F}, \mathcal{I})$.

(2) (soundness and completeness for $\mathrm{QKD4}^n$). *Let* A *be a formula in* $\mathcal{P}_{-\mathrm{C}}$. *Then* $\vdash_{\mathrm{QKD4}^n} A$ *if and only if* $\mathcal{M} \vDash A$ *for all Kripke models* $\mathcal{M}$.

The converse of (2.1) is a simple consequence from Theorem 2.3, which we write down explicitly, since the same type of comparisons will be made throughout the paper.

COROLLARY 2.4 (conservativeness of $\mathrm{QKD4}^n$ upon $\mathrm{KD4}^n$). *Let* A *be a formula in* $\mathcal{P}_{-\mathrm{CQ}}$. *Then* $\vdash_{\mathrm{KD4}^n} A$ *if and only if* $\vdash_{\mathrm{QKD4}^n} A$.

Before going to the next section, we state Wolter's [24] result, which is one of the main concerns of this paper. We write $\vDash A$ iff $\mathcal{M} \vDash A$ for all Kripke models $\mathcal{M}$.

THEOREM 2.5 (non-recursive-enumerability). *Suppose that the language* $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...]$ *contains at least nine unary predicate symbols. Then the set* $\{A \in \mathcal{P} : \models A\}$ *is not recursively enumerable.*

This theorem does not depend upon the particular choice of the assumptions of transitivity and seriality on frames. It would remain to hold even if we strengthen the assumptions for frames to, for example, the S5-assumption that each R_i is an equivalence relation. For details, see Wolter [24].

Throughout the remaining part of this paper to avoid repetitive qualifications, we *assume* that the language $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...]$ has at least nine unary predicate symbols. Wolter proved in [23] the above non-recursive-enumerability theorem under this assumption, though Wolter [24] gave a different proof under the assumption of an infinite countable number of unary predicate symbols.

The set of provable formulae in $\mathrm{QKD4}^n$ is recursively enumerable, since $\mathrm{QKD4}^n$ is recursively axiomatizable. Therefore, it follows from Theorem 2.3 that the set $\{A \in \mathcal{P}_{-\mathrm{C}} : \models A\}$ is recursively enumerable (and so is $\{A \in \mathcal{P}_{-\mathrm{CQ}} : \models A\}$). Therefore, Theorem 2.5 is a phenomenon caused by introducing common knowledge into $\mathrm{QKD4}^n$. In subsequent sections, we will discuss more exactly when such a phenomenon occurs.

Remark 2.6 (Barcan axiom and constant domains). The completeness of $\mathrm{QKD4}^n$ (Theorem 2.3.(2)) remains to hold when we drop the Barcan axiom $\forall$-B, and correspondingly when the assumption of constant domains is weakened to that if $(w, u) \in R_i$, then $M^w \subseteq M^u$, where M^w and M^u are the individual domains in worlds w and u. On the other hand, the present completeness proof for the system QCY, which is the key in this paper, relies upon the Barcan axiom $\forall$-B. Our present purpose is to draw a map of common knowledge logics for better understanding of the entire situation, but not to draw a technically detailed map of them. Therefore, we assume the Barcan axiom $\forall$-B throughout the paper.[3]

3. Common knowledge logics HM and QHM

Halpern-Moses [5] extended various multi-agent propositional epistemic logics to fixed-point logics in order to incorporate common knowledge. A variant is the fixed-point extension of $\mathrm{KD4}^n$. In this paper, since we focus on the KD4-type logics, we give the name HM to the fixed-point extension of $\mathrm{KD4}^n$,

[3] For other problems arising from the Barcan axiom, see, for example, Fitting-Mendelsohn [3].

and QHM to the predicate extension of HM. In fact, it follows from Wolter's non-recursive enumerability theorem that QHM is Kripke incomplete. We are going to consider how the incompleteness result may be understood.

Consider the following axiom schema and inference rule:

CA: $\mathrm{C}(A) \supset A \wedge \mathrm{B}_1\, \mathrm{C}(A) \wedge \cdots \wedge \mathrm{B}_n\, \mathrm{C}(A)$;

CI: $\dfrac{D \supset A \wedge \mathrm{B}_1(D) \wedge \cdots \wedge \mathrm{B}_n(D)}{D \supset \mathrm{C}(A)}$.

We define the logics HM and QHM, respectively, as follows:

HM: $\mathrm{KD4}^n + (\mathrm{CA} + \mathrm{CI})$ within $\mathcal{P}_{-\mathrm{Q}}$;

QHM: $\mathrm{QKD4}^n + (\mathrm{CA} + \mathrm{CI})$ within $\mathcal{P}$.

The axiom schema CA is often called the *fixed-point property* (with respect to A). This consists of the formulae of Lemma 2.2.(2) as well as $\mathrm{C}(A) \supset A$. Once this axiom schema is added to $\mathrm{QKD4}^n$ (or $\mathrm{KD4}^n$), it follows that $\mathrm{C}(A)$ implies $\mathrm{B}_e(A) = \mathrm{B}_{i_1} \ldots \mathrm{B}_{i_m}(A)$ for all $e = (i_1, \ldots, i_m) \in N^*$, which is explicitly written as Lemma 3.1. That is, the formulae derived show the intended meaning of the "common knowledge of A". The inference rule CI means that if a formula D has the fixed-point property with respect to A, then D implies $\mathrm{C}(A)$. In other words, $\mathrm{C}(A)$ is the deductively weakest formula having the fixed-point property. The inference rule CI is called the *fixed-point rule.* The logics HM and QHM are defined by the addition to the same axiom schema and inference rules to $\mathrm{KD4}^n$ within $\mathcal{P}_{-\mathrm{Q}}$ and to $\mathrm{QKD4}^n$ within $\mathcal{P}$.

Our main concern is to consider the predicate extension QHM, but not HM. Therefore, we will mention properties mainly on QHM, but talk about HM when necessary.

LEMMA 3.1. $\vdash_{\mathrm{QHM}} \mathrm{C}(A) \supset \mathrm{B}_e(A)$ *for all* $e \in N^*$.

The completeness theorem was proved for HM by Halpern-Moses [5].

THEOREM 3.2 (soundness and completeness for HM). *For any* $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{HM}} A$ *if and only if* $\models A$.

Let us return to the predicate QHM. It is straightforward to see that QHM is sound with respect to the Kripke semantics. However, it is incomplete, which is an implication of Wolter's non-recursive enumerability theorem (Theorem 2.5). Indeed, since CA and CI as well as the other axioms and inference rules for $\mathrm{QKD4}^n$ are finitary, the set $\{A \in \mathcal{P} : \vdash_{\mathrm{QHM}} A\}$ is recursively enumerable. However, the set $\{A \in \mathcal{P} : \models A\}$ is not by Theorem 2.5. We summarize these results as the following theorem.

THEOREM 3.3. (1) (soundness). *For any* $A \in \mathcal{P}$, *if* $\vdash_{\mathrm{QHM}} A$, *then* $\models A$.

(2) (incompleteness). *There exists a formula* $A \in \mathcal{P}$ *such that* $\models A$ *but* $\nvdash_{\mathrm{QHM}} A$.

We may see the difference between the above theorems from two points of view.

First, let us see the above two theorems from the finitary point of view. As stated in Section 2, both $\mathrm{KD4}^n$ and $\mathrm{QKD4}^n$ are Kripke complete and have recursively enumerable sets of provable formulae. In the propositional case, HM is complete and $\{A \in \mathcal{P}_{-\mathrm{Q}} : \models A\}$ is also recursively enumerable, while in the predicate case, QHM is incomplete and $\{A \in \mathcal{P} : \models A\}$ is not recursively enumerable. Although HM and QHM are obtained from $\mathrm{KD4}^n$ and $\mathrm{QKD4}^n$, respectively, by adding both finitary axiom schema CA and inference rule CI, only the logic QHM turns out to be incomplete. In this sense, this incompleteness seems to be an unexpected jump.

Second, let us see the above two theorems from the infinitary point of view. In the infinitary approach, various completeness results are known, which will be discussed in Sections 4 and 6. Indeed, Kripke completeness is recovered in the sense that the strengthened provability in an infinitary approach captures $\{A \in \mathcal{P} : \models A\}$. In the propositional case, Kaneko [8] proved that HM can be regarded as a fragment of infinitary propositional epistemic logics. Hence we can regard HM as well as QHM as having already an infinitary aspect in part. The non-recursive-enumerability of $\{A \in \mathcal{P} : \models A\}$ is better understood from this point of view. In this sense, from the infinitary point of view, we may regard the completeness of HM as an unexpected result, rather than the incompleteness of QHM.

In order to consider those views, we relate QHM and HM to other logical systems. Here, we mention two lemmas on QHM and HM. The first motivates us to introduce the two other common knowledge logics given in Section 4. The second will be used to consider the C-fragment of QHM, which is closely related to the predicate extension of the unimodal S4.

The first lemma is stated as a derived inference rule for the semantic validity $\models$. We omit the proof of the lemma.

LEMMA 3.4. *If* $\models D \supset \mathrm{B}_e(A)$ *for all* $e \in N^*$, *then* $\models D \supset \mathrm{C}(A)$.

By the soundness and completeness for HM (Theorem 3.3), we restate this lemma as follows: for $A, D \in \mathcal{P}_{-\mathrm{Q}}$,

(0_{HM}): if $\vdash_{\mathrm{HM}} D \supset \mathrm{B}_e(A)$ for all $e \in N^*$, then $\vdash_{\mathrm{HM}} D \supset \mathrm{C}(A)$.

We look at this claim together with Lemma 3.1. We regard the "common knowledge of A" as a formula having the content $\{\mathrm{B}_e(A) : e \in N^*\}$. From this point of view, the instances in Lemma 3.1 and the statement 0_{HM} are the required properties for the common knowledge A. Therefore, it looks natural to define a common knowledge logic by adding these to KD4^n and QKD4^n. However, since the infinite conjunction is implicit in the common knowledge operator C, we have a problem with the formulae: $\mathrm{C}(A) \supset \mathrm{B}_i\,\mathrm{C}(A)$ for $i = 1, \ldots, n$. By analogy with the Barcan axiom $\forall$-B, we call $\mathrm{C}(A) \supset \mathrm{B}_i\,\mathrm{C}(A)$ the *C-Barcan axiom* for each $i \in N$. In the propositional case, we would obtain a logic equivalent to HM by adding, to KD4^n, the instances in Lemma 3.1, the inference rule 0_{HM} and C-Barcan axiom. In the predicate case, we need stronger inference rules than 0_{HM}. These are the subjects of the next section.

The following lemma will be useful in considering the question of what kind of formulae make a discrepancy between the provability $\vdash_{\mathrm{QHM}}$ and validity $\models$. It enables us to make comparisons of the C-fragment of QHM with the predicate extension QS4(C) of the unimodal S4 with its modal operator C.

LEMMA 3.5. (1) $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B) \supset (\mathrm{C}(A) \supset \mathrm{C}(B))$;

(2) $\vdash_{\mathrm{QHM}} \mathrm{C}(A) \supset A$;

(3) $\vdash_{\mathrm{QHM}} \mathrm{C}(A) \supset \mathrm{C}\,\mathrm{C}(A)$;

(4) *if* $\vdash_{\mathrm{QHM}} A$, *then* $\vdash_{\mathrm{QHM}} \mathrm{C}(A)$;

(5) $\vdash_{\mathrm{QHM}} \forall x\,\mathrm{C}(A(x)) \supset \mathrm{C}(\forall x A(x))$.

PROOF. We prove (1), (4) and (5).

(1) It suffices to prove $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B) \wedge \mathrm{C}(A) \supset \mathrm{C}(B)$. Since $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B) \wedge \mathrm{C}(A) \supset B$ and $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B) \wedge \mathrm{C}(A) \supset \mathrm{B}_i(\mathrm{C}(A \supset B) \wedge \mathrm{C}(A))$ for all $i \in N$ by CA, we have, by CI, $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B) \wedge \mathrm{C}(A) \supset \mathrm{C}(B)$.

(4) Suppose $\vdash_{\mathrm{QHM}} A$. Then $\vdash_{\mathrm{QHM}} \mathrm{B}_i(A)$ for all $i \in N$ by Nec for B_i. Hence $\vdash_{\mathrm{QHM}} A \wedge \mathrm{B}_1(A) \wedge \cdots \wedge \mathrm{B}_n(A)$, which implies $\vdash_{\mathrm{QHM}} A \supset A \wedge \mathrm{B}_1(A) \wedge \cdots \wedge \mathrm{B}_n(A)$. By CI, $\vdash_{\mathrm{QHM}} A \supset \mathrm{C}(A)$. Hence $\vdash_{\mathrm{QHM}} \mathrm{C}(A)$.

(5) Since $\vdash_{\mathrm{QHM}} \mathrm{C}(A(a)) \supset A(a) \wedge \mathrm{B}_1\,\mathrm{C}(A(a)) \wedge \cdots \wedge \mathrm{B}_n\,\mathrm{C}(A(a))$ by CA, we have $\vdash_{\mathrm{QHM}} \forall x\,\mathrm{C}(A(x)) \supset \forall x A(x) \wedge \forall x\,\mathrm{B}_1\,\mathrm{C}(A(x)) \wedge \cdots \wedge \forall x\,\mathrm{B}_n\,\mathrm{C}(A(x))$. By $\forall$-B_i, we have $\vdash_{\mathrm{QHM}} \forall x\,\mathrm{C}(A(x)) \supset \forall x A(x) \wedge \mathrm{B}_1(\forall x\,\mathrm{C}(A(x))) \wedge \cdots \wedge \mathrm{B}_n(\forall x\,\mathrm{C}(A(x)))$. Regarding this as the upper formula of CI, we have $\vdash_{\mathrm{QHM}} \forall x\,\mathrm{C}(A(x)) \supset \mathrm{C}(\forall x A(x))$. ■

Lemma 3.5.(5) is the Barcan formula for C with respect to $\forall$.

Let QS4(C) be the predicate extension of the unimodal S4 with the Barcan axiom with respect to $\forall$, i.e., it is the logic within the set of formulae $\mathcal{P}_{-\mathrm{B}}$ defined by the axioms and inference rules of the classical logic and the formulae (1)–(3), (5), the inference rule (4) of Lemma 3.5 for the modal operator C. The provability relation of QS4(C) is denoted by $\vdash_{\mathrm{QS4(C)}}$. Lemma 3.5 implies that for any $A \in \mathcal{P}_{-\mathrm{B}}$, if $\vdash_{\mathrm{QS4(C)}} A$, then $\vdash_{\mathrm{QHM}} A$. In fact, the converse also holds, which will be discussed in Section 5.

4. Common knowledge logics CX, QCX and CY, QCY

As stated before, no finitary extensions of QKD4^n capture the semantic validity $\models$. In this section, we present extensions CX, CY and QCX, QCY of KD4^n and QKD4^n, respectively. These logics keep the sets $\mathcal{P}_{-\mathrm{Q}}$ and $\mathcal{P}$ of finitary formulae, but admit *infinitary* proofs. In the propositional case, both CX and CY turn out to be equivalent to HM. In the predicate case, QCY is Kripke complete, and thus differs from QHM, but QCX is not known to be equivalent to QHM, QCY or neither.

4.1. Logics CX and QCX

To define CX and QCX, we adopt the formulae in Lemma 3.1 as an axiom schema and the inference rule corresponding to Lemma 3.4:

CA*: $\mathrm{C}(A) \supset \mathrm{B}_e(A)$, where $e \in N^*$;

CI$_0^*$: $$\frac{\{D \supset \mathrm{B}_e(A) : e \in N^*\}}{D \supset \mathrm{C}(A)}.$$

The axiom CA* means that $\mathrm{C}(A)$ contains the "common knowledge of A" in the sense of $\{\mathrm{B}_e(A) : e \in N^*\}$. The inference rule CI$_0^*$ states that if D has the "common knowledge of A" in the same sense, then D contains $\mathrm{C}(A)$. In other words, $\mathrm{C}(A)$ is the deductively weakest formula having the "common knowledge of A". In this sense, CI$_0^*$ is regarded as the dual of the axiom CA*.[4]

Although the above two additions may look sufficient to determine $\mathrm{C}(A)$ to be the "common knowledge of A", it is stated in Lemma 2.2.(2) that the Kripke semantics has the C-Barcan property, i.e., $\models \mathrm{C}(A) \supset \mathrm{B}_i\,\mathrm{C}(A)$ for all $i \in N$. Correspondingly, we assume these C-Barcan formulae as an axiom schema:

C-B: $\mathrm{C}(A) \supset \mathrm{B}_i\,\mathrm{C}(A)$, for all $i \in N$

[4] Some reader may wonder if the assumption formula D in CI$_0^*$ is needed here. Probably, $\mathrm{C}(A \supset B) \wedge \mathrm{C}(A) \supset \mathrm{C}(B)$ could not be provable for some A and B in the system lacking D in CI$_0^*$. In this sense, D in CI$_0^*$ seems indispensable.

To have the balance between the syntactical system and Kripke semantics, the axiom C-B is indispensable. See Remark 4.3. Note that the axiom CA given in Section 3 has C-B as part.

We define CX and QCX as follows:

CX: $\mathrm{KD4}^n + (\mathrm{CA}^* + \mathrm{CI}_0^* + \text{C-B})$ within $\mathcal{P}_{-\mathrm{Q}}$;

QCX: $\mathrm{QKD4}^n + (\mathrm{CA}^* + \mathrm{CI}_0^* + \text{C-B})$ within $\mathcal{P}$.

The inference rule CI_0^* requires countably infinite numbers of upper formulae. Accordingly, the definition of a finitary proof should be modified into a countable tree where every path from the root is finite and a countably infinite branching occurs with CI_0^*. Infinitary proofs are allowed in CX and QCX.

The logic CX is given as a sequent calculus in Kaneko [8].

Before stating the soundness-completeness result for CX, we mention the relationship between CX, QCX and HM, QHM.

LEMMA 4.1. (1) (CA)*:* $\vdash_{\mathrm{QCX}} \mathrm{C}(A) \supset A \wedge \mathrm{B}_1\,\mathrm{C}(A) \wedge \cdots \wedge \mathrm{B}_n\,\mathrm{C}(A)$.

(2) (CI)*: if* $\vdash_{\mathrm{QCX}} D \supset A \wedge \mathrm{B}_1(D) \wedge \cdots \wedge \mathrm{B}_n(D)$*, then* $\vdash_{\mathrm{QCX}} D \supset \mathrm{C}(A)$.

These claims hold with the replacement of QCX *with* CX.

PROOF. (1) follows from CA* and C-B.

(2) Suppose $\vdash_{\mathrm{QCX}} D \supset A \wedge \mathrm{B}_1(D) \wedge \cdots \wedge \mathrm{B}_n(D)$. Then we can prove $\vdash_{\mathrm{QCX}} D \supset \mathrm{B}_e(A)$ for all $e \in N^*$. Therefore, by CI_0^*, $\vdash_{\mathrm{QCX}} D \supset \mathrm{C}(A)$. ∎

We have the following theorem.

THEOREM 4.2. (1) (equivalence of HM and CX). *For any* $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{HM}} A$ *if and only if* $\vdash_{\mathrm{CX}} A$.

(2) *For any* $A \in \mathcal{P}$, *if* $\vdash_{\mathrm{QHM}} A$, *then* $\vdash_{\mathrm{QCX}} A$.

PROOF. Lemma 4.1 implies (2) as well as the *only-if* part of (1). The *if* part of (1) follows from CA and 0_{HM} stated after Lemma 3.4. ∎

The completeness of CX is a by-product of Theorem 4.2.(1) and Theorem 3.2. Nevertheless, since we do not have the completeness for QHM as stated in Theorem 3.3, we cannot, at present, guarantee a parallel result in the predicate case.

Remark 4.3. The two logics $\mathrm{KD4}^n + (\mathrm{CA}^* + \mathrm{CI}_0^*)$ within $\mathcal{P}_{-\mathrm{Q}}$ and $\mathrm{QKD4}^n + (\mathrm{CA}^* + \mathrm{CI}_0^*)$ within $\mathcal{P}$ look natural. However, the following fact is known for the propositional case: The axiom C-B is not provable in $\mathrm{KD4}^n + (\mathrm{CA}^* + \mathrm{CI}_0^*)$.

This implies that $\mathrm{KD4}^n + (\mathrm{CA}^* + \mathrm{CI}_0^*)$ is Kripke incomplete which obtained as follows: The Gentzen-style sequent formulation of $\mathrm{KD4}^n + (\mathrm{CA}^* + \mathrm{CI}_0^*)$ enjoys cut-elimination, which implies the full subformula property. Using this subformula property, we can prove that C-B is not provable in $\mathrm{KD4}^n + (\mathrm{CA}^* + \mathrm{CI}_0^*)$. This method is not directly extended to the predicate case because of the Barcan axiom $\forall$-B for B_i.

4.2. Logics CY and QCY

The logic CX is complete as stated above, but we do not know whether or not its predicate extension QCX is complete. Nevertheless, we would obtain completeness if the inference rule CI_0^* is strengthened into the following form:

$$\mathrm{CI}^*: \quad \frac{\{D \supset T(\mathrm{B}_e(A)) : e \in N^*\}}{D \supset T(\mathrm{C}(A))},$$

where $T(E)$ is any (single) formula of the following form:

$$\mathrm{B}_{j_k}(D_k \supset \cdots \supset \mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset E))\dots). \tag{4.1}$$

Here $D_1, \dots, D_k$ are any formulae in $\mathcal{P}$ and that $(j_k, ..., j_1)$ is any sequence in N^*. Formula $T(\mathrm{B}_e(A))$ is obtained from $T(E)$ by substituting $\mathrm{B}_e(A)$ for E. When $k = 0$, CI^* becomes CI_0^*.

The inference rule CI^* states that if D implies $T(\mathrm{B}_e(A))$ for any $e \in N^*$, then D implies $T(\mathrm{C}(A))$. It looses the direct duality to CA^*. Although this looks artificial, the resulting logical systems become Kripke complete in both propositional and predicate cases. In the propositional case, the resulting system is deductively equivalent to CX, *a fortiori*, HM.

We define CY and QCY as follows:

CY: $\mathrm{KD4}^n + (\mathrm{CA}^* + \mathrm{CI}^*)$ within $\mathcal{P}_{-\mathrm{Q}}$.

QCY: $\mathrm{QKD4}^n + (\mathrm{CA}^* + \mathrm{CI}^*)$ within $\mathcal{P}$.

Proofs in these logics are defined in the similar manner as in CX and QCX.

It can be verified that CY and QCY are extensions of CX and QCX, as follows. First, observe that CI_0^* is a special case of CI^*, as already stated. Second, C-B is also provable in QCY. Indeed, since $\vdash_{\mathrm{QCY}} \mathrm{C}(A) \supset \mathrm{B}_i(\neg p \vee p \supset \mathrm{B}_e(A))$ for all $e \in N^*$ by CA^*, we have, using CI^*, $\vdash_{\mathrm{QCY}} \mathrm{C}(A) \supset \mathrm{B}_i(\neg p \vee p \supset \mathrm{C}(A))$, i.e., $\vdash_{\mathrm{QCY}} \mathrm{C}(A) \supset \mathrm{B}_i\,\mathrm{C}(A)$. We write this second fact explicitly.

LEMMA 4.4. (1) *For any* $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{CY}} \mathrm{C}(A) \supset \mathrm{B}_i\,\mathrm{C}(A)$ *for all* $i \in N$.

(2) *For any* $A \in \mathcal{P}$, $\vdash_{\mathrm{QCY}} \mathrm{C}(A) \supset \mathrm{B}_i\,\mathrm{C}(A)$ *for all* $i \in N$.

This lemma together with the fact that CI_0^* is included in CI^* implies the following.

LEMMA 4.5. (1) *For any* $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{CX}} A$ *implies* $\vdash_{\mathrm{CY}} A$.

(2) *For any* $A \in \mathcal{P}$, $\vdash_{\mathrm{QCX}} A$ *implies* $\vdash_{\mathrm{QCY}} A$.

We have the following soundness-completeness result for CY and QCY.

THEOREM 4.6 (soundness and completeness for CY and QCY).
(1) *For any formula* $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{CY}} A$ *if and only if* $\models A$.

(2) *For any formula* $A \in \mathcal{P}$, $\vdash_{\mathrm{QCY}} A$ *if and only if* $\models A$.

Since Lemma 4.5.(1) states that CY is an extension of CX, and since CX is Kripke complete, we would obtain Theorem 4.6.(1) if CI^* is sound in any Kripke frame. This verification is straightforward. Hence, the completeness part of Theorem 4.6.(2) is crucial here. Tanaka [19] discusses this completeness. His proof is given under the language with no function symbols. Although function symbols are unavoidable for future applications, a proof of (2) for the language with function symbols can be obtained by modifying Tanaka's [19] proof.[5] See also Section 7.2.

Since $\{A \in \mathcal{P} : \models A\}$ is not recursively enumerable by Theorem 2.5, Theorem 4.6.(2) implies that the set $\{A \in \mathcal{P} : \vdash_{\mathrm{QCY}} A\}$ is also not recursively enumerable. This non-recursive-enumerability may be regarded as caused by infinitary proofs. On the other hand, since $\{A \in \mathcal{P}_{-\mathrm{Q}} : \vdash_{\mathrm{CY}} A\}$ coincides with

$$\{A \in \mathcal{P}_{-\mathrm{Q}} : \models A\} = \{A \in \mathcal{P}_{-\mathrm{Q}} : \vdash_{\mathrm{HM}} A\} = \{A \in \mathcal{P}_{-\mathrm{Q}} : \vdash_{\mathrm{CX}} A\},$$

the set $\{A \in \mathcal{P}_{-\mathrm{Q}} : \vdash_{\mathrm{CY}} A\}$ remains recursively enumerable, even though we allow infinitary proofs in CY. Therefore, an infinitary proof is not solely a cause for the non-recursive-enumerability of $\{A \in \mathcal{P} : \vdash_{\mathrm{QCY}} A\}$.

In sum, completeness holds for $\mathrm{QKD4}^n$, becomes unavailable for QHM, and recovers again for QCY. In contrast to the predicate case, there are no such gaps in the propositional case. Now, we have an entire map of common knowledge logics except the infinitary ones. In the next section, we look at more detailed relationship between these logics.

We mention one corollary from Theorem 4.6.

COROLLARY 4.7 (conservativeness of QCY upon CY).
For any $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{CY}} A$ *if and only if* $\vdash_{\mathrm{QCY}} A$.

[5] A proof in the case with functions symbols is given in the discussion paper version of this paper. A copy would be provided on request.

5. Comparisons of common knowledge logics

In this section, we make comparisons of the logics QKD4^n, QHM, QCX, QCY and QS4(C) as well as their propositional fragments. It will be known from these comparisons except with QCX that if one is an extension of another, the extension is conservative upon the other. In addition to this, we will get a good but still partial answer to the question of what the difference between QHM and QCY is. It remains open whether QCX coincides with QHM or QCY (or neither). We obtain also the result from these comparisons that any formula that is valid but is not provable in QHM contains a belief operator B_i, the common knowledge operator C as well as a quantifier.

First, we compare the provabilities of our logics for propositional formulae. We remark that the semantical validity $\models A$ can be added to the list of the provability statements in the following theorems.

THEOREM 5.1 (propositional formulae). *For any formula $A \in \mathcal{P}_{-\mathrm{Q}}$, the following six statements are all equivalent:* (1) $\vdash_{\mathrm{HM}} A$; (1_Q) $\vdash_{\mathrm{QHM}} A$; (2) $\vdash_{\mathrm{CX}} A$; (2_Q) $\vdash_{\mathrm{QCX}} A$; *and* (3) $\vdash_{\mathrm{CY}} A$; (3_Q) $\vdash_{\mathrm{QCY}} A$.

If A is C-free, then we can add (0) $\vdash_{\mathrm{KD4}^n} A$ *and* (0_Q) $\vdash_{\mathrm{QKD4}^n} A$ *to the above list.*

PROOF. By the definition of each logic, we have $(1) \Rightarrow (1_\mathrm{Q})$, $(2_\mathrm{Q}) \Rightarrow (3_\mathrm{Q})$ and $(2) \Rightarrow (3)$. Theorem 4.2.(2) states $(1_\mathrm{Q}) \Rightarrow (2_\mathrm{Q})$. Corollary 4.7 states $(3) \Leftrightarrow (3_\mathrm{Q})$. Theorem 4.2.(1) states $(1) \Leftrightarrow (2)$. These are described as follows:

$$\begin{array}{ccccc}
(1) \vdash_{\mathrm{HM}} A & \Leftrightarrow & (2) \vdash_{\mathrm{CX}} A & \Rightarrow & (3) \vdash_{\mathrm{CY}} A \\
\Downarrow & & \Downarrow & & \Updownarrow \\
(1_\mathrm{Q}) \vdash_{\mathrm{QHM}} A & \Rightarrow & (2_\mathrm{Q}) \vdash_{\mathrm{QCX}} A & \Rightarrow & (3_\mathrm{Q}) \vdash_{\mathrm{QCY}} A
\end{array}$$

Finally, we get $(1) \Leftrightarrow (3)$ from Theorems 3.2 and 4.6.(1). Thus, we have the equivalences of all the six claims. ■

It is an implication of Theorem 5.1 that QHM, QCX, QCY and their propositional fragments are all conservative extensions of KD4^n.

Next, we compare the provabilities of these logics for C-free formulae.

THEOREM 5.2 (C–free formulae). *For any $A \in \mathcal{P}_{-\mathrm{C}}$, the following four are equivalent:* (1) $\vdash_{\mathrm{QKD4}^n} A$; (2) $\vdash_{\mathrm{QHM}} A$; (3) $\vdash_{\mathrm{QCX}} A$; *and* (4) $\vdash_{\mathrm{QCY}} A$.

PROOF. By definitions, we have $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4)$. By Theorem 4.2.(2), $(2) \Rightarrow (3)$ holds. Conversely, suppose (4). By Theorem 4.6.(2), we have $\models A$. Since A does not contain C, the completeness for QKD4^n (Theorem 2.3.(2)) implies $\vdash_{\mathrm{QKD4}^n} A$. Thus, (1), (2), (3) and (4) are all equivalent. ■

Although QHM is incomplete, Theorems 5.1 and 5.2 imply that QHM is a conservative extension of HM and KD4^n. Of course, QCY is a conservative extension of CY and also of QKD4^n.

Next, consider B-free formulae. That is, we consider the provabilities of formulae containing no $\mathrm{B}_1, \ldots, \mathrm{B}_n$ but, maybe, C in our predicate extensions of KD4^n. In this consideration, we focus on the predicate extension QS4(C) of the unimodal S4, since comparisons with it give good hints to understand common knowledge extensions of KD4^n.

In fact, the provabilities of QHM, QCX and QCY for B-free formulae collapse into that of the predicate extension QS4(C) of unimodal S4, whose modality is C.

THEOREM 5.3 (B-free formulae). *For any formula $A \in \mathcal{P}_{-\mathrm{B}}$, the following four statements are all equivalent:* (1) $\vdash_{\mathrm{QHM}} A$; (2) $\vdash_{\mathrm{QCX}} A$; (3) $\vdash_{\mathrm{QCY}} A$; *and* (4) $\vdash_{\mathrm{QS4(C)}} A$.

A proof of this theorem will be given below.

In the propositional case, we have the parallel result to Theorem 5.3: for any formula $A \in \mathcal{P}_{-\mathrm{BQ}}$, $(1_{-\mathrm{Q}})$ $\vdash_{\mathrm{HM}} A$; $(2_{-\mathrm{Q}})$ $\vdash_{\mathrm{CX}} A$; $(3_{-\mathrm{Q}})$ $\vdash_{\mathrm{CY}} A$; and $(4_{-\mathrm{Q}})$ $\vdash_{\mathrm{S4(C)}} A$ as well as (1)–(4) of Theorem 5.3 are all equivalent.

To prove Theorem 5.3, we introduce the translator $\psi\colon \mathcal{P} \to \mathcal{P}_{-\mathrm{B}}$ to associate with each A the formula $\psi(A)$ obtained from A by replacing all occurrences $\mathrm{B}_1, \ldots, \mathrm{B}_n$ in A by C. For example, $\psi(\mathrm{C}(A) \supset \mathrm{B}_{i_1} \ldots \mathrm{B}_{i_m}(A)) = \mathrm{C}(\psi A) \supset \mathrm{C} \ldots \mathrm{C}(\psi A)$. Note $\psi A = A$ for any $A \in \mathcal{P}_{-\mathrm{B}}$.

LEMMA 5.4. *For any $A \in \mathcal{P}$, if $\vdash_{\mathrm{QCY}} A$, then $\vdash_{\mathrm{QS4(C)}} \psi(A)$.*

PROOF. It suffices to prove that $\vdash_{\mathrm{QS4(C)}} \psi(D)$ for all axioms D for QCY, and that the inference rules translated by ψ from those for QCY are admissible in QS4(C). If D is an instance of L1–L7, then $\vdash_{\mathrm{QS4}(C)} \psi(D)$. Let D be an instance of the axiom K, i.e., $\mathrm{B}_i(A \supset B) \supset (\mathrm{B}_i(A) \supset \mathrm{B}_i(B))$. Then $\psi(D) = \mathrm{C}(\psi A \supset \psi B) \supset (\mathrm{C}(\psi A) \supset \mathrm{C}(\psi B)))$, which is an instance of an axiom in QS4(C). In the same manner, if D is an instance of the axiom D or 4, we have $\vdash_{\mathrm{QS4(C)}} \psi(D)$. Consider the Barcan axiom $\forall x\, \mathrm{B}_i(A(x)) \supset \mathrm{B}_i(\forall x A(x))$. The translation is $\psi(\forall x\, \mathrm{B}_i(A(x)) \supset \mathrm{B}_i(\forall x A(x))) = \forall x\, \mathrm{C}(\psi A(x)) \supset \mathrm{C}(\forall x \psi A(x))$, which is an instance of the Barcan axiom for the operator C. Consider an instance of CA*: $\mathrm{C}(A) \supset \mathrm{B}_e(A)$, where $e = (i_1, ..., i_m)$. Then $\psi(\mathrm{C}(A) \supset \mathrm{B}_e(A)) = \mathrm{C}(\psi A) \supset \mathrm{C} \ldots \mathrm{C}(\psi A)$. This is provable in QS4(C).

Regarding the inference rules, we consider only Nec and CI*.

Nec: Let $\vdash_{\mathrm{QS4(C)}} \psi(A)$. Then $\vdash_{\mathrm{QS4(C)}} \mathrm{C}(\psi A)$, which is equivalent to $\vdash_{\mathrm{QS4(C)}} \psi(\mathrm{B}_i(A))$.

CI*. Suppose $\vdash_{QS4(C)} \psi(D \supset T(\mathrm{B}_e(A)))$ for all $e \in N^*$. Recall that T is any formula of the form (4.1). Consider the specific one $\vdash_{QS4(C)} \psi(D \supset T(\mathrm{B}_1(A)))$. Since $\psi(D \supset T(\mathrm{B}_1(A)))$ is $\psi(D) \supset \psi T(\mathrm{B}_1(A))$, and since $\psi T(\mathrm{B}_1(A)) = \psi(\mathrm{B}_{j_m}(D_m \supset \ldots \mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{B}_1(A)))...)) = \mathrm{C}(\psi D_m \supset \ldots \mathrm{C}(\psi D_2 \supset \mathrm{C}(\psi D_1 \supset \mathrm{C}(\psi A)))...) = \psi\, \mathrm{B}_{j_m}(D_m \supset \ldots \mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{C}(A)))...)) = \psi T(\mathrm{C}(A))$, we have $\vdash_{QS4(C)} \psi(D \supset T(\mathrm{C}(A)))$. ■

PROOF OF THEOREM 5.3. By Theorem 4.2.(2), we have (1) $\Rightarrow$ (2). By definition, (2) $\Rightarrow$ (3). By Lemma 5.4, we have (3) $\Rightarrow$ (4). Suppose (4). Then there is a proof of A in QS4(C). It suffices to show that the logical axioms and inference rules in QS4(C) are admissible in QHM. These are stated in Lemma 3.3. ■

As stated in the beginning of this section, we can add the semantic validity $\vDash A$ to the list in the above three theorems. Since these theorems imply that all the sets $\{A \in \mathcal{P}_{-\mathrm{Q}} : \vDash A\}$, $\{A \in \mathcal{P}_{-\mathrm{B}} : \vDash A\}$ and $\{A \in \mathcal{P}_{-\mathrm{C}} : \vDash A\}$ are recursively enumerable, their union $\{A \in \mathcal{P}_{-\mathrm{Q}} \cup \mathcal{P}_{-\mathrm{B}} \cup \mathcal{P}_{-\mathrm{C}}\colon \vDash A\}$ is also recursively enumerable.

Let $\mathcal{P}_{\mathrm{QF}}$ be the set of all *quantifier-free* formulae in $\mathcal{P}$. We define the quantifier-free fragments $\mathrm{KD4}^n_{\mathrm{QF}}$, $\mathrm{HM}_{\mathrm{QF}}$, $\mathrm{CX}_{\mathrm{QF}}$ and $\mathrm{CY}_{\mathrm{QF}}$ in the same manners as $\mathrm{KD4}^n$, HM, CX and CY by adopting $\mathcal{P}_{\mathrm{QF}}$ rather than $\mathcal{P}_{-\mathrm{Q}}$. Then all the theorems for the propositional fragments remain true for $\mathrm{KD4}^n_{\mathrm{QF}}$, $\mathrm{HM}_{\mathrm{QF}}$, $\mathrm{CX}_{\mathrm{QF}}$ and $\mathrm{CY}_{\mathrm{QF}}$. The last conclusion of the above paragraph becomes that $\{A \in \mathcal{P}_{\mathrm{QF}} \cup \mathcal{P}_{-\mathrm{B}} \cup \mathcal{P}_{-\mathrm{C}} : \vDash A\}$ is recursively enumerable. Note that $\vdash_{\mathrm{QHM}} A$ holds for any formula A in $\{A \in \mathcal{P}_{\mathrm{QF}} \cup \mathcal{P}_{-\mathrm{B}} \cup \mathcal{P}_{-\mathrm{C}} : \vDash A\}$. Hence, we have the following theorem.

THEOREM 5.5 (difference between QCY and QHM). (1) *For any $A \in \mathcal{P}$, if $\vdash_{\mathrm{QCY}} A$,* a fortiori, *$\vDash A$, but $\nvdash_{\mathrm{QHM}} A$, then A contains a belief operator B_i for at least one i, the common knowledge operator C and a quantifier.*[6]

(2) *The set $\{A \in \mathcal{P} : \vdash_{\mathrm{QCY}} A$ and $\nvdash_{\mathrm{QHM}} A\}$ is not recursively enumerable.*

It is an important open problem to find a particular formula for (1). After all, such a formula contains B_i for some i, C as well as $\forall$ (or $\exists$). Conversely, for other formulae, the provability $\vdash_{\mathrm{QCY}}$ coincides with $\vdash_{\mathrm{QHM}}$. Although we have met quite complicated formulae containing C and quantifiers in the game theoretical applications in Kaneko-Nagashima [10] and [11], they are not ones for (1).

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[6] Tanaka [19] showed that the occurrence of C in A must be positive, by applying the method of tree-sequent calculus.

We have been focussed on extensions of QHM rather than on its fragments. On the other hand, Sturm-Wolter-Zakharyaschev [18] considered the monodic fragment of QHM under the *assumption* that the language has no m-ary function symbols for $m \geq 1$. We say that a formula A is *monodic* iff each of any subformulae $\mathrm{B}_j(D)$ $(j \in N)$ and $\mathrm{C}(D)$ of A contains at most one free variable. Without the assumption of no m-ary function symbols with $m \geq 1$, the formula obtained from a monodic formula by substitution of a term may not be monodic. They proved the Kripke completeness of the monodic fragment of QHM under the assumption of no m-ary function symbols with $m \geq 1$. This fragment is located between HM and QHM. The gap between QHM and QCY occurs after the monodic fragment of QHM.

6. Game logics GL_ω and QGL_ω: Infinitary approach

Since the common knowledge $\mathrm{C}(A)$ of A is naturally understood as the conjunction of the infinite set $\{\mathrm{B}_e(A) : e \in N^*\}$, it would be a direct approach than the fixed-point one to consider an infinitary extension of $\mathrm{KD4}^n$. Kaneko-Nagashima [10] and [11] took this approach and provided the infinitary epistemic logics GL_ω and QGL_ω, where common knowledge is explicitly formulated as an infinitary conjunctive formula. They developed these systems from the proof-theoretic point of view, which are now expected to be Kripke-complete from the results of Tanaka-Ono [22] and Tanaka [21]. In the propositional case, Kaneko [8] showed that HM is *faithfully* embedded into GL_ω (with a slight restriction).[7] In this section, we will give a connection from the logics CY and QCY given in Section 4 to GL_ω and QGL_ω.

First, we add the new conjunction and disjunction symbols $\bigwedge$ and $\bigvee$ to the list of primitive symbols in Section 2.1. These are applied to infinite sets of formulae.

Let $\mathcal{Q}$ be a given set of formulae. We define $\mathcal{E}(\mathcal{Q})$ as follows:

IF1: $\mathcal{Q}' = \mathcal{Q} \cup \{(\bigwedge \Phi), (\bigvee \Phi) : \Phi$ is a countably infinite subset of $\mathcal{Q}$ containing at most a finite number of free variables$\}$;

IF2: $\mathcal{E}(\mathcal{Q})$ is the set of formulae defined from $\mathcal{Q}'$ by the standard induction, that is, (1): any expression in $\mathcal{Q}'$ belongs to $\mathcal{E}(\mathcal{Q})$; (2): if $A, B \in \mathcal{E}(\mathcal{Q})$, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$ and $\mathrm{B}_1(A), \ldots, \mathrm{B}_n(A)$ belong to $\mathcal{E}(\mathcal{Q})$; and (3): if $A(a) \in \mathcal{E}(\mathcal{Q})$, then $(\forall x A(x))$ and $(\exists x A(x))$ belong to $\mathcal{E}(\mathcal{Q})$.

[7] Heifetz [6] discussed the infinitary approach and fixed-point approach in the propositional case, by unifying these approaches into one system and proving its Kripke completeness.

By replacing $\mathcal{Q}$ by $\mathcal{E}(\mathcal{Q})$, we define $\mathcal{E}^2(\mathcal{Q}) = \mathcal{E}(\mathcal{E}(\mathcal{Q}))$, and $\mathcal{E}^{m+1}(\mathcal{Q}) = \mathcal{E}(\mathcal{E}^m(\mathcal{Q}))$ for any nonnegative integer m. We adopt the set of formulae $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}}) := \bigcup_{m<\omega} \mathcal{E}^m(\mathcal{P}_{-\mathrm{C}})$, taking $\mathcal{P}_{-\mathrm{C}}$ as $\mathcal{Q}$. In the following, we call Φ an *allowable set* iff Φ is a countably infinite set of formulae in $\mathcal{E}^m(\mathcal{P}_{-\mathrm{C}})$ for some $m < \omega$ and contains a finite number of free variables. We denote the propositional fragment of $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$ by $\mathcal{E}^\omega_{-\mathrm{Q}}(\mathcal{P}_{-\mathrm{CQ}})$.

The large conjunction and disjunction symbols $\bigwedge$, $\bigvee$ are applied only to allowable sets of formulae. The set of formulae $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$ are closed with respect to $\neg$, $\wedge$, $\vee$, $\supset$ and $\bigwedge$, $\bigvee$. Since $\{\mathrm{B}_e(A) : e \in N^*\}$ is an allowable set for any $A \in \mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$, this set $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$ is sufficiently large to discuss the problem of common knowledge. The restriction that each allowable set has only a finite number of free variables is not restrictive for our purpose of incorporating common knowledge into a first-order theory.

In the above construction of the sets of formulae, we do not include the common knowledge operator symbol C, since common knowledge is now expressed as an infinitary conjunctive formula, that is,

$$\bigwedge\{\mathrm{B}_e(A) : e \in N^*\}, \tag{6.2}$$

which we denote by $\mathrm{C}_0(A)$.

We add the modifications of the axioms L4, L5 and inference rules $\wedge$-Rule, $\vee$-Rule: For any allowable set Φ of formulae,

L4$_\omega$: $\bigwedge \Phi \supset A$, where $A \in \Phi$;

L5$_\omega$: $A \supset \bigvee \Phi$, where $A \in \Phi$

and

$$\frac{\{A \supset B : B \in \Phi\}}{A \supset \bigwedge \Phi}\ (\bigwedge\text{-Rule}) \qquad \frac{\{A \supset B : A \in \Phi\}}{\bigvee \Phi \supset B}\ (\bigvee\text{-Rule}).$$

We denote the union of the axioms for QKD4^n and L4$_\omega$, L5$_\omega$, $\bigwedge$-Rule, $\bigvee$-Rule by QKD4$^n_\omega$. The propositional fragment of QKD4$^n_\omega$ is denoted by KD4$^n_\omega$.

Since an allowable set Φ is infinite, we need also the Barcan property on each belief operator B_i with respect to $\bigwedge$: For any allowable set Φ,

$\bigwedge$-B: $\bigwedge \mathrm{B}_i(\Phi) \supset \mathrm{B}_i(\bigwedge \Phi)$,

where $\mathrm{B}_i(\Phi) := \{\mathrm{B}_i(A) : A \in \Phi\}$. In $\bigwedge \mathrm{B}_i(\Phi)$, player i believes every formula in Φ, and in $\mathrm{B}_i(\bigwedge \Phi)$, he believes additionally the entirety of Φ. Hence, the latter is stronger than the former. Indeed, this is provable in QKD4$^n_\omega$, but the converse, $\bigwedge$-B, is not necessarily.[8] To make direct comparisons with QCY, we need the axiom $\bigwedge$-B.

We define GL_ω and QGL_ω by

$$\mathrm{GL}_\omega: \quad \mathrm{KD4}^n_\omega + \textstyle\bigwedge\text{-B within } \mathcal{E}^\omega_{-\mathrm{Q}}(\mathcal{P}_{-\mathrm{CQ}});$$

$$\mathrm{QGL}_\omega: \quad \mathrm{QKD4}^n_\omega + \textstyle\bigwedge\text{-B within } \mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}}).$$

Recall that $\mathcal{E}^\omega_{-\mathrm{Q}}(\mathcal{P}_{-\mathrm{CQ}})$ is the propositional fragment of $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$.

A *proof* in QGL_ω is defined to be a countable tree in the same manner as in QCX. We write $\vdash_{\mathrm{Q}\omega} A$ iff there is a proof of A in QGL_ω. The provability relation of GL_ω is denoted by $\vdash_\omega$.

First, we show that the operator $\mathrm{C}_0(\cdot)$ in QGL_ω has the same properties as C in QCY. The same assertions hold in GL_ω.

LEMMA 6.1. (1) $\vdash_{\mathrm{Q}\omega} \mathrm{C}_0(A) \supset \mathrm{B}_e(A)$ *for all* $e \in N^*$.

(2) *If* $\vdash_{\mathrm{Q}\omega} D \supset T(\mathrm{B}_e(A))$ *for all* $e \in N^*$, *then* $\vdash_{\mathrm{Q}\omega} D \supset T(\mathrm{C}_0(A))$, *where* $T(E)$ *is any formula, in* $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$, *of the form (4.1).*

PROOF. (1) follows from the definition of $\mathrm{C}_0(A)$.

Consider (2). We prove by induction on the structure of T that $\vdash_{\mathrm{Q}\omega} \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\} \supset T(\mathrm{C}_0(A))$. When T is the null symbol, the assertion holds.

Let $T(\mathrm{B}_e(A))$ be written as $\mathrm{B}_{j_m}(D_m \supset \ldots \mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{B}_e(A)))...)$. Suppose $\vdash_{\mathrm{Q}\omega} \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\} \supset T(\mathrm{C}_0(A))$. Then $\vdash_{\mathrm{Q}\omega} (D_{j_{m+1}} \supset \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\}) \supset (D_{j_{m+1}} \supset T(\mathrm{C}_0(A)))$. Hence $\vdash_{\mathrm{Q}\omega} \mathrm{B}_{j_{m+1}}(D_{m+1} \supset \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\}) \supset \mathrm{B}_{j_{m+1}}(D_{j_{m+1}} \supset T(\mathrm{C}_0(A)))$. On the other hand, since $\vdash_{\mathrm{Q}\omega} \bigwedge\{\mathrm{B}_{j_{m+1}}(D_{m+1} \supset T(\mathrm{B}_e(A))) : e \in N^*\} \supset \mathrm{B}_{j_{m+1}}(D_{m+1} \supset \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\})$, we have $\vdash_{\mathrm{Q}\omega} \bigwedge\{\mathrm{B}_{j_{m+1}}(D_{m+1} \supset T(\mathrm{B}_e(A))) : e \in N^*\} \supset \mathrm{B}_{j_{m+1}}(D_{m+1} \supset T(\mathrm{C}_0(A)))$. ∎

The semantic valuation $(\mathcal{M}, \sigma, w) \vDash$ of Section 2.3 can be applied to any formula in $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$ just by modifying E3 and E4 into the following manner: for any allowable sets Φ,

$$\mathrm{E3}_\omega: \quad (\mathcal{M}, \sigma, w) \vDash \textstyle\bigwedge \Phi \iff (\mathcal{M}, \sigma, w) \vDash A \text{ for all } A \in \Phi;$$

$$\mathrm{E4}_\omega: \quad (\mathcal{M}, \sigma, w) \vDash \textstyle\bigvee \Phi \iff (\mathcal{M}, \sigma, w) \vDash A \text{ for some } A \in \Phi.$$

[8] Exactly speaking, we have a proof of the unprovability of the converse in the propositional case $\mathrm{KD4}^n_\omega$ and in the predicate case $\mathrm{QKD4}^n_\omega$ without the axiom $\forall$-B. See also Remark 4.3.

Then we have the following soundness-completeness result, which is obtained by modifying the proof of the completeness result for QCY. See Section 7 for a remark on its proof.

THEOREM 6.2 (soundness and completeness for GL_ω and QGL_ω).
(1) *For any* $A \in \mathcal{E}^{\omega}_{-\mathrm{Q}}(\mathcal{P}_{-\mathrm{CQ}})$, $\vdash_\omega A$ *if and only if* $\mathcal{M} \vDash A$ *for all models* $\mathcal{M}$.

(2) *For any* $A \in \mathcal{E}^{\omega}(\mathcal{P}_{-\mathrm{C}})$, $\vdash_{\mathrm{Q}\omega} A$ *if and only if* $\mathcal{M} \vDash A$ *for all models* $\mathcal{M}$.

Now, we compare the infinitary approach with the finitary one. In the comparisons, the following formulae in $\mathcal{E}^{\omega}(\mathcal{P}_{-\mathrm{C}})$ are essential. We call a formula A in $\mathcal{E}^{\omega}(\mathcal{P}_{-\mathrm{C}})$ a *cc-formula* iff (1) no infinitary disjunctions occur in A and (2) if $\bigwedge \Phi$ is a subformula of A, then $\bigwedge \Phi$ is expressed as $\mathrm{C}_0(B)$ for some B.

We obtain cc-formulae by translating a formula in $\mathcal{P}$ by replacing $\mathrm{C}(\cdot)$ with $\mathrm{C}_0(\cdot)$. By this translation, $\mathcal{P}$ is embedded into $\mathcal{E}^{\omega}(\mathcal{P}_{-\mathrm{C}})$. We define the translator $\psi_\mathrm{C} : \mathcal{P} \to \mathcal{E}^{\omega}(\mathcal{P}_{-\mathrm{C}})$ inductively as follows:

T0: $\psi_\mathrm{C}(A) = A$ for all atomic A;

T1: $\psi_\mathrm{C}(\neg A) = \neg\psi_\mathrm{C}(A)$;

T2: $\psi_\mathrm{C}(A \supset B) = \psi_\mathrm{C}(A) \supset \psi_\mathrm{C}(B)$;

T3: $\psi_\mathrm{C}(A \wedge B) = \psi_\mathrm{C}(A) \wedge \psi_\mathrm{C}(B)$; and $\psi_\mathrm{C}(A \vee B) = \psi_\mathrm{C}(A) \vee \psi_\mathrm{C}(B)$;

T4: $\psi_\mathrm{C}(\forall x A(x)) = \forall x \psi_\mathrm{C}(A(x))$; and $\psi_\mathrm{C}(\exists x A(x)) = \exists x \psi_\mathrm{C}(A(x))$;

T5: $\psi_\mathrm{C}(\mathrm{B}_i(A)) = \mathrm{B}_i(\psi_\mathrm{C}(A))$;

T6: $\psi_\mathrm{C}(\mathrm{C}(A)) = \bigwedge\{\mathrm{B}_e(\psi_\mathrm{C}(A)) : e \in N^*\}$ $(= \mathrm{C}_0(\psi_\mathrm{C}(A)))$.

It is easy to see that $\psi_\mathrm{C}(A)$ is a cc-formula for any $A \in \mathcal{P}$. We can prove also the following lemmas.

LEMMA 6.3. ψ_C *is a bijection from* $\mathcal{P}$ *to the set of all cc-formulae.*

LEMMA 6.4. *For any* $A \in \mathcal{P}$, $\vDash A$ *if and only if* $\vDash \psi_\mathrm{C}(A)$.

These lemmas hold also in the propositional case. The translation ψ_C embeds CY and QCY into GL_ω and QGL_ω.

THEOREM 6.5 (faithful embedding). (1) *For any* $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{CY}} A$ *if and only if* $\vdash_\omega \psi_\mathrm{C}(A)$.

(2) *For any* $A \in \mathcal{P}$, $\vdash_{\mathrm{QCY}} A$ *if and only if* $\vdash_{\mathrm{Q}\omega} \psi_\mathrm{C}(A)$.

PROOF. We consider only (2). Suppose $\vdash_{\mathrm{QCY}} A$. Note that $\vdash_{\mathrm{Q}\omega} \psi_{\mathrm{C}}(B)$ for any instance B of the axioms for QCY, CA* and CI* are already verified in Lemma 6.1, and the classical inference rules translated by ψ_{C} are admissible in QGL_ω. Thus, a proof of A in QCY is translated into that of $\psi_{\mathrm{C}}(A)$ in QGL_ω. Therefore, $\vdash_{\mathrm{Q}\omega} \psi_{\mathrm{C}}(A)$.

Suppose $\vdash_{\mathrm{Q}\omega} \psi_{\mathrm{C}}(A)$. By Theorem 6.2, we have $\models \psi_{\mathrm{C}}(A)$. This is equivalent to $\models A$ by Lemma 6.4. Hence $\vdash_{\mathrm{QCY}} A$ by the completeness for QCY (Theorem 4.6). ■

The assertion (1) of this theorem is an improvement of the embedding theorem obtained in Kaneko [8] in that no restriction on the axiom $\bigwedge$-B is needed here, while the axiom $\bigwedge$-B is restricted only to the cc-formulae in [8].

In the propositional case, HM, CX and CY are all equivalent. It follows from this fact and Theorem 6.5.(1) that HM and CX are also faithfully embedded into GL_ω. Hence, the set

$$\{A \in \mathcal{E}^{\omega}_{-\mathrm{Q}}(\mathcal{P}_{-\mathrm{CQ}}) : A \text{ is a cc-formula and} \vdash_\omega A\}$$

is recursively enumerable, even though GL_ω is an infinitary logic.

The predicate case differs considerably from the propositional case. The set $\{A \in \mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}}) : \vdash_{\mathrm{Q}\omega} A$ and A is a cc-formula$\}$ is not recursively enumerable, since $\{A \in \mathcal{P} : \vdash_{\mathrm{QCY}} A\}$ is not recursively enumerable by Wolter's result and the soundness-completeness result for QCY. This result may look natural since QGL_ω is already an infinitary logic. Thus, there is a great difference between the propositional and predicate cases.

The above faithful embedding theorem does not hold between QHM and QGL_ω, since $\{A \in \mathcal{P} : \vdash_{\mathrm{QHM}} A\} \subsetneq \{A \in \mathcal{P} : \vdash_{\mathrm{QCY}} A\}$ and QCY is faithfully embedded into QGL_ω. We do not know whether or not it holds between QCX and QGL_ω. Nevertheless, when we restrict our attention to C–free formulae or B-free formulae, the faithful embedding theorem recovers for QHM and QCX to QGL_ω.

Remark 6.6 (relation between common knowledge and common belief).
In the infinitary logic GL_ω (or QGL_ω), the *common belief* of A is formulated as $\bigwedge\{\mathrm{B}_e(A) : e \in N^*$ and $e \neq \epsilon\}$. That is, it is defined by excluding $A = \mathrm{B}_\epsilon(A)$ from the common knowledge formula $\mathrm{C}_0(A)$. In GL_ω, this common belief formula is equivalent to

$$\mathrm{B}_1\,\mathrm{C}_0(A) \wedge \cdots \wedge \mathrm{B}_n\,\mathrm{C}_0(A)\,.$$

Using the above embedding theorem (Theorem 6.5), this formula is translated into $\mathrm{B}_1\,\mathrm{C}(\psi_{\mathrm{C}}^{-1}A) \wedge \cdots \wedge \mathrm{B}_n\,\mathrm{C}(\psi_{\mathrm{C}}^{-1}A)$ in CY. Since HM and CY are deductively equivalent, the common belief of proposition $A \in \mathcal{P}_{-\mathrm{Q}}$ is formulated as $\mathrm{B}_1\,\mathrm{C}(A) \wedge \cdots \wedge \mathrm{B}_n\,\mathrm{C}(A)$ in HM.

Another way around is to start with new common belief operator C_B as a primitive, and modify the logic HM into the *common belief logic* by replacing the axiom CA and inference rule CI by

$$\mathrm{C}_B\mathrm{A}: \quad \mathrm{C}_B(A) \supset \mathrm{B}_1(A \wedge \mathrm{C}_B(A)) \wedge \cdots \wedge \mathrm{B}_n(A \wedge \mathrm{C}_B(A));$$

$$\mathrm{C}_B\mathrm{I}: \quad \frac{D \supset \mathrm{B}_1(A \wedge D) \wedge \cdots \wedge \mathrm{B}_n(A \wedge D)}{D \supset \mathrm{C}_B(A)}.$$

Correspondingly, E8 of Section 2 is modified by assuming that reachability is defined by a sequence of length of at least 2. Then this common belief logic is also Kripke complete (see Halpern-Moses [5]). In this logic, the common knowledge is defined $\mathrm{C}_B(A) \wedge A$. This procedure can be done in all the constructs, e.g., HM, QHM and others in this paper.

In this sense, we can start either common knowledge or common belief, and then the other is defined as a derived concept.

Remark 6.7. The final remark is on a larger infinitary logics than GL_ω and QGL_ω. Possible candidates are $\mathrm{L}_{\omega_1\omega}(\mathrm{QKD4}^n)$ and $\mathrm{QL}_{\omega_1\omega}(\mathrm{KD4}^n)$, which is more standard in the literature of infinitary logics (cf., Karp [13]). As far as we assume proper Barcan axioms, we would have the same embedding theorems. For larger sets of formulae create no further difficulties. In this sense, GL_ω and QGL_ω are the smallest choices of infinitary extensions so that all of CY (and HM, CX) and QCY are faithfully embedded.

7. Conclusions

7.1. Map

From the considerations in Section 2 through Section 6, we draw Diagram 2. The map has four rows: The above two give the logics considered in the propositional and predicate cases. The two additional rows explain the characteristics of each logic, e.g., QHM has finitary formulae and finitary proofs, while QCX has finitary formulae but infinitary proofs. The arrow $\rightarrow$ denotes the relationship of deductive strength and $\rightsquigarrow$ that of being faithfully embedded.

No serious problems remain in the propositional case. All the logics from $\mathrm{KD4}^n$ to GL_ω are Kripke complete, and HM, CX, CY are deductively equivalent. They can be faithfully embedded into GL_ω. Also, GL_ω is a conservative extension of $\mathrm{KD4}^n$. On the other hand, we have met various problems in the predicate case. The logics $\mathrm{QKD4}^n$, QCY and QGL_ω are

$$
\begin{array}{llllllllll}
 & \mathrm{KD4}^n & \rightarrow & \mathrm{HM} & \longleftrightarrow & \mathrm{CX} & \longleftrightarrow & \mathrm{CY} & \rightsquigarrow & \mathrm{GL}_\omega \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \mathrm{QKD4}^n & \rightarrow & \mathrm{QHM} & \rightarrow & \mathrm{QCX} & \rightarrow & \mathrm{QCY} & \rightsquigarrow & \mathrm{QGL}_\omega \\
\text{formulae} & \text{finitary} & & \text{finitary} & & \text{finitary} & & & & \text{infinitary} \\
\text{proofs} & \text{finitary} & & \text{finitary} & & \text{infinitary} & & & & \text{infinitary}
\end{array}
$$

Diagram 2.

Kripke complete, but QHM is Kripke incomplete. The Kripke completeness of QCX remains open. The logic QCY is faithfully embedded into QGL_ω, and QGL_ω is a conservative extension of $\mathrm{QKD4}^n$. Though QHM is embedded into QGL_ω, its faithfulness does not hold. The logic QCX is embedded into QGL_ω but faithfulness remains also open.

In Diagram 2, the gap exists around QHM and QCX. There is a formula that is valid in the Kripke semantics but is not provable in QHM. Theorem 5.5 states that such a formula contains quantifiers, belief operators and the common knowledge operator. Nevertheless, it is also an open problem to find a concrete one.

7.2. Other remarks

(1) We have chosen $\mathrm{KD4}^n$ as the base logic, and have considered various extensions of it keeping the KD4-basic axioms. The choice of $\mathrm{KD4}^n$ is made because $\mathrm{KD4}^n$ (or KD^n) seems to be central for concrete applications such as game theoretical ones. For example, the distinction between true and false beliefs is possible by dropping the axiom T (truthfulness axiom), which could be crucial for future game theoretical applications (cf., Kaneko [9] and Kaneko-Suzuki [12]). Also, each S4-type logic is treated inside the corresponding KD4-type logic with the well known translation: $\mathrm{B}^*(A) = \mathrm{B}(A) \wedge A$. As far as the map of common knowledge logics is concerned, almost the same results holds for the other choice of a base logic, e.g., K^n, KD^n, $\mathrm{S4}^n$ as well as $\mathrm{S5}^n$. The non-recursive-enumerability result by Wolter [24] holds also for these logics. Only some comparisons such as Remark 4.3 may be difficult in the case of the choice of $\mathrm{S4}^n$ or $\mathrm{S5}^n$ as the base logic.

(2) The completeness theorem for QCY (Theorem 4.6) is proved in Tanaka [19]. The proof is based on an algebraic method along the line of Rasiowa-Sikorski [17]. The basic strategy of the completeness proof for QCY is standard in that when a formula is consistent in QCY, we construct a Kripke model where the formula is true in some world. The construction of a Kripke model is made by starting with the Lindenbaum algebra

and taking all the *Q-filters* for the set of possible worlds. Here we have to modify the definition of Q-filters suitable for QCY. Then we will use the Rasiowa-Sikorski lemma and a lemma given in Tanaka-Ono [22] on Q-filters, to show that the constructed Kripke model is a desired one.

The proof given in Tanaka [19] does not deal with function symbols, but can be modified to incorporate functions symbols in the standard manner.

To prove the completeness of QGL_ω (Theorem 6.2), we should modify the completeness proof of QCY. That is, the choice of Q-filters should be changed from those in the completeness proof for QCY.

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References

[1] Fagin, R., J. Y. Halpern, Y. Moses and M. Y. Verdi, (1995), *Reasoning about Knowledge*, The MIT Press, Cambridge.

[2] Chellas, B. F., (1980), *Modal Logic: An Introduction*, Cambridge University Press, Cambridge.

[3] Fitting, M., and R. L. Mendelsohn, (1998), *First-Oder Modal Logic*, Kluwer Academic Publisher, London.

[4] Garson, J. W., (1984), 'Quantification in modal logic', *Handbook of Philosophical Logic II*, eds. D. Gabbay and F. Guenthner. D. Reidel Publishing Company, Dordrecht, 249–307.

[5] Halpern, J. H. and Y. Moses, (1992), A Guide to Completeness and Complexity for Modal Logics of Knowledge and Beliefs, *Artificial Intelligence* 54, 319–379.

[6] Heifetz, A., (1999), 'Iterative and fixed point common belief', *Journal of Philosophical Logic* 28, 61–79.

[7] Hughes, G. E., and M. J. Cresswell, *A Companion to Modal Logic*, Methuen & Co. London.

[8] Kaneko, M., (1999), 'Common knowledge logic and Game Logic', *Journal of Symbolic Logic* 64, 685–700.

[9] Kaneko, M., (2002), 'Epistemic logics and their applications: Introduction'. The special issue "Logic and Economics" in *Economic Theory* 19, 7–62.

[10] Kaneko, M., and T. Nagashima, (1996), 'Game logic and its applications I', *Studia Logica* 57, 325–354.

[11] Kaneko, M., and T. Nagashima, (1997), 'Game logic and its applications II', *Studia Logica* 58, 273–303.

[12] Kaneko, M., and N.-Y. Suzuki, (2001), Bounded Interpersonal Inferences and Decision Making. The special issue "Logic and Economics" in *Economic Theory* 19, 63–103.

[13] Karp, C., (1964), *Languages with Expressions of Infinite Lengths*, North-Holland. Amsterdam.

[14] Lismont, L., and P. Mongin, (1994), 'On the logic of common belief and common knowledge', *Theory and Decision* 37, 75–106.

[15] Meyer, J.-J. Ch., and W. van der Hoek, (1995), *Epistemic Logic for AI and Computer Science*, Cambridge University Press.

[16] Segerberg, K., (1994), 'A model existence theorem in infinitary propositional modal logic', *Journal of Philosophical Logic* 23, 337–367.

[17] Rasiowa, H. and R. Sikorski, (1963), *The Mathematics of Metamathematics*, PWN, Warszawa.

[18] Sturm, H., F. Wolter and M. Zakharyaschev, 'Monodic epistemic predicate logic'. The special issue "Logic and Economics" in *Economic Theory* 17, 153-186.

[19] Tanaka, Y., (2000a), 'Some proof system for common knowledge predicate logic'. Mimeo.

[20] Tanaka, Y., (2001), 'Model existence in non-compact modal logic'. *Studia Logica* 67, 61–71.

[21] Tanaka, Y., (1999), 'Kripke completeness of infinitary predicate multi-modal logics'. To appear in *Notredame Journal of Formal Logic* 40, 326–340.

[22] Tanaka, Y., and H. Ono, (2001), 'The Rasiowa-Sikorski lemma and Kripke completeness of predicate and infinitary modal logics', pp. 401–419 in M. Zakharyaschev *et. al* (eds.), *Advances in Modal Logics 2*, CSLI Publications, Stanford.

[23] Wolter, F., (1998), 'First order common knowledge logics'. The proceedings of the Loft3 conference.

[24] Wolter, F., (2000), 'First order common knowledge logics', *Studia Logica* 65, 249–271. This is a revised version of [23].

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