

Housing Markets with Indivisibilities

MAMORU KANEKO

*Institute of Socio-Economic Planning, The University of Tsukuba, Ibaraki-ken 305,
Japan¹, and Cowles Foundation for Research in Economics, Yale University,
Box 2125, Yale Station, New Haven, Connecticut 06520*

Received February 20, 1981; revised September 8, 1981

A model of a rental housing market in which houses are treated as indivisible commodities is presented. A recursive equation that determines a competitive equilibrium is provided, and it is argued that this competitive equilibrium is representative of the set of all competitive equilibria. Using this representative equilibrium, several propositions on comparative statics are considered that have to do with how the competitive rents change when certain parameters of the model change.

1. INTRODUCTION

Many authors have considered housing markets from theoretical points of view and formulated several types of mathematical models.² Alonso studied a housing market in a linear or circular city in his classical work [1]. Beckmann [2], Montesano [13], Wheaton [20], Hartwick *et al.* [6], MacKinnon [8], Romanos [14], and others considered other varieties of the Alonso model. One common feature of these housing-market models is that households and homeowners choose house sizes freely. That is, houses are assumed to be perfectly divisible like the standard commodities in equilibrium analysis. Some divisibility does exist in housing markets, but it is also true that though households can select houses freely under their budgets, they cannot select sizes freely. If suppliers have already built houses, then households can select only from them. Then they encounter indivisibility. Existing housing markets have these two features, which are entangled with each other. We should also study the second extreme case, which would help our understanding of existing housing markets. The purpose of this paper is to provide a model of housing markets in which houses are treated as indivisible commodities.

A general theory for this phenomenon has been developed in the related field of game theory, see for example, Böhm-Bawerk [3], von Neumann and Morgenstern [21], Gale [4], Shapley and Shubik [16], Telser [19], Schotter

¹Current address.

²Of course, there are many more papers in which housing markets have been studied from empirical points of view, but this paper does not refer to them.

[15], and Kaneko [9, 11]. These authors have concerned themselves with the relations among the competitive equilibria, the core of market game, or the dual prices of assignment problem and their existence.³ The theory seems to be mature enough to be applied to the problem of housing markets. Indeed, this paper is an application of Kaneko [11] to a housing market.⁴

In this paper houses are indivisible and each household wants to rent at most one house. Other consumption commodities are treated as one composite commodity, which will be called money. To examine competitive equilibria in this market model, the paper provides a certain recursive equation that determines a competitive equilibrium, and it is argued that this equilibrium can be regarded as a representative of the set of all competitive equilibria. Using this result, several comparative statics propositions relating to how the competitive rents change when certain parameters of the model change are investigated.

The format of the paper is as follows. A model of a housing market is described in Section 2. Competitive equilibrium is defined in Section 3 and several theorems are provided that describe fully the structure of the set of all competitive equilibria in the market. In Section 4 several propositions on comparative statics are provided. Finally, in Section 5 a remark is offered on the introduction of externalities among households into the model. Throughout this paper numerical examples are provided that seem to aid the intuitive understanding of the model and the results.

2. A HOUSING MARKET (M, N)

A mathematical model of rental housing market (M, N) is considered in which s kinds of houses appear. A house is called an *apartment* in this paper. Let $M = \{1, \dots, m\}$ and $N = \{1', \dots, n'\}$. A member $i \in M$ is called a *landlord* and a member $j \in N$ a *household*. Each landlord $i \in M$ owns $w^i = (w_1^i, \dots, w_s^i)$ units of apartments that he can lease to households. It is assumed that each apartment is indivisible. Hence, each w_k^i ($k = 1, \dots, s$, $i \in M$) is a nonnegative integer. We assume

$$(A) \quad \sum_{i \in M} w_k^i > 0 \text{ for all } k = 1, \dots, s \text{ and } \sum_{k=1}^s \sum_{i \in M} w_k^i \geq n.$$

³Schotter [15] already applied the assignment game to a housing market and considered the nonemptiness of the core of a housing market with externalities among households. In Section 5 of this paper, we also briefly consider a housing market with externalities.

⁴Koopmans and Beckmann [12], Hartwick [5], Heffley [7], and others considered location problems of farms in terms of assignment problems. The orientation of these studies is different from that of the above works. The works of Smith [17] and Sweeney [18] are close to the subject of this paper, and the basic idea for this paper is the same as the works of Smith and Sweeney.

That is, the potential supply of each apartment is positive and the potential total supply is not smaller than the number of households. No household in N rents any apartment in the beginning. Each household $j \in N$ has $I_j > 0$ dollars for renting an apartment. Money should be interpreted as a composite commodity. I_i is the *income* of household i . It is assumed that money is perfectly divisible and is represented as a real number. We order the households as

$$I_1 \geq I_2 \geq \cdots \geq I_n. \quad (1)$$

A simple utility function is assumed for landlords. Each landlord $i \in M$ has the *evaluation function* $u^i(x)$ on the set $X^i \equiv \{x = (x_1, \dots, x_s): 0 \leq x_k \leq w_k^i \text{ and } x_k \text{ is an integer for all } k = 1, \dots, s\}$ such that for all $x \in X^i$

$$(B) \quad u^i(x) = \sum_{k=1}^s u^i(x_k e^k) \text{ and } u^i(x_k e^k) = a_k x_k \text{ for all } k = 1, \dots, s,$$

where e^k is the s -dimensional vector with $e_j^k = 1$ if $j = k$ and $e_j^k = 0$ if $j \neq k$ and a_k is a positive real number. Note that a_k is independent of landlords.

Notice that x in (B) designates the units of remaining apartments, that is, $w^i - x$ is the units of apartments that landlord i leases to households. The term a_k is the least value at which a landlord can lease one unit of the k th apartment without decreasing his utility below that of not leasing it when the price of money equals 1. Hence, if landlord i leases $w^i - x$ units of apartments at rents r_1, r_2, \dots, r_s , his utility (profit) is

$$u^i(x) + \sum_{k=1}^s r_k (w_k^i - x_k) = \sum_{k=1}^s a_k x_k + \sum_{k=1}^s r_k (w_k^i - x_k). \quad (2)$$

It should be noted that a monotone transformation of (2) has the same meaning as (2). For example, subtracting $c = \sum_{k=1}^s a_k w_k^i$ from (2), we get

$$u^i(x) + \sum_{k=1}^s r_k (w_k^i - x_k) - \sum_{k=1}^s a_k w_k^i = \sum_{k=1}^s (r_k - a_k) (w_k^i - x_k).$$

This formula really means landlord i 's profit when he leases $w^i - x$ units of apartments. Kaneko [11] explained the conditions for a preference relation

to be represented as (2) (see also [10]). We call a_k the *evaluation value of the k th apartment*.⁵

In this paper the price of money is always taken to be 1. Each household $j \in N$ is assumed to have the same consumption set $Y \equiv \{0, e^1, \dots, e^s\} \times R_+$, where R_+ is the set of all nonnegative real numbers. For convenience, we may use $e^{s+1} = 0$ and $a_{s+1} = 0$ in the following. The expression $(x, m) \in Y$ means that if $x = e^k$ ($k \leq s$), then household j rents one unit of the k th apartment and his available money is m after paying the rent; and if $x = e^{s+1} = 0$, he does not rent any apartment and, of course, $m = I_j$. This formulation assumes that a household never rents more than one apartment.⁶

Each household $j \in N$ has the same preference relation R on Y . $(x, m_1)R(y, m_2)$ means that each household prefers (x, m_1) to (y, m_2) or is indifferent between them. The term R is assumed to be a weak ordering.⁷ We define the *strict preference* P and the *indifference relation* Q by

$$\begin{aligned} (x, m_1)P(y, m_2) & \quad \text{iff not} \quad (y, m_2)R(x, m_1), \\ (x, m_1)Q(y, m_2) & \quad \text{iff} \quad (x, m_1)R(y, m_2) \\ & \quad \text{and} \quad (y, m_2)R(x, m_1). \end{aligned}$$

We make the following assumptions.

- (C) For all $(x, m) \in Y$, if $\delta > 0$, then $(x, m + \delta)P(x, m)$.
- (D) If $(x, m_1)P(y, m_2)$, then there is an $m_3 \geq 0$ such that $(x, m_1)Q(y, m_3)$.
- (E) If $(x, m_1)Q(y, m_2)$, $m_1 < m_2$ and $\delta > 0$, then $(x, m_1 + \delta)P(y, m_2 + \delta)$.

Assumptions C and D are mild and need no particular comment. Assumption E deserves a comment. It means that the marginal utility of

⁵The evaluation function means the substitutional relation between apartments and money but not any relation between apartments and nominal profits. One possible interpretation of this evaluation function is as follows. When landlord i builds w_k^i units of the k th apartment and the cost is M measured in terms of the composite commodity, a_k is determined by $M/w_k^i = \sum_{t=1}^T a_k/(1+\gamma)^t$, where γ is the interest rate and T is the rental term. If we employ this interpretation, we should reinterpret the initial endowment w_k^i as a potential supply that is not built before contract. This is the simplest interpretation. We could provide a more complicated dynamic interpretation. Anyway, a_k is measured in terms of money (the composite commodity) but not nominal value. If the price of money p is not equal to 1, the landlords' objective function is written as $u^i(x) + \sum_{k=1}^s r_k(w_k^i - x_k)/p$ or $pu^i(x) + \sum_{k=1}^s r_k(w_k^i - x_k)$.

⁶In Kaneko [11] a more general consumption set is employed and this property is assumed on preference relations.

⁷ R is said to be a *weak ordering* iff $(x, m_1)R(y, m_2)$ or $(y, m_2)R(x, m_1)$ for all $(x, m_1), (y, m_2) \in Y$ and $(x, m_1)R(y, m_2)$ and $(y, m_2)R(z, m_3)$ imply $(x, m_1)R(z, m_3)$.

money is diminishing or, in other words, that housing is a normal commodity; that is, as income rises the demand of a better apartment increases. If the conclusion of it is replaced by $(x, m_1 + \delta)Q(y, m_2 + \delta)$, this condition together with (C) and (D) become a condition for transferable utility; in other words, a condition for consumer's surplus to be well defined.⁸ As is well known, this case does not permit any income effect. In a housing market, one cannot neglect the income effect because rents are not negligible relative to incomes. That is, as the income level becomes lower, an additional income δ becomes more important.

The following example provides a preference relation satisfying the above assumptions. In the succeeding examples we will use the same terminology.

EXAMPLE 1. Let h_1, h_2, \dots, h_s be positive real numbers but $h_{s+1} = 0$, and let $g(m)$ be a strictly concave, continuous, and increasing function on R_+ with $\lim_{m \rightarrow \infty} g(m) = +\infty$. The utility function $U(e^k, m) = h_k + g(m)$ for $k = 1, \dots, s, s+1$ gives a preference relation R on Y , that is, $U(e^k, m_1) \geq U(e^t, m_2)$ iff $(e^k, m_1)R(e^t, m_2)$. This preference relation satisfies the above assumptions. Of course, any monotone transformation of this $U(x, m)$ is also a utility function representing R .⁹

LEMMA 1. (i). If $(x, m_1)Q(y, m_2)$ and $0 < \delta \leq m_2 < m_1$, then $(x, m_1 - \delta)P(y, m_2 - \delta)$. (ii) If $(e^k, 0)P(e^t, 0)$ and $(e^k, m_1)Q(e^t, m_2)$, then $m_1 < m_2$.

Proof. See the Appendix.

It is assumed for simplicity that when households consume no money, there is no indifference relation between any pair of apartments, that is, for a any k and t with $k \neq t$, $(e^k, 0)P(e^t, 0)$ or $(e^t, 0)P(e^k, 0)$. Furthermore we reorder $1, 2, \dots, s$ such that

$$(F) \quad (e^1, 0)P(e^2, 0)P \cdots P(e^s, 0).^{10}$$

Assumption F is equivalent to

$$(e^1, m)P(e^2, m)P \cdots P(e^s, m) \quad \text{for all } m \geq 0. \quad (3)$$

⁸See Kaneko [10].

⁹Sweeney [18] introduced a concept of convexity of R in the usual sense of equilibrium analysis as follows. If $(x, m_1), (y, m_2) \in Y$, $0 \leq \gamma \leq 1$, and $(\gamma(x, m_1) + (1 - \gamma)(y, m_2)) \in Y$, then $(\gamma(x, m_1) + (1 - \gamma)(y, m_2))R(x, m_1)$ or $(\gamma(x, m_1) + (1 - \gamma)(y, m_2))R(y, m_2)$. But we can easily find an example that satisfies the convexity but not assumption E. For example, the preference relation generated by $U(e^k, m) = h_k + m^2$ satisfies the convexity but not (E). Indeed, let $h_1 = 21$ and $h_2 = 10$. Then $U(e^1, 5) = 21 + 25 = 46$, $U(e^2, 6) = 10 + 36 = 46$, but $U(e^1, 6) = 57 < 59 = U(e^2, 7)$.

¹⁰Note that we do not necessarily assume that the apartments are ordered according to the distances from the center of the city. The apartments are ordered according to the qualities. See Example 2.

which gives us a clearer meaning of assumption F. That is, if the amounts of consumable money are the same, the households have the same preference relation on the apartments as that given by (F). The proof is easy: if $(e^{k+1}, m_1)Q(e^k, m)$, then $m_1 > m$ by Lemma 1(ii), which implies $(e^k, m)P(e^{k+1}, m)$.

We define a function $G(k)$ ($k = 1, \dots, s$) by

$$G(k) = \sum_{i=1}^k \sum_{i \in M} w_i^i. \quad (4)$$

If $G(k) \leq n$, then it corresponds to the household $G(k)$. In this case we call $G(k)$ the k th *marginal household*. Assumption A implies

$$G(f-1) < n \leq G(f) \quad \text{for some } f \leq s. \quad (5)$$

We call f the *marginal apartment*. These concepts, the marginal households and the marginal apartments will play essential roles in this paper.¹¹

We define a vector (p_1, \dots, p_{f-1}) backward recursively by

$$\begin{aligned} & (e^f, I_{G(f-1)} - a_f)Q(e^{f-1}, I_{G(f-1)} - p_{f-1}) \\ & (e^{f-1}, I_{G(f-2)} - p_{f-1})Q(e^{f-2}, I_{G(f-2)} - p_{f-2}) \\ & \vdots \\ & (e^2, I_{G(1)} - p_2)Q(e^1, I_{G(1)} - p_1). \end{aligned} \quad (6)$$

Note that $I_{G(k)}$ is the income of the marginal household $G(k)$ ($k = 1, \dots, f-1$).

In fact, we shall construct a competitive equilibrium using this rent vector (p_1, \dots, p_{f-1}) , that is, we shall show that this rent vector forms a competitive rent vector under appropriate assumptions. The following lemma provides a condition for (6) to have a unique solution.

LEMMA 2. *If $I_n \geq a_f$ and $(e^f, I_n - a_f)P(e^1, 0)$, then there is a unique vector (p_1, \dots, p_{f-1}) that satisfies (6) and has the property*

$$p_1 > p_2 > \dots > p_{f-1} > a_f. \quad (7)$$

Proof. See the Appendix.

¹¹The households are ordered according to the income levels and attached with larger incomes to better apartments. But these are just notations, not an *a priori* assumption. In Theorem 3, this is derived as a result. This is different from the works of Beckmann [2], Montesano [13], or Hartwick *et al.* [6].

We assume the supposition of Lemma 2 and the conditions

- (G) $I_n \geq a_f, (e^f, I_n - a_f)P(e^1, 0)$ and $(e^f, I_n - a_f)P(e^k, I_n - a_k)$ for all k ($f < k \leq s+1$) with $I_n \geq a_k$.
 (H) $p_k > a_k$ for all $k = 1, \dots, f-1$.

Assumption G means that the household with the least income can rent the marginal apartment if the rent is the least, that is, the landlords' evaluation value a_f , is the least; that the renter prefers it to the best apartment with zero consumption; and that he chooses the f th apartment among ones that are worse than the marginal apartment, even if the rents are the landlords' evaluation values. The term p_{f-1} means the maximal amount that household $G(f-1)$ pays for the $(f-1)$ th apartment if he can also rent the f th apartment at a_f . The term p_{f-2} is the maximal amount that household $G(f-2)$ pays for the $(f-2)$ th apartment if he can also rent the $(f-1)$ th one at p_{f-1} . The values for p_{f-3}, \dots, p_1 are defined recursively. Assumption H means that these values are greater than the landlords' evaluation values. Under these assumptions it will be shown in the next section that these values form a competitive equilibrium.

Let j be a household with $I_{G(k-1)+1} \geq I_j \geq I_{G(k)}$ and let $p^j = (p_1^j, \dots, p_s^j)$ be defined by

$$(e^t, I_j - p_t^j)Q(e^k, I_j - p_k) \quad \text{for all } t = 1, \dots, s,$$

where p is the vector given by (6). The term p^j is the rent vector such that if the rent of the k th apartment is p_k , then it makes the household j indifferent between the k th apartment and every other. The following lemma implies that p is the envelope curve of p^1, \dots, p^n .¹² (See Fig. 1.) From this fact, we know that p has a shape like a convex curve, though it is not necessarily exactly true.

LEMMA 3. *Let I be an income level such that $I_{G(k-1)+1} \geq I \geq I_{G(k)}$ ($1 \leq k \leq f-1$). Then the following propositions hold:*

- (i) $(e^k, I - p_k)R(e^{k+1}, I - p_{k+1})R \cdots R(e^{f-1}, I - p_{f-1})R(e^f, I - a_f)$ and $(e^f, I - a_f)P(e^t, I - a_t)$ for all $t > f$ with $I \geq a_t$.
 (ii) $(e^k, I - p_k)R(e^{k-1}, I - p_{k-1})R \cdots R(e^{t+1}, I - p_{t+1})R(e^t, I - p_t)$ if $p_t \leq I$.

Proof. See the Appendix.

Before discussing the main subject of this paper, we check the consistency of our model by the following example, which also helps our understanding of the mathematical model.

¹² Lemma 3 implies $p_t^j \leq p_t$ for all $t \leq f$ and it holds by definition that $p_k^{G(k)} = p_k$ & $p_{k+1}^{G(k)} = p_{k+1}$ for all $k \leq f-1$.

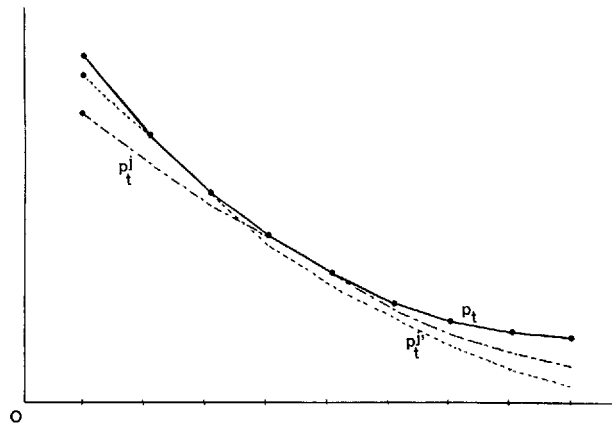


FIGURE 1

EXAMPLE 2. Consider a model of a city with one center. All households are employed by factories, banks, universities, etc. that are located in the center. Everyone uses the same transportation system. Let the potential supplies of apartments $\sum_{i \in M} w^i$ be represented by Table 1. A rectangle with a diagonal line means that there is no supply of that kind of apartment. Note that it is not necessary to specify exact numbers of supplies. Let the utility function of the households be

$$V(s, t, m) = 6\sqrt{10s} + 5\sqrt{80 - t} + 2\sqrt{m}$$

where s is the number of rooms, for example, $s = 3$ means a 3-room apartment; t is the distance from the center measured in terms of minutes, for example, $t = 30$ means that it takes 30 min to get to the center; and m is

TABLE 1

		k		h_k				
$s \backslash t$								
		1		2		3		4
10	10	60.80	4	68.66	/		/	
20	13	57.70	6	65.56				
30	14	54.33	9	62.19	5	68.22	1	73.31
40	16	50.59	12	58.45	8	64.48	3	69.57
50	17	46.36	15	54.22	11	60.25	7	65.34

Note. t is distance (min), s the number of rooms, and k the number of apartments.

the amount of available money after paying rent, for example, $m = 750$ means that a household can spend \$750 on consumption for a month. The h_k 's in Example 1 are reached by calculating $6\sqrt{10s} + 5\sqrt{80 - t}$, which gives us Table 1. Let the landlords' evaluation values of apartments be given by Table 2. Furthermore, it is assumed that the marginal apartment is the 13th, and that the incomes of the marginal households are given by Table 3 and $I_n = \$700/\text{month}$. Finally, in Table 4 we solve the backward equation

$$\begin{aligned} h_{13} + 2\sqrt{800 - 160} &= h_{12} + 2\sqrt{800 - p_{12}} \\ h_{12} + 2\sqrt{900 - p_{12}} &= h_{11} + 2\sqrt{900 - p_{11}} \\ &\vdots \\ h_2 + 2\sqrt{1900 - p_2} &= h_1 + 2\sqrt{1900 - p_1}. \end{aligned}$$

TABLE 2

		k		a_k					
s	t	1		2		3		4	
		10	180	4	220				
10	10	10	180	4	220				
20	13	160	6	200	2	240			
30	14	140	9	180	5	220	1	260	
40	16	120	12	160	8	200	3	240	
50	17	100	15	140	11	180	7	220	

Note. a_k is the evaluation value (\$/month).

TABLE 3

		k		I _G (k)					
s	t	1		2		3		4	
10	10	1000	4	1600					
20	13	700	6	1400					
30	14	ϕ	9	1100	5	1500	1	1900	
40	16	ϕ	12	800	8	1200	3	1700	
50	17	ϕ	15	ϕ	11	900	7	1300	

Note. $I_{G(k)}$ is the income of household $G(k)$ and $I_n = 700$.

TABLE 4

		k		p _k					
s	t	1		2		3		4	
10	10	241.6	4	485.1					
20	13	160	6	383.2	2	588.0			
30	14		9	281.8	5	470.4	1	649.6	
40	16		12	178.8	8	349.9	3	516.6	
50	17		15		11	226.3	7	376.2	

Note. p_k (\$/month) and $p_{13} = 160$.

It is not difficult to verify that this example satisfies assumptions A–H. Note that, as will be shown, (p_1, \dots, p_{f-1}) forms a competitive equilibrium of this market.¹³

From this example we know that our model is applicable to complicated housing markets to a certain extent, but that it cannot be applicable to housing markets in multiple-center cities. For example, market models like Romanos' [14] cannot be analyzed.

3. THE COMPETITIVE EQUILIBRIA IN THE MARKET (M, N)

We are now in a position to discuss the main subject of this paper, that is, to define competitive equilibrium and to provide theorems that describe the structure of the set of all competitive equilibria.

DEFINITION. $(r, x) = (r_1, \dots, r_s, x^1, \dots, x^m, x^{1'}, \dots, x^{n'})$ is said to be a *competitive equilibrium* iff

$$x^i \in X^i \quad \text{for all } i \in M, \quad x^j \in \{e^1, \dots, e^s, e^{s+1}\} \quad \text{for all } j \in N, \\ \text{and} \quad r_k \geq 0 \quad \text{for all } k = 1, \dots, s, \quad (8)$$

$$\sum_{i \in M} x^i + \sum_{j \in N} x^j = \sum_{i \in M} w^i, \quad (9)$$

$$\text{for all } i \in M, \quad u^i(x^i) + r(w^i - x^i) = \max_{z \in X^i} (u^i(z) + r(w^i - z)) \quad (10)$$

$$\text{for all } j \in N, \quad rx^j \leq I_j \quad \text{and} \quad (x^j, I_j - rx^j)R(z, m) \\ \text{for all } (z, m) \in Y \quad \text{such that} \quad rz + m \leq I_j. \quad (11)$$

¹³In Beckmann [2], Montesano [13], and others, the transportation costs are taken into account explicitly. If it is assumed ([2], [13]) that all households go the same number of times to the center, it is easy to introduce the transportation costs into our example.

We call $r = (r_1, \dots, r_s)$ a *competitive rent vector* and each r_k ($k = 1, \dots, s$) a *competitive rent* of the k th apartment.

Condition (8) means that each commodity bundle belongs to the appropriate set; (9) is the equivalence of total supplies and total demands of apartments; (10) is the utility maximization of the landlords; (11) is the budget constraint and the utility maximization of the households under the budget constraint. Note that the equivalence of total demand and total supply of money is automatically satisfied.

First we show the existence of a competitive equilibrium.

THEOREM 1. *The rent vector $r = (r_1, \dots, r_s)$ is a competitive rent vector in both of the following cases.*

(i) *If $G(f-1) < n < G(f)$, then*

$$\begin{aligned} r_k &= p_k & \text{if } k < f \\ &= a_k & \text{otherwise} \end{aligned} \quad (12)$$

where (p_1, \dots, p_s) is the vector defined by (6).

(ii) *If $G(f) = n$, then r is defined by*

$$\begin{aligned} r_k &= a_k \quad \text{for all } k \geq f+1, \quad a_f \leq r_f \leq p_f^*, \\ &\text{and } (e^k, I_{G(k)} - r_k) Q(e^{k+1}, I_{G(k)} - r_{k+1}) \\ &\text{for all } k = 1, \dots, f-1, \end{aligned} \quad (13)$$

where p_f^* is defined by $(e^f, I_n - p_f^*) Q(e^t, I_n - a_t)$ for some t such that $t \geq f+1$ and $(e^t, I_n - a_t) R(e^k, I_n - a_k)$ for all k with $k \geq f+1$ and $I_n \geq a_k$ if such a p_f^* exists and otherwise $p_f^* = I_n$.¹⁴

In this theorem we give not only the existence of a competitive equilibrium, but also the procedure for computing a competitive rent vector. In each case the backward recursive equation (6) provides a competitive equilibrium. Hence, a competitive rent vector of the market of Example 2 is provided in Table 4. In case (ii), there exist multiple competitive rent vectors. We consider this case further below.

In Kaneko [11], the existence of a competitive equilibrium is proved under more general conditions and with an additional one¹⁵: a variety of landlords' and households' preferences is permitted. Therefore, the existence theorem can be applied to a multicenter model with multitransporta-

¹⁴Note that we permit $k = s+1$ or $t = s+1$.

¹⁵That is, assumption D of [11] is not assumed in this paper and Example 2 does not satisfy it. But this assumption concerns the exclusion of corner solutions and is not so strong. It is easy to modify Example 2 to satisfy assumption D.

tion systems. The existence theorem, however, provides no explicit shape of the rent vector.

Proof. We prove in case (i) that the rent vector is competitive. Because the proof for case (ii) is similar we omit it.

Since Lemma 2 ensures the existence and the uniqueness of (p_1, \dots, p_{f-1}) , it is sufficient to construct a vector $x = (x^1, \dots, x^m, x^{1'}, \dots, x^{n'})$ and to show that (r, x) is a competitive equilibrium. We define $(x^1, \dots, x^m, x^{1'}, \dots, x^{n'})$ by

$$\begin{aligned} x_k^i &= 0 \text{ if } k < f \\ &= w_k^i \text{ if } k > f \end{aligned} \quad (14)$$

$$\begin{aligned} \sum_{i \in M} x_f^i + \sum_{j \in N} x_f^j &= \sum_{j \in M} w_f^j \quad \text{and} \quad 0 \leq x_f^i \leq w_f^i \quad \text{for all } i \in M, \\ x^j &= e^k \quad \text{if } G(k-1) < j \leq G(k) \\ &\text{and } k = 1, \dots, f. \end{aligned} \quad (15)$$

It is easily verified that x satisfies (8) and (9), but we should check (10) and (11).

Since $r_k > a_k$ for all $k = 1, \dots, f-1$ by assumption H, and $r_k = a_k$ for all $k = f, \dots, s$ by (12), it holds that for all $i \in M$, $u^i(x^i) + r(w^i - x^i) = \max_{z \in X^i} (u^i(z) + r(w^i - z))$.

Let j be a household such that $G(k-1) + 1 \leq j \leq G(k)$. Lemma 3 says that if $r_t \leq I_j$, then $(e^k, I_j - r_k)R(e^t, I_j - r_t)$. Q.E.D.

THEOREM 2. Consider case (ii) of Theorem 1. Each r_f with $a_f \leq r_k \leq p_f^*$ determines exactly one competitive rent vector satisfying (13). If $r_f > r'_f$ and r, r' are the competitive rent vectors determined by r_f and r'_f , then

$$0 < r_1 - r'_1 < r_2 - r'_2 < \dots < r_{f-1} - r'_{f-1} < r_f - r'_f \quad (16)$$

$$1 < r_1/r'_1 < r_2/r'_2 < \dots < r_{f-1}/r'_{f-1} < r_f/r'_f. \quad (17)$$

Proof. It can be proved in the same as in Lemma 2 that each r_f determines exactly one vector. Since it can be shown similarly to Lemma 2 that $r'_1 > r'_2 > \dots > r'_f$, (17) follows from (16). We omit the proof of (16) because it is almost the same as that of Theorem 5, for which a proof is offered below. Q.E.D.

EXAMPLE 3. Let us consider the housing market of Example 2, and let $n = G(13)$. Then the competitive rent vector r^* determined by p_{13}^* is

TABLE 5

k		r_k^*	$r_k^* - r_k$
		r_k	r_k^*/r_k

s		1		2		3			4			
t												
10	10	295.2	53.6	4	532.2	47.1						
		241.6	1.222		485.1	1.097						
20	13	216.9	56.9	6	432.7	49.5	2	633.1	45.1			
		160	1.356		383.2	1.129		588.0	1.077			
30	14	140	0	9	334.1	52.3	5	517.8	47.4	1	693.6	44.0
		140	1		281.8	1.186		470.4	1.101		649.6	1.068
40	16	120	0	12	234.9	56.1	8	400.2	50.3	3	563.1	46.5
		120	1		178.8	1.314		349.9	1.144		516.6	1.090
50	17	100	0	15	140	0	11	280.5	54.2	7	425.8	49.6
		100	1		140	1		226.3	1.240		376.2	1.132

provided in Table 5. Also in Table 5 the vector determined by $r_{13} = a_{13}$, the differences and the proportions between all pairs of their rents, are given.

This result can be explained as follows. The competitive rent vectors we think of are determined by the marginal households. Each marginal household $G(k)$ is indifferent between $(e^k, I_{G(k)} - r'_k)$ and $(e^{k+1}, I_{G(k)} - r'_{k+1})$. In this case he should pay a higher rent for the k th apartment than for the $(k+1)$ th because the k th is better. So the remaining income is lower in the case of renting the k th one than in that of the other. If both rents rise a little, but if the household also is indifferent between them, then the increment of the better one would be smaller than that of the other because he is paying the additional payment from his smaller budget. This is the implication of assumption E.

We considered the competitive rent vector of the special type, that defined by (12) or (13). But this can be representative of all competitive rent vectors. That is, an arbitrary one has similar properties and is not so different from that defined by (12) or (13). To show this, we need two theorems.

THEOREM 3. *Let (r, x) be an arbitrary competitive equilibrium. Then the following propositions hold.*

- (i) For every $j \in N$, $x^j = e^k$ for some $k \leq f$, and for every $i \in M$, $x_k^i = 0$ for all $k \leq f-1$.
- (ii) If $I_{j_1} < I_{j_2}$ ($j_1, j_2 \in N$), $x^{j_1} = e^{k_1}$ and $x^{j_2} = e^{k_2}$, then $k_1 \geq k_2$.
- (iii) $r_k \geq a_k$ for all $k = 1, \dots, f$ and $r_k \leq a_k$ for all $k = f+1, \dots, s$.
- (iv) $r_1 > r_2 > \dots > r_{f-1} > r_f$.

Proof. Note that if $r_k < a_k$, then landlords never lease any unit of the k th apartment and that if $r_k > a_k$, then landlords lease all units of the k th apartment.

First we show (ii). Suppose $k_1 < k_2$. By assumption F, $(e^{k_1}, 0)P(e^{k_2}, 0)$. Since (r, x) is a competitive equilibrium, we have $(e^{k_1}, I_{j_1} - r_{k_1})R(e^{k_2}, I_{j_1} - r_{k_2})$. By assumptions C and D there is a $\delta \geq 0$ such that $(e^{k_1}, I_{j_1} - r_{k_1})Q(e^{k_2}, I_{j_1} - r_{k_2} + \delta)$. By Lemma 1(ii), we have $I_{j_1} - r_{k_1} < I_{j_1} - r_{k_2} + \delta$. Then we have $(e^{k_1}, I_{j_2} - r_{k_1})P(e^{k_2}, I_{j_2} - r_{k_2} + \delta)R(e^{k_2}, I_{j_2} - r_{k_2})$ by assumptions E and C. This contradicts the assumption that (r, x) is a competitive equilibrium.

Next we show (i). Suppose that there is a household j with $x^j = 0$. If $x^{j'} = e^k$ for some $j' \in N$ with $I_{j'} < I_j$ and some $k \leq s$, then $(e^k, I_{j'} - r_k)R(0, I_j)$. Hence, we get $(e^k, I_j - r_k)P(0, I_j)$ by assumption E, which is a contradiction. So we have shown that if $x^j = 0$, then $x^{j'} = 0$ for all j' with $I_{j'} < I_j$. Hence, we can assume $x^{n'} = 0$ without loss of generality. If $r_f \leq a_f$, then $(e^f, I_n - r_f)P(0, I_n)$ by assumption G, which is a contradiction. So $r_f > a_f$. The total supply of the f th apartment is $\sum_{i \in M} w_f^i$. Hence there are $\sum_{i \in M} w_f^i$ number of households who rent the f th apartments. This and the supposition $x^{n'} = 0$ imply that some household j with $I_j \geq I_{G(f-1)}$ rents the f th apartment. Hence,

$$(e^f, I_j - r_f)R(e^{f-1}, I_j - r_{f-1}).$$

Since $(e^{f-1}, I_{G(f-1)} - p_{f-1})Q(e^f, I_{G(f-1)} - a_f)$ by (6) and $p_{f-1} > a_f$ by Lemma 2, we have $(e^{f-1}, I_j - p_{f-1})R(e^f, I_j - a_f)$ by assumption E. Since $r_f > a_f$, $(e^{f-1}, I_j - p_{f-1})R(e^f, I_j - a_f)P(e^f, I_j - r_f)R(e^{f-1}, I_j - r_{f-1})$, which implies $r_{f-1} > p_{f-1} > a_{f-1}$ by assumptions C and G. Repeating this argument, we get $r_t > a_t$ for all $t = 1, \dots, f$. Then the total supply $\sum_{k=1}^f \sum_{i \in M} w_k^i$ exceeds the number of households who rent one unit of apartment, because n' does not rent any apartment. This is a contradiction. Hence, every household rents one unit of apartment.

If some household j rents the k th apartment with $k > f$, then $(e^k, I_j - r_k)R(e^f, I_j - r_f)$. But since $(e^f, I_j - a_f)P(e^k, I_j - a_k)$ by Lemma 3 and $r_k \geq a_k$, $(e^f, I_j - a_f)P(e^k, I_j - a_k)R(e^k, I_j - r_k)R(e^f, I_j - r_f)$ which implies $a_f < r_f$. So the argument above is applicable to this case and a contradiction is derived. Hence, $x^j = e^k$ for some $k \leq f$. If $x_k^i > 0$ for some $i \in M$ and $k \leq f-1$, then there is a $j \in N$ by (ii) such that $I_j \geq I_{G(f-1)}$ and $x^j = e^f$. In this case we can prove analogously to the paragraph above that

$r_k > p_k > a_k$ for all $k \leq f-1$. Hence, we have $x_k^i = 0$ for all $k \leq f-1$ by the first remark of the proof. This is a contradiction.

Proposition (iii) follows proposition (i) and the note at the beginning of this proof.

Finally, we show (iv). Suppose $r_k \leq r_{k'}$ for some k and $k' \leq f$ with $k < k'$. Then no household rents the k' th apartment because $(e^k, I_j - r_k)P(e^{k'}, I_j - r_{k'})R(e^{k'}, I_j - r_{k'})$ by (3) and assumption C. This is a contradiction to (i). Q.E.D

To investigate the structure of the set of all competitive rent vectors, we define two concepts. We call $r = (r_1, \dots, r_s)$ the *maximal (minimal) competitive rent vector* iff

$$r \text{ is a competitive rent vector,} \quad (18)$$

$$\begin{aligned} &\text{for any competitive rent vector } r', \quad r_k \geq (\leq) r'_k \\ &\text{for all } k = 1, \dots, s. \end{aligned} \quad (19)$$

It should be noted that if the maximal (minimal) competitive rent vector exists, then it is unique.

THEOREM 4. (i)¹⁶ Let $G(f-1) < n < G(f)$. Then the maximal competitive rent vector is given by (12). Next let $r = (r_1, \dots, r_s)$ be defined by

$$\begin{aligned} &(e^f, I_{G(f-1)+1} - a_f)Q(e^{f-1}, I_{G(f-1)+1} - r_{f-1}) \\ &(e^{f-1}, I_{G(f-2)+1} - r_{f-1})Q(e^{f-2}, I_{G(f-2)+1} - r_{f-2}) \\ &\quad \vdots \\ &(e^2, I_{G(1)+1} - r_2)Q(e^1, I_{G(1)+1} - r_1) \end{aligned} \quad (20)$$

$r_f = a_f$ and $r_k = \max(0, p_k)$ for all $f < k \leq s$, where p_k 's are the numbers such that $(e^f, I_n - a_f)Q(e^k, I_n - p_k)$. If $r_k \geq a_k$ for all $k = 1, \dots, f-1$, then this (r_1, \dots, r_s) is the minimal competitive rent vector.

(ii) Let $G(f) = n$. The rent vector r determined by (13) and p_f^* is the maximal rent vector. The rent vector given by (20) is also the minimal competitive rent vector under the assumption that $r_k \geq a_k$ for all $k = 1, \dots, f-1$.

From Theorem 4 we get the following corollary.

COROLLARY 1. Let $G(f-1) < n < G(f)$. Suppose $I_{G(k)} = I_{G(k)+1}$ for all $k = 1, 2, \dots, f-1$. Then it holds that for any arbitrary competitive rent

¹⁶In fact, the maximal and minimal competitive rent vectors correspond to, respectively, the imputations (u^*, v_*) and (u_*, v^*) in the core of the assignment game given in Shapley and Shubik [16, Theorem 3].

vector r' ,

$$r'_k = r_k \quad \text{for all } k = 1, \dots, f \quad (21)$$

where r is the maximal competitive rent vector.

If n is large and if the potential supply $\sum_{i \in M} w_k^i$ of each apartment is large relative to the number of the kinds of apartments, f or s , then it seldom happens that $G(f) = n$. Moreover, it often happens in this case that $I_{G(k)} = I_{G(k)+1}$ for all $k = 1, \dots, f-1$. Although this does not hold exactly, it does hold approximately. These assumptions are plausible when we consider housing markets in towns that are not too small. Further, the rents of k th apartments ($k > f$) are not important because they are not rented. Therefore, we can regard the maximal competitive rent vector as a representative. In the next section we assume

$$(I) \quad G(f-1) < n < G(f).$$

Further, we employ the maximal competitive rent vector as a representative of the set of all competitive rent vectors to consider comparative statics.

Proof of Theorem 4. We show that r given by (12) of Theorem 1 is the maximal competitive rent vector in the case (i). We can prove the other propositions in a similar fashion, so we omit the proofs of them.

Let r' be an arbitrary competitive rent vector. Theorem 3(iii) says that $r'_k \leq a_k$ for all $k \geq f+1$. Suppose $r'_f > a_f$. Then the total supply of the f th apartment is $\sum_{i \in M} w_f^i$. Since r' is a competitive rent vector and $G(f-1) < n < G(f)$, there is a household j with $I_j \geq I_{G(f-1)}$ who rents one unit of the f th apartment. This implies $(e^f, I_j - r'_f)R(e^{f-1}, I_j - r'_{f-1})$. By Lemma 3 we have $(e^{f-1}, I_j - p_{f-1})R(e^f, I_j - a_f)$, where $p = (p_1, \dots, p_{f-1})$ is the vector defined by (5). Hence we get $(e^{f-1}, I_j - p_{f-1})R(e^f, I_j - a_f)P(e^f, I_j - r'_f)R(e^{f-1}, I_j - r'_{f-1})$ by assumption C, which implies $p_{f-1} < r'_{f-1}$. Repeating this argument we can get $r'_k > p_k > a_k$ for all $k \leq f$. This means that the total supply is at least $\sum_{k=1}^f \sum_{i \in M} w_k^i$, which exceeds n . This is a contradiction. Hence, we have shown that $r'_f = a_f$.

By Theorem 3 we can assume without loss of generality that in an arbitrary competitive equilibrium (r', x) each k th marginal household $G(k)$ ($k \leq f-1$) rents one unit of the k th apartment, that is, $x^{G(k)} = e^k$. Since r' is a competitive rent vector, it holds that $(e^{f-1}, I_{G(f-1)} - r'_{f-1})R(e^f, I_{G(f-1)} - a_f)$. But since $(e^{f-1}, I_{G(f-1)} - p_{f-1})Q(e^f, I_{G(f-1)} - a_f)$ by (5), we have $I_{G(f-1)} - r'_{f-1} \geq I_{G(f-1)} - p_{f-1}$ by assumption C, that is, $r'_{f-1} \leq p_{f-1} = r_{f-1}$. Therefore, we have $(e^{f-2}, I_{G(f-2)} - r'_{f-2})R(e^{f-1}, I_{G(f-2)} - r'_{f-1})R(e^{f-1}, I_{G(f-2)} - r_{f-1})Q(e^{f-2}, I_{G(f-2)} - r_{f-2})$ by

(C), (12), and the supposition that r' is a competitive rent vector. This implies $r'_{f-2} \leq r_{f-2}$ by (C). Similarly, we get $r'_k \leq r_k$ for all $k \leq f-1$.

Q.E.D.

4. COMPARATIVE STATICS

In this section effects of changes of certain parameters of the housing market (M, N) upon the competitive rent vectors are considered. Let (M^*, N^*) be a new housing market that is yielded by changes of certain parameters from the original housing market (M, N) , and that is also assumed to satisfy assumptions A-I. The expression $M^* = \{1^*, \dots, m^*\}$ is the set of all landlords, and $N^* = \{1^*, \dots, n^*\}$ is the set of all households. Of course, members of M^* or N^* may be different from those of M or N , respectively. But the kinds of apartments do not change, that is, there are s kinds of apartments. The landlords in M^* have the evaluation functions in the form given by (B) with the evaluation values a_1^*, \dots, a_s^* . The households in N^* have the same preference relation R as that of the households in N . As in Section 2, we denote the k th marginal household in (M^*, N^*) by $G^*(k)$, that is,

$$G^*(k) = \left(\sum_{i=1}^k \sum_{i \in M^*} w_i^i \right)^*. \quad (21)$$

Let f and f^* be the marginal apartments in (M, N) and (M^*, N^*) , respectively, that is, $G(f-1) < n < G(f)$ and $G^*(f^*-1) < n^* < G^*(f^*)$.

Here we should give a brief interpretation to our comparative statics. The two markets (M, N) and (M^*, N^*) may exist at different times, t and t^* , respectively ($t < t^*$). Each apartment appearing in (M, N) and (M^*, N^*) has a lease. Suppose that an apartment owned by a landlord in M is leased for T years to a householder at time t . If $t^* < t + T$, then the apartment does not appear in the market (M^*, N^*) unless the lease has been canceled by t^* . Hence, if the apartment appears in (M^*, N^*) , then $t^* \geq t + T$ or the lease is canceled before t^* and it has not been engaged between $t + T$ (or the time when the lease is canceled) and time t^* . Apartments appearing in (M^*, N^*) may not appear in (M, N) . They may be newly built or ones whose leases were made before t and have expired by t^* or have been canceled. Thus, we provided one interpretation of our comparative statics. But it is easier and does not yield any conceptual difficulty to interpret (M, N) and (M^*, N^*) as two different cities.¹⁷

The main result of this section is the following theorem.

¹⁷It is conceptually difficult to construct an argument that makes our interpretation complete or consistent in a dynamic situation. Still, the author thinks that the first interpretation is valuable and helps in the understanding of housing markets.

THEOREM 5. Let $f^* \geq f$, and assume

$$I_{G^*(1)} - I_{G(1)} \geq I_{G^*(2)} - I_{G(2)} \geq \cdots \geq I_{G^*(f-1)} - I_{G(f-1)}.^{18} \quad (22)$$

Let r and r^* be the maximal competitive rent vectors in (M, N) and (M^*, N^*) respectively. Let k be a number with $1 \leq k \leq f-1$. Then the following propositions hold.

- (i) $I_{G^*(k)} - I_{G(k)} \geq r_k^* - r_k$ if and only if $r_k^* - r_k \leq r_{k+1}^* - r_{k+1}$.
- (ii) If $r_k^* - r_k > r_{k+1}^* - r_{k+1}$, then $r_1^* - r_1 > r_2^* - r_2 > \cdots > r_k^* - r_k$.
- (iii) If $r_k^* - r_k < r_{k+1}^* - r_{k+1}$, then $r_{k+1}^* - r_{k+1} < \cdots < r_f^* - r_f$.

Proof. (i) Let $I_{G^*(k)} - I_{G(k)} < r_k^* - r_k$. Let $b_k = (r_k^* - r_k) - (I_{G^*(k)} - I_{G(k)})$. Since $(e^k, I_{G^*(k)} - r_k^*)Q(e^{k+1}, I_{G^*(k)} - r_{k+1}^*)$ by (6) and $r_k^* > r_{k+1}^*$, we get $(e^{k+1}, I_{G(k)} - r_{k+1})Q(e^k, I_{G(k)} - r_k) = (e^k, I_{G^*(k)} - r_k^* + b_k)P(e^{k+1}, I_{G^*(k)} - r_{k+1}^* + b_k)$ by assumption E and (6). This implies $I_{G(k)} - r_{k+1} > I_{G^*(k)} - r_{k+1}^* + b_k$, that is, $r_{k+1}^* - r_{k+1} > r_k^* - r_k$. Let $I_{G^*(k)} - I_{G(k)} > r_k^* - r_k$. Then we put $b_k = (I_{G^*(k)} - I_{G(k)}) - (r_k^* - r_k)$. Since $(e^k, I_{G(k)} - r_k)Q(e^{k+1}, I_{G(k)} - r_{k+1})$ by (6) and $r_k > r_{k+1}$, we get $(e^{k+1}, I_{G^*(k)} - r_{k+1}^*)Q(e^k, I_{G^*(k)} - r_k^*) = (e^k, I_{G(k)} - r_k + b_k)P(e^{k+1}, I_{G(k)} - r_{k+1} + b_k)$ by assumption E and (6). This implies $I_{G^*(k)} - r_{k+1}^* > I_{G(k)} - r_{k+1} + b_k$, that is, $r_k^* - r_k > r_{k+1}^* - r_{k+1}$. Similarly we can prove that $I_{G^*(k)} - I_{G(k)} = r_k^* - r_k$ implies $r_k^* - r_k = r_{k+1}^* - r_{k+1}$.

(ii) Let $r_k^* - r_k > r_{k+1}^* - r_{k+1}$. Then we get, by (i) and (4.2), $I_{G^*(k-1)} - I_{G(k-1)} \geq I_{G^*(k)} - I_{G(k)} > r_k^* - r_k$. Let $b_k = (I_{G^*(k-1)} - I_{G(k-1)}) - (r_k^* - r_k)$. Since $(e^{k-1}, I_{G(k-1)} - r_{k-1})Q(e^k, I_{G(k-1)} - r_k)$ by (6) and $r_{k-1} > r_k$, we get $(e^{k-1}, I_{G(k-1)} - r_{k-1} + b_k)P(e^k, I_{G(k-1)} - r_k + b_k) = (e^k, I_{G^*(k-1)} - r_k^*)Q(e^{k-1}, I_{G^*(k-1)} - r_{k-1}^*)$ by assumption E and (6). This implies $I_{G(k-1)} - r_{k-1} + b_k > I_{G^*(k-1)} - r_{k-1}^*$, that is $r_{k-1}^* - r_{k-1} > r_k^* - r_k$. We can repeat the above argument. So we get the result of (ii).

(iii) Let $r_k^* - r_k < r_{k+1}^* - r_{k+1}$. Then we get, by (i) and (22), $I_{G^*(k+1)} - I_{G(k+1)} \leq I_{G^*(k)} - I_{G(k)} < r_k^* - r_k < r_{k+1}^* - r_{k+1}$. Hence it follows from (i) that $r_{k+1}^* - r_{k+1} < r_{k+2}^* - r_{k+2}$. We repeat this argument, and so we get the result of (iii). Q.E.D.

Although this theorem provides a general criterion for changes in rents, the causal relations between changes in parameters and those in rents are not clear. So, we should consider the causal relations.

COROLLARY 2. Assume that the marginal apartments in (M, N) and (M^*, N^*) are the same, that is, $f = f^*$ and $a_f = a_f^*$. Assume that

$$I_{G^*(1)} - I_{G(1)} \geq \cdots \geq I_{G^*(f-1)} - I_{G(f-1)} > 0. \quad (23)$$

Let r and r^* be the maximal competitive rent vectors in (M, N) and

¹⁸In the case of $f^* < f$, this theorem remains true, replacing f by f^* .

(M^*, N^*) , respectively. Then

$$I_{G^*(k)} - I_{G(k)} > r_k^* - r_k \quad \text{for all } k = 1, \dots, f-1, \quad (24)$$

$$r_1^* - r_1 > \dots > r_{f-1}^* - r_{f-1} > 0.^{19} \quad (25)$$

Proof. It is clear that $r_f = r_f^* = a_f = a_f^*$. Let $r_{f-1}^* \leq r_{f-1}$. Then since $(e^{f-1}, I_{G(f-1)} - r_{f-1})Q(e^f, I_{G(f-1)} - a_f)$ and $r_{f-1} > a_f$, we get, by assumption E, $(e^{f-1}, I_{G^*(f-1)} - r_{f-1})P(e^f, I_{G^*(f-1)} - a_f)$. Further we have $(e^{f-1}, I_{G^*(f-1)} - r_{f-1}^*)R(e^{f-1}, I_{G^*(f-1)} - r_{f-1})$ by assumption C. Hence, $(e^{f-1}, I_{G^*(f-1)} - r_{f-1}^*)P(e^f, I_{G^*(f-1)} - a_f)$, which contradicts (6). Hence, we get $r_{f-1}^* > r_{f-1}$. Hence, $r_{f-1}^* - r_{f-1} > r_f^* - r_f = 0$. Therefore, we get (24) and (25) by Theorem 5(i) and (ii). Q.E.D.

This corollary states the following. Assume that the marginal apartment and the landlords' evaluation value for it are the same, and that for each $k < f$, the income of the k th marginal household rises more than that of the $(k+1)$ th marginal household. Then the increments in the maximal competitive rents of more preferred apartments are larger than those of less preferred ones. According to (24) the increments in the rents do not exceed those in the incomes. These results depend upon assumption E. The dependence can be explained intuitively as follows. Consider the situation where household j is indifferent between $(e^k, I_j - r_k)$ and $(e^{k+1}, I_j - r_{k+1})$. Since the k th apartment is better than the $(k+1)$ th, r_k is higher than r_{k+1} . Hence, the consumption level is lower in $(e^k, I_j - r_k)$ than in $(e^{k+1}, I_j - r_{k+1})$. If his income rises, but if both the rents rise in the same magnitude, which is less than the increment in income, then he chooses the k th apartment, because the increment in $(e^k, I_j - r_k)$ makes him more comfortable than in $(e^{k+1}, I_j - r_{k+1})$. Hence, the demand for the better apartment increases and violates the market clearing condition. So the rent of the better apartment rises more than that of the worse (see Fig. 2).

¹⁹Consider the case where the price of money (the composite commodity) changes into $p > 1$. Let $a_k^* = a_k$ for all $k = 1, \dots, s$ and $f = f^*$. Note that the evaluation values a_k or a_k^* reflect the costs of building the apartments measured in terms of money. See footnote 3. Furthermore we assume that the nominal incomes of marginal households do not change, that is, $I_{G(k)} = I_{G^*(k)}$ for all $k = 1, \dots, f-1$. Corollary 2 is applicable to these markets. Transforming the price p into 1, we get a housing market (M^{**}, N^{**}) such that $a_k^{**} = a_k$ for all $k = 1, \dots, s$ and $I_{G^{**}(k)} = I_{G(k)}/p$ for all $k = 1, \dots, f-1$. Of course, (M^{**}, N^{**}) is the same as (M^*, N^*) except the units of the prices. If $r_1^{**}, \dots, r_s^{**}$ are the maximal competitive rents, then it holds that $r_1 - r_1^{**} > \dots > r_{f-1} - r_{f-1}^{**} > 0$. That is, all of the real rents decrease and further the real rents of better apartments decrease more. But the nominal rents $pr_1^{**}, pr_2^{**}, \dots, pr_s^{**}$ have the property that $r_k < pr_k^{**}$ for all $k \geq f$. Therefore, there may exist the possibility that the nominal rents of worse apartments rise, but simultaneously those of better ones decrease. Regretfully, the author has not found any example for this possibility. In all his examples, it holds that $r_k < pr_k^{**}$ for all $k = 1, \dots, s$.

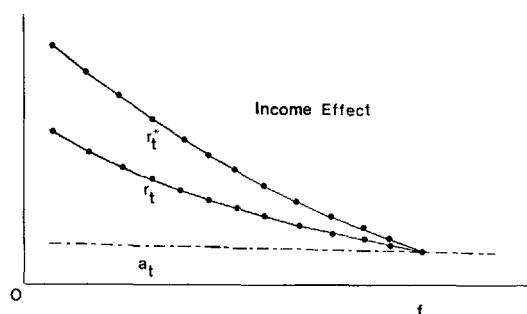


FIGURE 2

This corollary is applicable to the case where the income levels increase proportionally, that is, for some a , $I_{G^*(k)} = (1 + a)I_{G(k)}$ for all $k \leq f - 1$. This case is considered in the following example.

EXAMPLE 4. Let us consider the housing market (M, N) given in Example 2. Let $I_{G^*(k)} = 1.1 I_{G(k)}$ for all $k \leq 12$, which are written in Table 6. Let the other parameters be fixed. Then the maximal competitive rent vectors are given in Table 7.

Although the income of each marginal household rises uniformly by 10%, the increments in the rents are not equal. The increment of the most preferred apartment is 4.6% but that of the 12th one is 0.7%. In this example we get the interesting result that the maximal competitive rent of a more preferred apartment rises more than that of a lesser one, both absolutely and proportionally. But the proposition with respect to proportion is not generally true.

This corollary seems to contradict the result of Wheaton [20] and Hartwick, *et al.* [6] that the rent curve becomes flatter as income increases.

TABLE 6

<table><tr><td>k</td><td>$I_{G^*(k)}$</td></tr></table>										k	$I_{G^*(k)}$
k	$I_{G^*(k)}$										
<table><tr><td>s</td><td>t</td></tr></table>	s	t	1		2		3		4		
s	t										
10	10	1100	4	1760							
20	13	770	6	1540					2	1980	
30	14	ϕ	9	1210	5	1650	1	2090			
40	16	ϕ	12	880	8	1320	3	1870			
50	17	ϕ	15	ϕ	11	990	7	1430			

TABLE 7

		r_k^*		$r_k^* - r_k$		r_k		r_k^*/r_k	
k									
s	t	1		2		3		4	
10	10	246.5	4.9	4	504.8	19.7			
		241.6	1.020		485.1	1.041			
20	13	160	0	6	396.8	13.6	2	613.9	25.9
		160	1		383.2	1.0355		588.0	1.044
30	14	140	0	9	289.2	7.4	5	489.2	18.8
		140	1		281.8	1.026		470.4	1.040
40	16	120	0	12	180.0	1.2	8	361.4	11.5
		120	1		178.8	1.007		349.9	1.033
50	17	100	0	15	140	0	11	230.4	4.1
		100	1		140	1		226.3	1.018
								7	389.3
									376.2
									1.035

But since our rent curve is drawn with respect to quality but theirs to distances, there is no contradiction. Further, note that ours are rents for apartments but theirs are land rents per unit square.

LEMMA 4. Let $f \leq f^*$. Assume that $I_{G^*(k)} = I_{G(k)}$ for all $k \leq f - 1$. If $r_k^* > r_k$ for some $k \leq f$, then

$$r_t^* > r_t \text{ for all } t \leq k. \quad (26)$$

Proof. Let $r_k^* > r_k$. Since $(e^{k-1}, I_{G(k-1)} - r_{k-1})Q(e^k, I_{G(k-1)} - r_k)P(e^k, I_{G(k-1)} - r_k^*)Q(e^{k-1}, I_{G(k-1)} - r_{k-1}^*)$ by (6) and (C), we have $r_{k-1} < r_{k-1}^*$. Repeating this argument, we get (26). Q.E.D.

COROLLARY 3. Assume that $f = f^*$, and that $I_{G^*(k)} = I_{G(k)}$ for all $k = 1, \dots, f - 1$. And assume that $a_f^* > a_f$. Let r and r^* be the maximal competitive rent vectors in (M, N) and (M^*, N^*) , respectively. Then it holds that

$$0 < r_1^* - r_1 < \dots < r_{f-1}^* - r_{f-1} < a_f^* - a_f, \quad (27)$$

$$1 < r_1^*/r_1 < \dots < r_{f-1}^*/r_{f-1} < a_f^*/a_f. \quad (28)$$

Proof. Since $r_f = a_f < a_f^* = r_f^*$, we get, by Lemma 4, $r_t^* > r_t$ for all $t \leq f$. Hence, $r_1^* - r_1 > I_{G^*(1)} - I_{G(1)} = 0$, which implies that Theorem 5(iii)

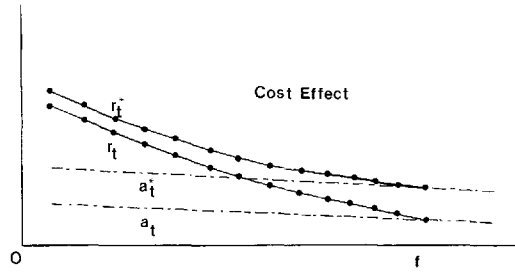


FIGURE 3

is applicable to this case. Therefore, we get (27). Since $r_1 > r_2 > \dots > r_f$ by Lemma 2, we get (28). Q.E.D.

If the marginal apartment and the income levels of the marginal households do not change, but if the evaluation value a_f of the marginal apartment rises, the maximal competitive rents also rise. In this case, however, the increments in rents have a different and converse tendency from those of Corollary 2. That is, the rent increment of a less preferred apartment is larger than that of a more preferred one. This is the same as Theorem 2. As the evaluation value a_f reflects the cost of building one unit of the marginal apartment f , we may think that the increment of the evaluation value a_f is that of the cost of building the apartment.²⁰ An increment of the cost of building the marginal apartment is one reason for the rises in the rents given in Corollary 3 (see Fig. 3).

This result could be explained in the same way as Theorem 2. Example 3 can become an example for this corollary with little change. So, we do not give any example here.

Wheaton [20] and Hartwick *et al.* [6] showed also in their model that when the population increases the rents rise. But they did not provide any explicit shape of increments in rents.

COROLLARY 4. Assume that $a_f^* \geq a_f$, and that $I_{G(k)} = I_{G^*(k)}$ for all $k \leq f-1$. And assume that $f < f^*$. Let r and r^* be the maximal competitive rent vectors in (M, N) and (M^*, N^*) , respectively. Then it holds that

$$0 < r_1^* - r_1 < \dots < r_f^* - r_f, \quad (29)$$

$$1 < r_1^*/r_1 < \dots < r_f^*/r_f. \quad (30)$$

Proof. Since $r_f^* > a_f^* \geq a_f = r_f$ by assumption G, we have, by Lemma 4, $r_t^* > r_t$ for all $t \leq f$. Hence, $r_1^* - r_1 > I_{G^*(1)} - I_{G(1)} = 0$, which implies that

²⁰See footnote 5.

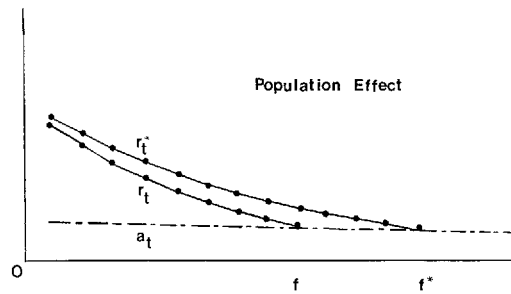


FIGURE 4

Theorem 5(iii) is applicable to this case. Therefore, we get (29). Since $r_1 > r_2 > \dots > r_f$, by Lemma 2, we get (30). Q.E.D.

If the incomes of the marginal households and the landlords' evaluation values do not change, but the marginal apartment becomes a worse one, the increments in rents have the same tendency as those of Corollary 3. That is, the rent of a less preferred apartment rises more than that of a more preferred one. This may occur when the population of an urban area participating in the housing market increases, that is, when the demand of apartments increases more than the supply (see Fig. 4). The crucial assumption of this corollary is that $I_{G^*(k)} = I_{G(k)}$ for all $k > f$, that is, as the population of the area increases, then increments of the population are in the group of households with lower income levels. Hence, this corollary can say nothing about the exact outcome in the case where the population increases uniformly on all groups. If we want to consider such a case, then we should take into account the effect of changes in the marginal households' incomes as in Corollary 2.

Wheaton [20] considered the same problem as that of Corollary 4 in his model. He showed only that if the land rent at the boundary of the city rises, the land rent of every point in the city also rises.

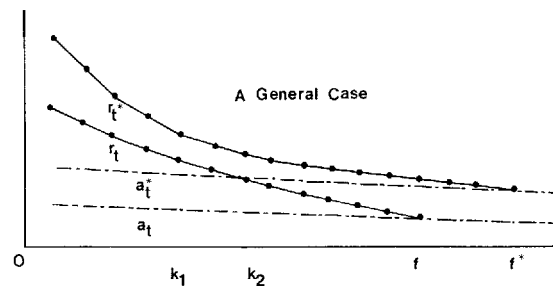


FIGURE 5

The effect of a change of each parameter upon the maximal competitive rent vector has been investigated. But Theorem 5 can be applied to the case where several parameters change simultaneously. Corollary 5 says that there are three possibilities, (25), (27), and a compromise between these two (see Fig. 5).

COROLLARY 5. *Let $f \leq f^*$. Assume (22). Let r and r^* be the maximal competitive rent vectors in (M, N) and (M^*, N^*) . Then there are k_1 and k_2 such that $1 \leq k_1 \leq k_2 \leq f$ and*

$$r_1^* - r_1 > r_2^* - r_2 > \cdots > r_{k_1}^* - r_{k_1}, \quad (31)$$

$$r_{k_1}^* - r_{k_1} = r_{k_1+1}^* - r_{k_1+1} = \cdots = r_{k_2}^* - r_{k_2} \quad (32)$$

$$r_{k_2}^* - r_{k_2} < r_{k_2}^* - r_{k_2} < \dots < r_f^* - r_f. \quad (33)$$

Proof. Obvious from Theorem 5. \square

EXAMPLE 5. Let us consider the housing market (M, N) given in Example 2. Let $I_{G^*(k)} = (1.1)I_{G(k)}$ for all $k \leq 12$, which are given in Table 6. Let $f = f^*$ and $a_{13}^* = 260$. Then we get Table 8. Approximately, it holds in this example that $k_1 = 10$ and $k_2 = 11$.

Smith [17] considered similar comparative statics in his “filtering model,” though his model is a numerical example and very restricted. Since our

TABLE 8

		k		r_k^*		$r_k^* - r_k$						
				r_k		r_k^*/r_k						
s	t	1		2		3		4				
10	10	341.0	99.4	4	588.4	103.3						
		241.6	1.411		485.1	1.213						
20	13	260	100	6	484.2	101.0	2	694.2	106.2			
		160	1.625		383.2	1.264		588.0	1.181			
30	14	240	100	9	381.5	99.7	5	573.3	102.9	1	757.7	108.1
		140	1.714		281.8	1.353		470.4	1.219		649.6	1.166
40	16	220	100	12	278.5	99.7	8	450.3	100.4	3	620.8	104.2
		120	1.833		178.8	1.558		349.9	1.287		516.6	1.202
50	17	200	100	15	240	100	11	325.7	99.4	7	477.1	100.9
		100	2.00		140	1.714		226.3	1.439		376.2	1.268

model is slightly different and his comparative statics are more special than ours, we cannot directly compare our result with his. But it would be interesting to consider comparative statics in more precise numerical examples in terms of our theory. Then it would be valuable to consider Smith's problem in numerical examples.

5. A REMARK ON HOUSING MARKETS WITH EXTERNALITIES

In the housing-market model of the previous sections, direct interactions among households in the form of externalities were not investigated. Such externalities may be an important element of housing consumption. If externalities among households are introduced into our model, then a competitive equilibrium may not exist. The market model is not already an assignment market and Kaneko's [11] existence theorem of the core cannot be applied to this case. Therefore, even the core of the market may be empty. Schotter [15] considered this kind of problem and provided a necessary condition for the nonemptiness of the core of a simple housing market with externalities. In this section we introduce a special type of externalities among households into our market model without loss of the essence of the previous sections.

Let τ be a partition of $\{1, 2, \dots, s+1\}$. The expression $S \in \tau$ means a housing area in which apartments in S are located. Every householder's preference ordering R depends upon his apartment, consumption, and the environment of $S \in \tau$ in which his apartment is located. We assume that the environment of S is represented by the average income of the householders' income levels who rent apartments in S . Therefore, householders' preference ordering R is defined as $Y = \{0, e^1, \dots, e^s\} \times R_+ \times R_+$. We make the following assumptions.

- (C') for all $(x, m, \bar{I}) \in Y$, if $\delta > 0$, then $(x, m + \delta, \bar{I})P(x, m, \bar{I})$,
- (D') if $(x, m_1, \bar{I})P(y, m_2, \bar{I})$, then there is an $m_3 \geq 0$ such that $(x, m_1, \bar{I})Q(y, m_3, \bar{I})$,
- (E') if $(x, m_1, \bar{I})Q(y, m_2, \bar{I})$, $m_1 < m_2$ and $\delta > 0$, then $(x, m_1 + \delta, \bar{I})P(y, m_2 + \delta, \bar{I})$.

Let $x = (x^1, \dots, x^m, x^{1'}, \dots, x^{n'})$ be a vector which satisfies (8) and (9). We call x an *allocation*. The *average income* \bar{I}^{xS} of area $S \in \tau$ with respect to x is defined by

$$\bar{I}^{xS} = \sum_{i \in S} \sum_{x^j = e^i} I_j / \sum_{i \in M} \sum_{i \in S} w_i^i. \quad (34)$$

²¹ Schotter [15] also considered the relations between his model and Koopmans–Beckmann's and Heffley's. See [15, p. 324, footnotes 2 and 3].

²² It is assumed for notational simplicity that $w_{s+1}^i = 0$ for all $i \in M$ and $\{s+1\} \notin \tau$.

For notational simplicity, we define $\bar{I}^x = (\bar{I}^{x^1}, \dots, \bar{I}^{x^s}, \bar{I}^{x^{(s+1)}})$ by

$$\bar{I}^{xt} = \bar{I}^{xs} \quad \text{for all } t \in S. \quad (35)$$

Here let us consider a special allocation $\bar{x} = (\bar{x}^1, \dots, \bar{x}^m, \bar{x}^{1'}, \dots, \bar{x}^{n'})$ which is defined by (14) and (15). Using this allocation \bar{x} , we assume

$$(F') \quad (e^1, 0, \bar{I}^{\bar{x}^1})P(e^2, 0, \bar{I}^{\bar{x}^2})P \dots P(e^s, 0, \bar{I}^{\bar{x}^s}).$$

Similar to (6), we define (p_1, \dots, p_{f-1}) by

$$\begin{aligned} (e^f, I_{G(f-1)} - a_f, \bar{I}^{\bar{x}^f})Q(e^{f-1}, I_{G(f-1)} - p_{f-1}, \bar{I}^{\bar{x}^{(f-1)}}) \\ \vdots \\ (e^2, I_{G(1)} - p_2, \bar{I}^{\bar{x}^2})Q(e^1, I_{G(1)} - p_1, \bar{I}^{\bar{x}^1}). \end{aligned} \quad (6')$$

Analogously to Lemma 2, we can prove that (6') has a unique solution under the following condition G':

$$\begin{aligned} (G') \quad I_n \geq a_f, (e^f, I_n - a_f, \bar{I}^{\bar{x}^f})P(e^1, 0, \bar{I}^{\bar{x}^1}) \text{ and} \\ (e^f, I_n - a_f, \bar{I}^{\bar{x}^f})P(e^k, I_n - a_k, \bar{I}^{\bar{x}^k}) \\ \text{for all } k (f < k \leq s+1) \text{ with } I_n \geq a_k. \end{aligned}$$

Here we modify the definition of competitive equilibrium as follows.

DEFINITION $(r, x) = (r_1, \dots, r_s, x^1, \dots, x^m, x^{1'}, \dots, x^{n'})$ is said to be a *competitive equilibrium in the housing market with externalities* iff

$$x = (x^1, \dots, x^m, x^{1'}, \dots, x^{n'}) \quad \text{is an allocation,} \quad (36)$$

$$\text{for all } i \in M, u^i(x^i) + r(w^i - x^i) = \max_{z \in X^i} (u^i(z) + r(w^i - z)), \quad (37)$$

$$\begin{aligned} \text{for all } j \in N, rx^j \leq I_j, (x^j, I_j - rx^j, \bar{I}^{xt})R(e^{t'}, I_j - re^{t'}, \bar{I}^{xt'}) \\ \text{for all } t' \text{ with } I_j \geq re^{t'}, \text{ where } x^j = e^{t'}. \end{aligned} \quad (38)$$

The important point of this definition is (38). We assume that each household's preference depends upon the average income of the area in which his apartment is located but that he has no influence to the average income. This is justified by the assumption of "perfect competitive market." In other words, we assume that the number of apartments potentially supplied in an area is very large, and so, each household's externalities are negligible.

We can get the following theorem.

THEOREM 1'. Assume conditions A, B, C', D', E', F', G', and H. Then the analog of Theorem 1 is true, that is, the rent vector $r = (r_1, \dots, r_s)$ which is given in the following (i) or (ii) is a competitive rent vector in each case.

$$(i) \quad \begin{aligned} & \text{If } G(f-1) < n < G(f), \text{ then} \\ & r_k = p_k \quad \text{if } k < f \\ & \quad = a_k \quad \text{otherwise,} \end{aligned} \quad (39)$$

where (p_1, \dots, p_{f-1}) is the vector defined by (6').

(ii) If $G(f) = n$, then r is defined by

$$\begin{aligned} & r_k = a_k \text{ for all } k \geq f+1, a_f \leq r_f \leq p_f^* \text{ and } (e^k, I_{G(k)} - r_k, \bar{I}^{\bar{x}k})Q \\ & (e^{k+1}, I_{G(k)} - r_{k+1}, \bar{I}^{\bar{x}(k+1)}) \text{ for all } k = 1, 2, \dots, f-1, \end{aligned} \quad (40)$$

where p_f^* is defined by $(e^f, I_n - p_f^*, \bar{I}^{\bar{x}f})Q(e^t, I_n - a_t, \bar{I}^{\bar{x}t})$ for some t such that $t \geq f+1$ and $(e^t, I_n - a_t, \bar{I}^{\bar{x}t})R(e^k, I_n - a_k, \bar{I}^{\bar{x}k})$ for all k with $k \geq f+1$ and $I_n \geq a_k$ if such a p_f^* exists and otherwise $p_f^* = I_n$.

Since we can prove this theorem in almost the same way as the proof of Theorem 1, we omit the proof.²³

Thus the essence of the previous sections is valid even if we introduce externalities among households, though the externalities are very special. The author thinks that this kind of model with externalities is important in housing markets, and plans to investigate it in another paper.

6. CONCLUSION

We constructed a simple mathematical model of the housing market in which apartments are treated as indivisible commodities and all the other commodities are treated as a single composite commodity. We got the constructive proof, the recursive equation (6), of the existence of a competitive equilibrium and argued that the equilibrium given by the equation could be regarded as representative of all competitive equilibria. Further we showed that there are several but limited tendencies of variations in rents when parameters change. As indicated, this is due primarily to the assumption of diminishing marginal utility, assumption E.

As pointed out, our analysis made many assumptions for simplification. For example, the theory of this paper is applicable to the case where there is a unique transportation system and all households go the same number of times to the center. Without these the analysis becomes much more difficult. Although Kaneko [11] provided several general properties, for example, the

²³The analog of Theorem 2 is also true in the market model of this section, but the other theorems do not necessarily have analogs.

existence of a competitive equilibrium, it cannot answer explicitly what happens in the market. For the purpose of analysis of a general housing market, we would need the algorithm for calculation of a competitive equilibrium because we cannot expect to get an explicit equation such as (6) that determines a competitive rent vector. In future work the author plans to construct such an algorithm.

ACKNOWLEDGMENTS

The author thanks K. Iwai, E. S. Mills, J. Beggs, and a referee of this journal for valuable discussions and comments on earlier drafts. The research in this paper was partially sponsored by the Office of Naval Research Contract N00014-77-C-0518.

APPENDIX

Proof of Lemma 1. (i) Suppose $(y, m_2 - \delta)R(x, m_1 - \delta)$. By assumptions C and D, there is a $b \geq 0$ such that $(y, m_2 - \delta)Q(x, m_1 - \delta + b)$. Since $m_1 - \delta + b > m_2 - \delta$, we have $(y, m_2)P(x, m_1 + b)$ by assumption E. But by assumption C we have $(x, m_1 + b)R(x, m_1)Q(y, m_2)$, that is, $(x, m_1 + b)R(y, m_2)$, which is a contradiction.

(ii) By assumptions C and D there is a $b > 0$ such that $(e^k, 0)Q(e^t, b)$. By assumption E, $(e^k, m_1)P(e^t, m_1 + b)$. Hence, $(e^t, m_2)Q(e^k, m_1)P(e^t, m_1 + b)$. This implies $m_2 > m_1 + b > m_1$ by assumption C. Q.E.D.

Proof of Lemma 2. Since $(e^f, I_{G(f-1)} - a_f)R(e^f, I_n - a_f)P(e^1, 0)P \dots P(e^{f-1}, 0)$ by assumptions C, F, and the supposition of this lemma, there is a b_{f-1} such that $(e^f, I_{G(f-1)} - a_f)Q(e^{f-1}, b_{f-1})$. This b_{f-1} is unique by assumption C. Let $p_{f-1} = I_{G(f-1)} - b_{f-1}$. Since $(e^{f-1}, I_{G(f-2)} - p_{f-1})R(e^{f-1}, I_{G(f-1)} - p_{f-1})Q(e^f, I_{G(f-1)} - a_n)P(e^1, 0)P \dots P(e^{f-2}, 0)$ by assumptions C, F, and the supposition of this lemma, there is a b_{f-2} such that $(e^{f-1}, I_{G(f-2)} - p_{f-1})Q(e^{f-2}, b_{f-2})$. Let $p_{f-2} = I_{G(f-2)} - b_{f-2}$. This b_{f-2} is also unique. Repeating this argument we get (p_1, \dots, p_{f-1}) . Lemma 1(ii) implies that if $(e^k, I_{G(k)} - p_k)Q(e^{k+1}, I_{G(k)} - p_{k+1})$, then $I_{G(k)} - p_k < I_{G(k)} - p_{k+1}$, that is, $p_k > p_{k+1}$. This is (6). Q.E.D.

Proof of Lemma 3. Let $k \leq t \leq f - 1$. Since $(e^t, I_{G(t)} - p_k)Q(e^{t+1}, I_{G(t)} - p_{k+1})$ and $p_k > p_{k+1}$, we have, by assumption E, $(e^t, I - p_k)R(e^{t+1}, I - p_{k+1})$. This is the first proposition. Let $t > f$ and let $I_n \geq a_t$. Since $(e^f, I_n - a_f)P(e^t, I_n - a_t)$ by assumption G, there is a $b_t > 0$ by assumptions C and D such that $(e^f, I_n - a_f)Q(e^t, I_n - a_t + b_t)$. By assumption H and Lemma 2, $I_n - a_f < I_n - a_t + b_t$. Hence, we have, by assumption E, $(e^f, I - a_f)R(e^t, I - a_t + b_t)P(e^t, I - a_t)$. Let $I \geq a_t > I_n$. Since $(e^f, 0)P(e^t, 0)$ by assumption F, there is a $b_t > 0$ such that $(e^f, 0)Q(e^t, b_t)$. Hence, $(e^f, I - a_t)R(e^t, I - a_t + b_t)P(e^t, I - a_t)$. Since $I - a_f \geq I_n - a_f \geq 0$ but $a_t > I_n$, we have $I - a_f > I - a_t$, which implies $(e^f, I - a_f)P(e^f, I - a_t)$. So we have $(e^f, I - a_f)P(e^t, I - a_t)$.

Let $t < k$. Then, since $(e^t, I_{G(t)} - p_t)Q(e^{t+1}, I_{G(t)} - p_{t+1})$ and $p_t > p_{t+1}$, we have, by Lemma 2(i), $(e^{t+1}, I - p_{t+1})R(e^t, I - p_t)$ if $p_t \leq I$. Q.E.D.

REFERENCES

1. W. Alonso, "Location and Land Use," Harvard Univ. Press, Cambridge (1964).
2. M. J. Beckmann, On the distribution of urban rent and residential density, *J. Econ. Theory*, **1**, 60-67 (1969).
3. E. von Böhm-Bawerk, "Positive Theory of Capital" (W. Smart, translator) Arno, New York (1923) (original publication, 1891).
4. D. Gale, "Theory of Linear Economic Models," McGraw-Hill, New York (1960).
5. J. M. Hartwick, Price sustainability of location assignments, *J. Urban Econ.*, **1**, 147-160 (1974).
6. J. M. Hartwick, U. Schweizer, and P. Varaiya, Comparative statics of a residential economy with several classes, *J. Econ. Theory*, **13**, 396-413 (1976).
7. D. Heffley, Efficient spatial allocation in the quadratic assignment problem, *J. Urban Econ.*, **3**, 309-322 (1976).
8. J. MacKinnon, Urban general equilibrium models and simplicial search algorithms, *J. Urban Econ.*, **1**, 161-183 (1974).
9. M. Kaneko, On the core and competitive equilibria of a market with indivisible goods, *Naval Res. Logistics. Quart.*, **23**, 321-337 (1976).
10. M. Kaneko, Note on transferable utility, *Int. J. Game Theory*, **5**, 183-185 (1976).
11. M. Kaneko, The Central Assignment Game and the Assignment Markets, *J. Math. Economics*, **10** (1982).
12. T. C. Koopmans and M. J. Beckmann, Assignment problems and the location of economic activities, *Econometrica*, **25**, 53-76 (1957).
13. A. Montesano, A restatement of Beckmann's model on the distribution of urban rent and residential density, *J. Econ. Theory*, **4**, 329-354 (1972).
14. M. Romanos, Household location in a linear multi-center metropolitan area, *Reg. Sci. Urban Econ.*, **7**, 233-250 (1977).
15. A. Schotter, On urban residential stability and the core, *Regional Sci. Urban Econ.*, **7**, 321-337 (1977).
16. L. S. Shapley and M. Shubik, The assignment game I: The core, *Int. J. Game Theory*, **1**, 111-130 (1972).
17. W. F. Smith, A Theory of Filtering, "Readings in Urban Economics" (M. Edel and J. Rothenberg, Eds., Macmillan Co., New York, pp. 193-204 (1972).
18. J. L. Sweeney, Quality, commodity hierarchies, and housing markets, *Econometrica*, **42**, 147-167 (1974).
19. L. Telser, "Competition, Collusion and Game Theory," Aldine, Chicago (1972).
20. W. C. Wheaton, A comparative static analysis of urban spatial structure, *J. Econ. Theory*, **9**, 223-237 (1974).
21. J. von Neumann and O. Morgenstern, "Theory of Games and Economic Behavior," 3rd ed., Princeton Univ. Press, Princeton (1953).