

# INFORMATION PROTOCOLS AND EXTENSIVE GAMES IN INDUCTIVE GAME THEORY \*

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## 1. Introduction

In this paper, we introduce a new mathematical representation of an extensive game situation, called an *information protocol*, without using the hypothetical concepts such as nodes and branches. The necessity of introducing such a new construct has been emerging in our study of inductive game theory. Inductive game theory was initiated in Kaneko-Matsui [6] and was developed more extensively in Kaneko-Kline [4] and [5]. Here, we first review the basic motivation of inductive game theory and how we have found the necessity of a new representation in the research in inductive game theory.

The basic motivation of inductive game theory is to ask how a player obtains his initial beliefs/knowledge of the game situation from experiences accumulated by playing the game. Its salient feature is the treatment of initial beliefs about the structure, and not the adjustment or convergence of parameter values in response to his experiences. The latter is a typical question in the game theory/economics literature, but this needs the presumption that each player already has beliefs/knowledge of the game structure. Contrary to this, it is the basic assumption of inductive game theory that any player has no *a priori* beliefs/knowledge on the structure of the game.<sup>1</sup>

In Kaneko-Kline [4] and [5], extensive games in the sense of Kuhn [7] are modified and adopted for the description of the objective social situation as well as for that of inductively derived personal views. For this adoption of extensive games, we met some conceptual difficulties and needed certain modifications. The difficulties are in the treatments of:

- (i) information;
- (ii) memory capability of a player.

Since their treatments are crucial in this paper, we discuss them separately.

In the traditional formulation of an extensive game, “information” is described by an information set consisting of some nodes in the game tree. If a player receives such a set as information, he would identify it by looking at its elements. With this interpretation, the implicit assumption of the understanding of the game tree sneaks into our theory. To avoid this implicit assumption, we replace “*an information set*” by “*an information piece*”, which is a symbolic expression such as a gesture, a sentence in the ordinary language or a formula in the sense of mathematical logic <sup>2</sup>.

Also in the traditional formulation of an extensive game, the memory capability of a player is described by an information set. For the consideration of the basic beliefs of a player and his learning by past experiences, the formulation of memory in terms of an information set is inadequate and insufficient. In inductive game theory, we cannot go further without a more explicit and structured concept to express the memory capability of a player. Therefore, we separately formulate memory by means of a *memory function*: a local memory at a point of time in an extensive game consists of the sequence of the information pieces received and actions taken. This was already done in Kaneko-Kline [4] in the context of an extensive game. In Section 5 of this paper, we explain the notion of memory in the context of an information protocol. With the memory described in such a manner, we have the source for an inductive derivation of a personal view.

From now on, we will refer to the extensive game with the replacement of information sets by information pieces, simply as an extensive game, and will treat the memory function as an additional concept.

The basic principle for the above replacement and introduction is to adopt tangible elements such as information pieces and actions in the theory of extensive games, while avoiding intangible elements such as hypothetical concepts of nodes and branches. In [4] and [5], however, we still used intangible tree structures and the question of whether we can fully expel them from our theory was naturally arising. This question follows the *principle of Occam’s razor* that a theory should cut unnecessary components. In our case, we avoid the use of nodes and branches to describe a game tree by reformulating an extensive situation as an *information protocol*.

The constituents of an information protocol are listed explicitly as:

- (a) *information pieces*;
- (b) *actions*;
- (c) *causality relation*.

The causality relation is directly described by information pieces and actions. The triple does not include the *player assignment* and *payoff assignment*, but they can be additionally given and will be introduced in Section 5. Then, we give two *basic axioms* and three *nonbasic axioms* for an information protocol. We will show that when an information protocol satisfies all of those axioms, it can be transformed into an extensive game with a tree structure, and we will show also the converse.

The concept of an information protocol enjoys various merits. One merit is the successful expulsion of intangible elements from the theory. Another merit is that the new theory is

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<sup>2</sup> For the definition of formulae, see, e.g., Mendelson [9] and Kaneko [3].

substantively simpler and is better suited for heuristic purposes than the standard theory of extensive games. The reader will notice the simplicity of the theory of information protocols, compared to the theory of extensive games, once he tries to spell out a full set of conditions defining an extensive game as will be done in Section 3. In game theoretical practices, basic notions such as a tree are typically borrowed without giving a complete specification.<sup>3</sup> This indolence has not been problematic since the definition of an extensive game has never been an object of game theoretical research. Rather, game theorists have focused on the resulting outcomes and/or equilibria in a given game. In inductive game theory, however, the precise definition of an extensive game matters since it is an object of player’s inductive thought.

The axiomatic formulation of an information protocol manifests its power even more when we consider the appropriate weakening of an extensive game in order to capture the inductive derivation of a personal view. Limitations on experiences and memory capacity imply that typically the inductively derived view will not be an extensive game in the strict sense. It is very difficult to think about how the definition of an extensive game should be weakened to capture these limitations. Kaneko-Kline [4] and [5] fumbled around with extensive games to obtain weakenings, but their definitions were not complete. Using the theory of information protocols, the classifications for the choices of such weakenings of extensive games becomes clear-cut and straightforward.

In order to show how an information protocol differs from an extensive game, we consider the absent-minded driver game (1-person game) situation given by Isbell [2] and Piccione-Rubinstein [10]. It is usually represented geometrically such as in Fig.1.1. The standard story behind this game is that when the player arrives at an exit  $E$ , he cannot recall if he has been to the exit  $E$  before or not. Since the decision nodes  $x_0$  and  $x_1$  lie in the same information set corresponding to  $E$ , the player is regarded as being unable to distinguish between the two situations. In the situation represented by  $x_0$ , the player has not yet been to an exit in the game, while in the situation  $x_1$ , he has been to an exit once. Since he cannot recall at  $x_1$  that he was already at  $x_0$ , this situation is interpreted as involving a player with imperfect memory. Nevertheless, it is the underlying assumption that the player is fully cognizant of the game structure given in Fig.1.1.

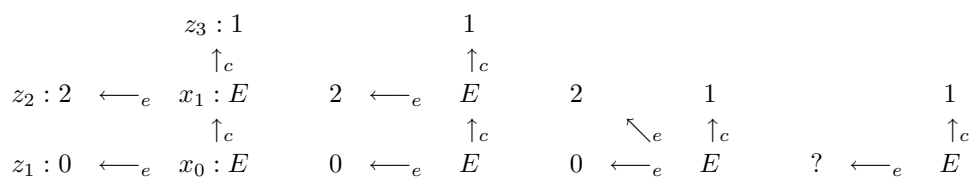


Fig.1.1

Fig.1.2

Fig.1.3

Fig.1.4

We drop the full cognizance assumption in our analysis. Also, we drop nodes and branches, and describe the possible structure by sequences of information pieces and actions, where  $E$  is now interpreted as a piece “Here is an exit”. The information protocol representation of this extensive game is represented as Fig.1.2. The protocol of Fig.1.2 can be considered as the objective description of the game situation or as a subjective view made in the mind of the player.

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<sup>3</sup> For example, definitions of extensive games are found in von Neumann-Morgenstern [13], Kuhn [7], Selten [12], Dubey-Kaneko [1] and Osborne-Rubinstein [11].

Without the full cognizance assumption, if a player constructs his own separate view based on his experiences, he may obtain the one described in Fig.1.3, which is consistent with his forgetfulness. Observe that this personal view could not be expressed as an extensive game in the standard sense, but it is naturally arising in this inductive setting. The replacement of the full cognizance assumption by the partial experiences of a player may be even more destructive. To see this, suppose that the absent driver always takes “ $c$ ” and has never experienced “ $e$ ”. Every time he arrives as an “exit” he has no idea of what will happen if he takes “ $e$ ”. Fig.1.4 describes the information protocol that such a player could derive from this situation. It can be expressed as an information protocol, but not as an extensive game without some modification. One final point to make is that the protocols of Fig.1.3 and Fig.1.4 are actually much simpler than the one described by Fig.1.2.

The main results of this paper are in showing the correspondences between the various types of information protocols and the various weakenings of extensive games. These are given in Section 4. We give the strongest results which connect an information protocol with all the basic and non-basic axioms to an extensive game in Section 3. The basic definitions for an information protocol and axioms for it are given in Section 2. We show how the exposition and applications of inductive game theory are simplified by the use of information protocols in Section 5. Finally, we give conclusions and directions for further research in Section 6.

## 2. An Information Protocol and Axioms for It

An information protocol is a description of a game situation with multiple players. It is used to describe such a situation objectively and/or subjectively. The definition of an information protocol is given in terms of primitives that are observable by players, as was discussed in Section 1.

In Section 2.1, we give a formal definition of an information protocol in terms of sequences of information pieces received and actions taken. It has its own diagrammatic representation which is similar to, but different from that of an extensive game. We provide two basic axioms for it in Section 2.2, and three nonbasic axioms in Section 2.3. The first two axioms determine an information protocol as a forest, and the other three determine it to be equivalent to an extensive game. An explicit comparison with extensive games will be given in Sections 3 and 4.

### 2.1. An Information Protocol

An *information protocol* is given as a triple  $\Pi = (W, A, \prec)$  :

IP1:  $W$  is a nonempty set of *information pieces*;

IP2:  $A$  is a nonempty set of *actions*;

IP3:  $\prec$  is a nonempty subset of  $\bigcup_{m=0}^{\infty} ((W \times A)^m \times W)$ .

Here  $(W \times A)^0 \times W$  is stipulated to be  $W$ . Throughout the paper, we assume  $W \cap A = \emptyset$  to avoid unnecessary complications. Condition IP3 means that  $\prec$  is the union of a unary relation on  $(W \times A)^0 \times W = W$ , a binary relation on  $(W \times A)^1 \times W$ , a ternary relation on  $(W \times A)^2 \times W, \dots$ , etc. In this paper, we consider only *finite* information protocols, i.e.,  $W, A$  and  $\prec$  are all finite sets.

The subset  $\prec$  of  $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$  is called a *causality relation*. We call each element  $\langle (w_1, a_1), \dots, (w_m, a_m), w \rangle \in \prec$  a *feasible sequence*. We sometimes write

$$[(w_1, a_1), \dots, (w_m, a_m)] \prec w$$

for  $\langle (w_1, a_1), \dots, (w_m, a_m), w \rangle \in \prec$ . We will use  $\langle \xi, w \rangle$  to denote a generic element of  $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$ . When a feasible  $\langle \xi, w \rangle$  is in  $(W \times A)^0 \times W$ , it is just  $\langle w \rangle$  and is denoted by  $\prec w$ .

An information protocol  $\Pi = (W, A, \prec)$  is intended to describe objectively and/or subjectively a finite game situation. In  $\Pi$ ,  $[(w_1, a_1), \dots, (w_m, a_m)] \prec w$  means that the information pieces  $w_1, \dots, w_m$  have successively occurred and the action  $a_t$  was taken at each  $w_t$  ( $t = 1, \dots, m$ ), and then the information piece  $w$  occurs. Note that  $[(w_1, a_1), \dots, (w_m, a_m)]$  is not necessarily an exhaustive history before  $w$ . The concept of an exhaustive history to  $w$  will be defined later.

We have various interpretations of an element  $w \in W$ . As already mentioned in Section 1,  $w$  is interpreted as a symbolic expression like a gesture, a sentence in an ordinary language, or a formula in the sense of mathematical logic. The other one is that  $w$  is an information set in the sense of von Neumann-Morgenstern [13] and Kuhn [7]. Our intention is to take the former interpretation rather than the latter.

When we use an information protocol  $\Pi = (W, A, \prec)$  for a game theoretical analysis, we need two more concepts: (i) the *player assignment*  $\pi$  and (ii) the *payoff assignments*  $h = (h_1, \dots, h_n)$ . However, the purpose of this paper is to study the possibility of an alternative formulation of an extensive game without using an underlying tree structure as discussed in Section 1. Therefore, we will ignore these additional components, except in Section 5.

Here, we remark that an information protocol may be regarded as a concept similar to a graph. One might wonder why feasible sequences of lengths more than 2 are needed. The necessity is caused by using one information piece for various “nodes” in a graph. Effectively, we replace the concept of a node in a game tree by the concept of a sequence of information pieces and actions.

Since the introduction of “information pieces” is very basic to our approach and is already a big departure from the standard extensive game, we illustrate “information pieces” by giving a simple formulation of the absent-minded driver game.

**Example 2.1. (Absent-Minded Driver Game)** Consider the diagram of Fig.1.2. One representation of this situation as an information protocol is:  $W = \{E, 0, 1, 2\}$ ,  $A = \{c, e\}$  and the set of feasible sequences is given as:

$$\prec^{2o} = \{\langle E \rangle, \langle (E, e), 0 \rangle, \langle (E, c), E \rangle, \langle (E, c), (E, e), 2 \rangle, \langle (E, c), (E, c), 1 \rangle\}. \quad (1)$$

Notice that 0, 1, 2 are treated as information pieces. Thus, we may sometimes regard end information pieces as including the information of payoffs. In (1), each feasible sequence is listed by counting from the root  $\langle E \rangle$ .

This is not the only representation, and another possibility is to include the set of all nonempty subsequences in  $\prec^{2o}$  as feasible sequences, that is, we define  $\prec^2$  to be the union of  $\prec^{2o}$  and

$$\{\langle 0 \rangle\} \cup \{\langle (E, c), 2 \rangle, \langle (E, e), 2 \rangle, \langle 2 \rangle\} \cup \{\langle (E, c), 1 \rangle, \langle (E, c), 1 \rangle, \langle 1 \rangle\}. \quad (2)$$

We will take the second approach, which is required by one basic axiom B1 to be introduced in Section 2.2. The choice of this axiom is not substantive, but is rather for simplicity in the mathematical treatment.

The diagram of Fig.1.3 is formulated in this way with the same  $W$ ,  $A$  and

$$\prec^3 = \{\langle E \rangle, \langle (E, e), 0 \rangle, \langle (E, e), 2 \rangle, \langle (E, c), 1 \rangle, \langle 0 \rangle, \langle 2 \rangle, \langle 1 \rangle\}. \quad (3)$$

Finally, the diagram of Fig.1.4 is formulated with the same  $W$ ,  $A$  and

$$\prec^4 = \{\langle E \rangle, \langle (E, c), 1 \rangle, \langle 1 \rangle\}. \quad (4)$$

Here no sequence with  $e$  at  $E$  is included, since the player has never experienced the outcome of the choice  $e$  at  $E$ .

To develop more concepts and to state axioms, we introduce decision pieces and endpieces. Let  $\Pi = (W, A, \prec)$  be an information protocol. We partition  $W$  into:

(Decision Pieces):  $W^D = \{w \in W : [(w, a)] \prec u \text{ for some } a \in A \text{ and } u \in W\}$ ;

(Endpieces):  $W^E = W - W^D$ .

These may be regarded as corresponding to the information sets and endnodes in the extensive game. In the protocol of Fig.1.2,  $W^D = \{E\}$  and  $W^E = \{0, 1, 2\}$ .

## 2.2. Basic Axioms for an Information Protocol

Here, we give two basic axioms, B1 and B2, for an information protocol  $\Pi = (W, A, \prec)$ .

We need the notion of a subsequence of an sequence in  $\bigcup_{m=0}^{\infty} ((W \times A)^m \times W)$ . For this purpose, we regard each  $(v_t, a_t)$  as a component in the sequence,  $[(v_1, a_1), \dots, (v_m, a_m)] \in \bigcup_{m=1}^{\infty} (W \times A)^m$ . We say that

$\langle (v_1, a_1), \dots, (v_m, a_m), v_{m+1} \rangle$  is a *subsequence* of  $\langle (u_1, b_1), \dots, (u_k, b_k), u_{k+1} \rangle$  iff  
 $[(v_1, a_1), \dots, (v_m, a_m), (v_{m+1}, a)]$  is a subsequence of  $[(u_1, b_1), \dots, (u_k, b_k), (u_{k+1}, b)]$   
for some  $a$  and  $b$ .

For example,  $\langle u_{k+1} \rangle$  is a subsequence of  $\langle (u_1, b_1), \dots, (u_k, b_k), u_{k+1} \rangle$  since  $[(u_{k+1}, a)]$  is a subsequence of  $[(u_1, b_1), \dots, (u_k, b_k), (u_{k+1}, a)]$  for any  $a$ . The *supersequence* relation is defined likewise.

The first basic axiom states that all subsequences of feasible sequences are also feasible.

**Axiom B1 (Contraction).** *Let  $\langle \xi, v \rangle$  be a feasible sequence, and  $\langle \xi', v' \rangle$  a subsequence of  $\langle \xi, v \rangle$ . Then  $\langle \xi', v' \rangle$  is a feasible sequence.*

The second basic axiom guarantees that the decision pieces can be distinguished from the endpieces.

**Axiom B2 (Weak Extension).** *If  $\xi \prec w$  and  $w \in W^D$ , then there are  $a \in A$  and  $v \in W$  such that  $[\xi, (w, a)] \prec v$ .*

Any protocol that satisfies Axioms B1 and B2 is called a *basic protocol*.

In Example 2.1, we described the information protocols  $(W, A, \prec^2)$  to  $(W, A, \prec^4)$  corresponding to Fig.1.2 to Fig.1.4. These are basic protocols, but  $(W, A, \prec^{2o})$  given by (1) violates Axiom B1. While Axiom B1 is mathematically convenient, it gives too many sequences for illustrations. Presently, we will give a lemma which enables us to focus on some subset of  $\prec$ . For that we need to discuss maximal sequences and initial segments.

We say that a feasible sequence  $\langle \xi, v \rangle$  is *maximal* iff there is no proper feasible supersequence  $\langle \xi', v' \rangle$  of  $\langle \xi, v \rangle$ . It is an exhaustive sequence to an endpiece. We say that  $\langle (w_1, a_1), \dots, (w_k, a_k), w_{k+1} \rangle$  for  $k = 0, \dots, m$  are *initial segments* of  $\langle \xi, w_{m+1} \rangle = \langle (w_1, a_1), \dots, (w_m, a_m), w_{m+1} \rangle$ . When  $k = m$ ,  $\langle \xi, w_{m+1} \rangle$  itself is an initial segment of  $\langle \xi, w_{m+1} \rangle$ . A *position*  $\langle \xi, v \rangle$  is defined to be an initial segment of some maximal feasible sequence. Each position can be regarded as an exhaustive history up to  $v$ . Axiom B1 guarantees that a position is always a feasible sequence.

We denote the set of positions by  $\Xi$ , and we partition  $\Xi$  into

the set of *end positions* - -  $\Xi^E = \{\langle \xi, w \rangle \in \Xi : w \in W^E\}$ ;

the set of *decision positions* - -  $\Xi^D = \{\langle \xi, w \rangle \in \Xi : w \in W^D\}$ .

Note that this partition is based on the last piece  $w$  being in  $W^E$  or  $W^D$ , and that it may be possible that for some  $w \in W$ , there is no feasible sequence  $\langle \xi, w \rangle$ , in which case no position with  $w$  exists.

We have the following results about positions for basic protocols.

**Lemma 2.1.** *Let  $\Pi = (W, A, \prec)$  be a basic protocol.*

**(a):** *If  $\langle \xi, w \rangle$  is a feasible sequence, then there is a position  $\langle \eta, w \rangle$  and  $\eta$  is a supersequence of  $\xi$ .*

**(b):**  *$\langle \xi, w \rangle$  is a feasible sequence if and only if  $\langle \xi, w \rangle$  is a subsequence of a maximal feasible sequence.*

**(c):**  *$\langle \xi, w \rangle$  is a maximal feasible sequence if and only if  $\langle \xi, w \rangle$  is an end position.*

**(d):** *For any  $\langle \xi, w \rangle \in \Xi$ ,  $w \in W^D$  if and only if there is a position  $\langle \xi, (w, a), v \rangle \in \Xi$ .*

*In fact, Axiom B2 is not used for (a) and (b).*

*Proof.* **(a):** Let  $\langle \xi, w \rangle$  be a feasible sequence. If it is maximal, then it is a position. Suppose that there is a proper feasible supersequence  $\langle \eta', v \rangle$  of  $\langle \xi, w \rangle$ . We can take  $\langle \eta', v \rangle$  as a maximal feasible sequence, since  $(W, A, \prec)$  is finite. Then, there is an initial segment (position)  $\langle \eta, w \rangle$  of  $\langle \eta', v \rangle$  and  $\eta$  is a supersequence of  $\xi$ .

**(b):** By Axiom B1, every subsequence of a maximal feasible sequence is a feasible sequence. Conversely, let  $\langle \xi, w \rangle$  be a feasible sequence. Then by (a), there is a position  $\langle \eta, w \rangle$  and  $\eta$  is a supersequence of  $\xi$ . Since  $\langle \eta, w \rangle$  is a position, it is an initial segment of a maximal feasible sequence  $\langle \eta', w' \rangle$ . Hence,  $\langle \xi, w \rangle$  is a subsequence of the maximal feasible sequence  $\langle \eta', w' \rangle$ .

**(c):** Suppose that  $\langle \xi, w \rangle$  is a maximal feasible sequence. Then it is a position. If it were a decision position, then by Axiom B2, there would be a feasible sequence  $\langle \xi, (w, a), v \rangle$  for some  $a \in A$  and  $v \in W$ . But this is impossible, since  $\langle \xi, w \rangle$  is a maximal feasible sequence. Hence,  $\langle \xi, w \rangle$  is an end position.

Conversely, let  $\langle \xi, w \rangle$  be an end position. Then it is an initial segment of a maximal feasible sequence  $\langle \eta, v \rangle$ . If  $\langle \xi, w \rangle$  is strictly shorter than  $\langle \eta, v \rangle$ , then there is an initial segment  $\langle \xi, (w, a), u \rangle$  of  $\langle \eta, v \rangle$ . Then,  $[(w, a)] \prec u$  by Axiom B1, which means  $w \in W^D$  contradicting that  $\langle \xi, w \rangle$  is an end position. Hence,  $\langle \xi, w \rangle$  is a maximal feasible sequence.

**(d):** Let  $\langle \xi, w \rangle \in \Xi$ . Suppose  $w \in W^D$ . Then,  $\langle \xi, w \rangle$  is not a maximal feasible sequence by (c). But since it is a position, it is an initial segment of some maximal feasible sequence  $\langle \eta, u \rangle$ . Thus, we can take the initial segment of  $\langle \eta, u \rangle$  which is one-component longer than

$\langle \xi, w \rangle$  which is the position we look for. The converse follows by B1 using the converse of (c). ■

As noted, the set of feasible sequences is quite big, since Axiom B1 requires it to be subsequence-closed. However, Lemma 2.1 suggests that we can simplify our description of a basic protocol by just listing the set of end positions, rather than the entire list of feasible sequences. *A fortiori*, the set of positions is enough as well. We will use this simplification throughout the paper. In Example 2.1,  $\prec^{2o}$  is the set of positions of  $\prec^2$ .

We consider three more examples to illustrate basic protocols. Fig.2.1 and Fig.2.3 are represented as basic protocols, but Fig.2.2 is not be since the sequence  $\langle (u, a), w \rangle$  cannot be extended and thus B2 is violated.

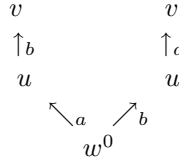


Figure 2.1

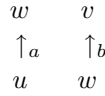


Figure 2.2

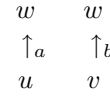


Figure 2.3

### 2.3. Non-Basic Axioms

We regard the basic axioms as too weak to describe an objectively given game theoretic situation, but perhaps not for a subjective view of a player. We will consider three other axioms. All of them together with the basic axioms determine an extensive game. Our presentation of those axioms are planned so that some choices of those non-basic axioms correspond to weak forms of extensive games adopted for subjective views of players in Kaneko-Kline [4] and [5].

**Axiom N1 (Root).** *There is a distinguished root element  $w^0 \in W$  such that  $\langle w^0 \rangle$  is an initial segment of every position.*

This axiom implies that all positions start with  $w^0$ . Without this axiom, an information protocol may have various starts such as Fig.2.2 and Fig.2.3.

The next axiom states that each position has the exhaustive history to determine the present piece.

**Axiom N2 (Determination).** *Let  $\langle \xi, u \rangle$  and  $\langle \eta, v \rangle$  be positions so that  $\xi$  and  $\eta$  are nonempty sequences. If  $\xi = \eta$ , then  $u = v$ .*

In Example 2.1, the protocol of Fig.1.3 violates Axiom N2, since the same exhaustive history  $(E, e)$  determines two different pieces, 0 and 2.

The next axiom states that the set of available actions depends upon the last piece of a position. It is a strengthening of Axiom B2. Fig.2.1 is excluded by this axiom.

**Axiom N3 (History Independent Extension).** *If  $\langle \xi, w \rangle$  is a position and  $[(w, a)] \prec u$ , then there is a  $v \in W$  such that  $\langle \xi, (w, a), v \rangle$  is a position.*

This is satisfied even by Fig.1.3 and Fig.1.4. The information protocol for Fig.1.4 satisfies all of these axioms.



The next lemma states that under axiom B1, N3 is a strengthening of B2.

**Lemma 2.2.** *If an information protocol satisfies B1 and N3, then it satisfies B2.*

*Proof.* Let  $\langle \xi, w \rangle$  be a feasible sequence and  $a \in A_w$ . By Lemma 2.1.(a), there is a position  $\langle \eta, w \rangle$  such that  $\eta$  is a super sequence of  $\xi$ . By N3, we have a  $v \in W$  so that  $\langle \eta, (w, a), v \rangle$  is a position, which is a feasible sequence. Using B1,  $\langle \xi, (w, a), v \rangle$  is a feasible sequence. ■

Thus, B2 and N3 are dependent. Since we will sometimes use B2 only and sometimes N3 additionally, it would be cumbersome to take this dependence into account in each result. Thus, we will ignore this dependence result in the following.<sup>4</sup>

For the transformation of an information protocol into an extensive game we will use the following notion. For an information piece  $v \in W$ , we define the set of *available actions* at  $w$  by:

$$A_w = \{a \in A : [(w, a)] \prec u \text{ for some } u \in W\}. \quad (5)$$

This is the set of actions that are used at some occurrence of  $w$ . When Axiom N3 holds, the set  $A_w$  would be the same as the set of available actions at a position  $\langle \xi, w \rangle$  described as:

$$\{a \in A : \langle \xi, (w, a), v \rangle \text{ is a position for some } v\}. \quad (6)$$

Without Axiom N3, however, these sets may differ and some actions in  $A_w$  may be available only at some positions ending in  $w$ . For example, in the protocol of Fig.2.1, the set  $A_u = \{a, b\}$ , but the set of actions available at the position  $\langle (w_0, a), u \rangle$  is reduced to  $\{b\}$ , and it is only  $\{a\}$  at the position  $\langle (w_0, b), u \rangle$ .

### 3. Comparisons with Extensive Games

In this section, we will compare information protocols to extensive games in the most strong sense with the replacement of information sets by information pieces. We will postpone, to Section 4, comparisons between information protocols in weak senses and the corresponding extensive games used in Kaneko-Kline [4] and [5].

**Definition 3.1. (Extensive Games)** *An extensive game  $\Gamma = ((X, <), (\lambda, W), \{(\varphi_x, A_x)\}_{x \in X})$  is defined as follows:*

*K1(Game Tree):  $(X, <)$  is a finite forest (in fact, a tree by K13);*

*K11:  $X$  is a finite non-empty set of nodes, and  $<$  is a partial ordering over  $X$ ;*

*K12:  $\{x \in X : x < y\}$  is totally ordered with  $<$  for any  $y \in X$ ;<sup>5</sup>*

*K13(Root):  $X$  has the smallest element  $x_0$ , called the root.<sup>6</sup>*

*Now,  $X$  is partitioned into the set  $X^D$  of decision nodes and the set  $X^E$  of endnodes so that every node in  $X^D$  has at least one successor, and every node in  $X^E$  has no successors;<sup>7</sup>*

<sup>4</sup> The axioms other than B2 and N3 are independent, which is verified by our examples.

<sup>5</sup> The binary relation  $<$  is called a *partial ordering* on  $X$  iff it satisfies (i)(irreflexivity):  $x \not< x$ ; and (ii)(transitivity):  $x < y$  and  $y < z$  imply  $x < z$ . It is a *total ordering* iff it is a partial ordering and satisfies (iii)(totality):  $x < y$ ,  $x = y$  or  $y < x$  for all  $x, y \in X$ .

<sup>6</sup> A node  $x$  is called the *smallest* element in  $X$  iff  $x < y$  or  $x = y$  for all  $y \in X$ .

<sup>7</sup> We say that  $y$  is a *successor* of  $x$  iff  $x < y$ , and that  $y$  is an *immediate* successor of  $x$  iff  $x < y$  and there is no  $z \in X$  such that  $x < z$  and  $z < y$ .

*K2(Information Function):*  $W$  is a finite set of information pieces and  $\lambda : X \rightarrow W$  is a function with  $\lambda(x) \neq \lambda(z)$  for any  $x \in X^D$  and  $z \in X^E$ ;

*K3(Available Action Sets):*  $A_x$  is a finite set of available actions for each  $x \in X$ ;

*K31:*  $A_x = \emptyset$  for all  $x \in X^E$ ;

*K32:* for all  $x, y \in X^D$ ,  $\lambda(x) = \lambda(y)$  implies  $A_x = A_y$ ;

*K33(Bijective):* for any  $x \in X$ ,  $\varphi_x$  is a bijection from the set of immediate successors of  $x$  to  $A_x$ .

Usually, the above set of conditions is regarded as the definition of an extensive game. However, we need one more condition, the reason for which will be given after stating the condition and the main theorems of this section. For this additional condition to be stated, we need the notion of a history of information pieces and actions up to a node  $x$ .

For any  $x \in X$ , we define the history up to  $x = x_{m+1}$  by

$$\theta(x) = \langle (\lambda(x_1), a_1), \dots, (\lambda(x_m), a_m), \lambda(x_{m+1}) \rangle, \quad (7)$$

where  $\langle x_1, \dots, x_m, x_{m+1} \rangle$  is the exhaustive sequence to  $x = x_{m+1}$  in  $\Gamma$  with  $x_t < x_{t+1}$  and  $a_t = \varphi_{x_t}(x_{t+1})$  for all  $t = 1, \dots, m$ . This notion is uniquely defined by K12. We use  $\Theta(\Gamma) = \{\theta(x) : x \in X\}$  to denote the set of histories in an extensive game  $\Gamma$ . If there is no history such that it is a proper supersequence of  $\theta(x)$ , then  $\theta(x)$  is called a *maximal history*. We formulate the last condition for an extensive game as:

**K4:** for any  $x \in X$ ,  $\theta(x)$  is an initial segment of a maximal history.

Since this condition looks different from the other conditions, we will consider the nature of K4 using a few examples after presenting the main theorems of this section.

The main difference between the above formulation of an extensive game and that of Kuhn [7] is the use of information pieces rather than information sets. Then the function  $\lambda$  assigns an information piece to each node. This means that when a player reaches a node  $x$ , he receives an information piece  $\lambda(x)$ , but we should not interpret this as meaning that the player perceives  $\lambda(\cdot)$  as a function. The latter needs the assumption that he knows the underlying structure of the extensive game, which should be avoided. A motivation for introducing an information protocol is to avoid this unintended but prevailing interpretation of an extensive game.

As in the formulation of an information protocol, the above formulation of an extensive game does not have a *player assignment* and *payoff assignments*. These eliminations are made to facilitate the purpose at hand: the comparison between an extensive game and an information protocol.

Now we show how to transform an information protocol into an extensive game. In this section, we give the results for the case of an information protocol with B1-B2, N1-N3 and an extensive game with K1-K4. The proofs of these results will follow from the theorems given in Section 4.

Let us recall that given an information protocol  $\Pi = (W, A, <)$ , the set of positions  $\Xi$  is defined without any axioms for  $\Pi$ . Then, we define the *induced extensive game*  $\Gamma(\Pi) = ((X, <), (\lambda, W), \{\varphi_x, A_x\}_{x \in X})$  as follows:

**G1:**  $X = \Xi$ , and for any  $\langle \xi, w \rangle, \langle \eta, v \rangle \in X = \Xi$ ,

$$\langle \xi, w \rangle < \langle \eta, v \rangle \iff \langle \xi, w \rangle \text{ is a proper initial segment of } \langle \eta, v \rangle \quad (8)$$

**G2:**  $\lambda(x) = w$  for any  $x = \langle \xi, w \rangle \in X = \Xi$ ;

**G3:** for any  $x = \langle \xi, w \rangle \in X = \Xi$ ,

**G3a:**  $A_x = A_{\langle \xi, w \rangle} = \{a : [(w, a)] \prec v \text{ for some } v\}$ ;

**G3b:**  $\varphi_x(\langle \xi, (w, a), v \rangle) = a$  for any immediate successor  $\langle \xi, (w, a), v \rangle \in \Xi$ .

In particular, we note that those definitions do not require any conditions on the information protocol  $\Pi = (W, A, \prec)$ .

Now, we can state the transformation from an information protocol with all the axioms to an extensive game.

**Theorem 3.1. (From an Information Protocol to an Extensive Game)** *Let  $\Pi = (W, A, \prec)$  be an information protocol satisfying Axioms B1-B2 and N1-N3. Then,  $\Gamma(\Pi)$  satisfies K1-K4, and the set of histories  $\Theta(\Gamma(\Pi))$  coincides with the positions  $\Xi(\Pi)$ .*

Thus, we can transform an information protocol satisfying all the axioms into an extensive game with all the conditions, and the set of positions in  $\Pi$  is preserved as the set of histories in  $\Gamma(\Pi)$ .

Let us consider the converse of Theorem 3.1. Let an extensive game  $\Gamma = ((X, <), (\lambda, W), \{\varphi_x, A_x\}_{x \in X})$  be given. Then we define the *induced information protocol*  $\Pi(\Gamma) = (W, A, \prec)$  as follows:

**P1:**  $A = \bigcup_{x \in X} A_x$ ;

**P2:**  $\prec = \{\langle \xi, w \rangle : \langle \xi, w \rangle \text{ is a subsequence of some history in } \Theta(\Gamma)\}$ .

We remark that the set  $W$  in  $\Pi(\Gamma)$  is simply taken from  $\Gamma$ , and also that some of K1-K4 are unnecessary for P1 and P2. These remarks will be used in Section 4.

Now, we have the converse of Theorem 3.1.

**Theorem 3.2. (From an Extensive Game to an Information Protocol)** *Let  $\Gamma = ((X, <), (\lambda, W), \{\varphi_x, A_x\}_{x \in X})$  be an extensive game satisfying K1-K4. Then,  $\Pi(\Gamma)$  satisfies Axioms B1-B2 and N1-N3, and the set of positions  $\Xi(\Pi(\Gamma))$  coincides with the set of histories  $\Theta(\Gamma)$ .*

By the above two theorems, the set of information protocols satisfying B1-B2, N1-N3 and extensive games satisfying K1-K4 can be regarded as equivalent structures. For this equivalence to be complete, we have stated that the set of positions coincides with the set of histories, in which sense the structure is preserved in the transformations.

In fact, it is a difference between  $\Pi$  and  $\Pi(\Gamma(\Pi))$  that  $\Pi$  may contain superfluous actions, but they disappear in  $\Pi(\Gamma(\Pi))$ . Hence, when  $\Pi = (W, A, \prec)$  itself has no such superfluous actions, we have the exact equivalence between  $\Pi$  and  $\Pi(\Gamma(\Pi))$ . Technically, we say that  $\Pi = (W, A, \prec)$  has *no superfluous actions* iff for any  $a \in A$ , there are  $w, v \in W$  such that  $[(w, a)] \prec v$ . This means simply that there are no actions in  $A$  that do not appear in any feasible sequence. In this case, we have the following result about multiple applications of the transformations.

**Theorem 3.3. (Reversibility)** *Let  $\Pi = (W, A, \prec)$  be an information protocol with no superfluous actions and with Axioms B1-B2 and N1-N3. Let  $\Gamma(\Pi)$  be the induced extensive*

game from  $\Pi$  and  $\Pi(\Gamma(\Pi))$  the induced protocol from  $\Gamma(\Pi)$ . Then, we have  $\Pi(\Gamma(\Pi)) = \Pi$ , diagrammatically,

$$\Pi \xrightarrow{\text{Th.3.1}} \Gamma(\Pi) \xrightarrow{\text{Th.3.2}} \Pi(\Gamma(\Pi)) = \Pi \quad (9)$$

*Proof.* By definition,  $W$  does not change from  $\Pi$  to  $\Pi(\Gamma(\Pi))$ . Let us see that the action set  $A$  does not change. Since now  $A$  has no superfluous actions in  $\Pi$ , each  $a$  is used at some node  $x = \langle \xi, w \rangle$  in  $\Gamma(\Pi)$  by G3a, i.e.,  $a \in A_x$ . By P1,  $a$  is included in  $A$  of  $\Pi(\Gamma(\Pi))$ . Finally, we should see that  $\prec$  does not change from  $\Pi$  to  $\Pi(\Gamma(\Pi))$ , but this follows from Theorem 3.1, P2 and Axiom B1. ■

Theorem 3.3 states that the two transformations of taking the induced game and induced protocol are reversible. However, if we start with an extensive game  $\Gamma$ , this relation needs some modification, since  $\Gamma(\Pi(\Gamma))$  may have different names of nodes from  $\Gamma$ . However, after one transformation, we have the reversibility since:

$$\Gamma \xrightarrow{\text{Th.3.2}} \Pi(\Gamma) \xrightarrow{\text{Th.3.1}} \Gamma(\Pi(\Gamma)) \xrightarrow{\text{Th.3.2}} \Pi(\Gamma) \xrightarrow{\text{Th.3.1}} \Gamma(\Pi(\Gamma)) \quad (10)$$

The point is that the third and fifth induced extensive games are identical. Starting with an extensive game  $\Gamma$  with the arbitrary names of nodes, we transform  $\Gamma$  into the induced protocol  $\Pi(\Gamma)$ , where the initially given nodes disappear. After this transformations, the induced protocol and induced game are constructed from the same ingredients. Thus, we have the reversibility in (10). In fact,  $\Pi$  and  $\Pi(\Gamma(\Pi))$  are isomorphic, i.e., only the names of nodes are different.

Now, we are in the state to see the necessity of condition K4. Condition K4 is related to the following condition given by Kuhn [7]:

(*Irreflexivity*): For any two nodes  $x, y \in X$ , if  $x < y$ , then  $\lambda(x) \neq \lambda(y)$ .

This means that the same information piece does not occur more than once in one play. Irreflexivity implies Condition K4. The extensive game of Fig.1.1 violates Irreflexivity but satisfies Condition K4. Condition K4 excludes the game of Fig.3.1, where the same information piece  $u$  is attached to the endnodes  $z_1$  and  $z_2$ .

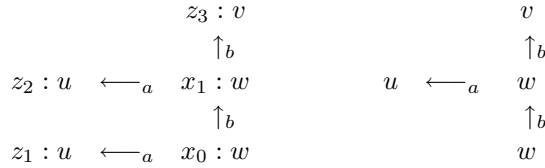


Fig.3.1

Fig.3.2

Let us see the induced protocol  $\Pi(\Gamma)$  defined by P1 and P2 from the game  $\Gamma$  of Fig.3.1, which is described by Fig.3.2. While this transformation is well defined, the history  $\theta(z_1) = \langle (w, a), u \rangle$  is not a position in the induced protocol. Hence, something in the basic structure is lost in the transformation. Incidentally, the induced protocol also violates Axiom N3. On the other hand, the game of Fig.1.1 can be transformed without losing any of the basic structure since it satisfies K4. In this sense, K4 but not irreflexivity, is needed for the successful transformation of an extensive game into an information protocol.

In the next section we will show that the basic structure in terms of the histories in a game and the positions in a protocol are preserved by the transformations described in Theorems 3.1 and 3.2 even for weak structures.

#### 4. Information Protocols and Extensive Games in Weak Forms

Kaneko-Kline [4] and [5] described the objective social situation as an extensive game satisfying conditions K1-K4. In their inductive game theory, a player accumulates some experiences by making trial deviations and then constructs his personal view from his accumulated experiences. Thus, his view may be smaller than the objective extensive game and it may violate some part of K1-K3. In [4] and [5], therefore, some weak forms of extensive games were adopted for personal views. In this section, we discuss how to weaken the theory of information protocols when we use information protocols for inductively derived personal views.

Let us recall that a full form of an extensive game  $\Gamma$  is defined by K1-K4. Now, we weaken K13, K33 into K13<sup>o</sup>, K33f as follows:

K13<sup>o</sup>: if  $x$  and  $y$  are minimal nodes in  $(X, <)$ , then  $\lambda(x) \neq \lambda(y)$ ;

K33f (Function): for any  $x \in X$ ,  $\varphi_x$  is a function from the set of immediate successors of  $x$  to  $A_x$ .

We say that  $\Gamma$  is a *basic extensive game* iff it is defined by K1-K4 with the replacements of K13, K33 by K13<sup>o</sup>, K33f. In this section, we show that these choices of relaxations lead to a basic information protocol satisfying Axioms B1 and B2, but not necessarily N1-N3.

In Kaneko-Kline [4], a basic extensive game was defined without K13<sup>o</sup>. The addition of K13<sup>o</sup> changes nothing essential in the analysis of [4], but only simplifies the results. We adopt K13<sup>o</sup> for a basic extensive game since it makes the connection to a basic information protocol more straightforward. On the other hand, in [5], the strong root assumption K13(root) is maintained, since the memory function and domain of accumulation considered there permit it. Also in [5], the condition K33f was strengthened to K33i given as

K33i(*Injective*): for any  $x \in X$ ,  $\varphi_x$  is an injective function from the set of immediate successors of  $x$  to  $A_x$ .

By a simple parallelism, we have another strengthening of K33f:

K33s(*Surjective*): for any  $x \in X$ ,  $\varphi_x$  is a surjective function from the set of immediate successors of  $x$  to  $A_x$ .

Although K33s did not appear in [4], [5], it will be shown in Section 5 that it has some status in inductive game theory. Now, we show how these distinctions correspond to different axioms for an information protocol. For example, we will see that K33i and K33s for extensive game exactly correspond to Axioms N2 and N3.

We will make comparisons in the same manner as in Theorems 3.1 and 3.2, but the present comparisons are made between weak forms of extensive games and information protocols. Given an information protocol  $\Pi$ , the game structure  $\Gamma(\Pi)$  is defined by G1, G2 and G3, and conversely, given an extensive game  $\Gamma$ , the information protocol  $\Pi(\Gamma)$  is defined by P1 and P2. As noted in Section 3, the definition of  $\Gamma(\Pi)$  in G1, G2 and G3 does not need any condition on  $\Pi$ , but the definition of the induced protocol  $\Pi(\Gamma)$  in P1 and P2 needs some conditions for  $\Gamma$ . However, when  $\Gamma$  is a basic extensive game, the definition in P1 and P2 has no problem.

We have the four theorems for these correspondences. The first result is the correspondence between a basic information protocol and a basic extensive game:

Th.4.1 - - a basic information protocol  $\longleftrightarrow$  a basic extensive game.

Then, we can make comparisons between the axioms. The results are summarized as follows:

Th.4.2 - -Axiom N1(Root)  $\longleftrightarrow$  K13 (Root)

Th.4.3 - -Axiom N2 (Determination)  $\longleftrightarrow$  K33i (Injective)

Th.4.4- -Axiom N3 (Independent Extension)  $\longleftrightarrow$  K33s (Surjective)

Let us start with the first correspondence. We mention the additional equivalence results between the set of histories and the set of positions only in the first theorem. These equivalence results hold in the other three theorems, too.

**Theorem 4.1. (a Basic Protocol -vs- a Basic Game)**

(1): Let  $\Gamma$  be a basic extensive game. Then the induced protocol  $\Pi(\Gamma)$  is a basic information protocol, and the set of positions  $\Xi(\Pi(\Gamma))$  coincides with the set of histories  $\Theta(\Gamma)$ .

(2): Let  $\Pi$  be a basic information protocol. Then the induced game  $\Gamma(\Pi)$  is a basic extensive game, and the set of histories  $\Theta(\Gamma(\Pi))$  coincides with the set of positions  $\Xi(\Pi)$ .

*Proof.* (1): It suffices to check Axioms B1, B2 for  $\Pi(\Gamma) = (W, A, \prec)$ . Axiom B1 follows the definition P2 of feasible sequences. Consider Axiom B2. Suppose that (i)  $\langle \xi, w \rangle$  is a feasible sequence and (ii)  $[(w, a)] \prec v$  for some  $v$ . Then, using P2 on (i),  $\langle \xi, w \rangle$  is a subsequence of some history  $\langle \xi', w \rangle = \theta(x) \in \Theta(\Gamma)$ . Using P2 on (ii),  $\langle (w, a), v \rangle$  is a subsequence of some  $\langle \eta', v \rangle = \theta(y) \in \Theta(\Gamma)$ . Then there is some  $y' \in X^D$  such that  $\lambda(y') = w$  and  $\theta(y')$  is an initial segment of  $\theta(y)$ . Since  $\lambda(y') = \lambda(x) = w$  and  $y' \in X^D$ , we have  $x \in X^D$  by K2. Hence,  $x$  has an immediate successor  $x'$  and  $\theta(x') = \langle \xi', (w, \varphi_x(x')), \lambda(x') \rangle$ . Then, by B1,  $\langle \xi, (w, \varphi_x(x')), \lambda(x') \rangle$  is a feasible sequence and B2 is satisfied.

Now let us prove  $\Xi(\Pi(\Gamma)) = \Theta(\Gamma)$ . Let  $\langle \xi, v \rangle \in \Xi(\Pi(\Gamma))$ . Then  $\langle \xi, v \rangle$  is an initial segment of a maximal feasible sequence  $\langle \xi', v' \rangle$ . By P2 and the maximality of  $\langle \xi', v' \rangle$  in  $\Xi(\Pi(\Gamma))$ , it follows that  $\langle \xi', v' \rangle = \theta(x)$  for some  $x \in X$ . Since  $\langle \xi, v \rangle$  is an initial segment of  $\theta(x)$ , it must be that  $\langle \xi, v \rangle = \theta(y)$  for some  $y \in X$ . Conversely, let  $\langle \xi, v \rangle \in \Theta(\Gamma)$ . Then  $\langle \xi, v \rangle = \theta(y)$  for some  $y \in X$ . By K4,  $\theta(y)$  is an initial segment of some maximal history in  $\Theta(\Gamma)$ . By P2, every maximal history in  $\Theta(\Gamma)$  is a maximal feasible sequence, *a fortiori*, a position in  $\Xi(\Pi(\Gamma))$ .

(2): We show that  $\Gamma(\Pi)$  satisfies K1-K4 with K13<sup>o</sup> and K33f instead of K13 and K33.

K1: By (8),  $(X, <)$  is a partially ordered set - K11, and it satisfies K12. Consider K13<sup>o</sup>. If two connected parts have the same initial information piece  $\lambda(x) = \lambda(y) = w$ , we have, by G1 and G2,  $x = y = \langle w \rangle$ .

K2: Since  $\Pi$  is a protocol, the set  $W$  is partitioned into  $W^D$  and  $W^E$ . By G2,  $\lambda$  is a function from  $X = \Xi$  to  $W$ . Since  $\Pi$  is basic, by Lemma 2.1.(d) and the definition of G1,  $\langle \xi, v \rangle \in X^D$  if and only if  $v \in W^D$ . Hence,  $\lambda(x) \neq \lambda(z)$  for any  $x \in X^D$  and  $z \in X^E$ .

K3: If  $\langle \xi, v \rangle \in X^E$ , then  $\langle \xi, v \rangle$  is an endposition and  $A_{\langle \xi, v \rangle} = \emptyset$  - K31. By G2 and G3a,  $A_x = A_y$  if  $\lambda(x) = \lambda(y)$  - K32. Let us see K33f. Let  $x = \langle \xi, w \rangle \in X^D$  and let  $y$  be an immediate successor of  $x$ . Then, by G1,  $y$  takes the form of  $y = \langle \xi, (v, a), w \rangle$  for some

$a \in A_x$  and  $w \in W$ . By G3a and G2b,  $\varphi_x(y) = a \in A_x$ . Thus, we proved K33f.

K4: Since  $\Pi$  is a protocol, each  $x \in X = \Xi(\Pi)$  is an initial segment of a maximal feasible sequence  $y \in X = \Xi(\Pi)$ . Since  $\Theta(\Gamma) = \Xi(\Pi(\Gamma))$  and  $\theta(x) = x$  and  $\theta(y) = y$ , it follows that  $\theta(x)$  is an initial segment of a maximal history  $\theta(y)$ .

Finally, let us see  $\Theta(\Gamma(\Pi)) = \Xi(\Pi)$ . For this, it is enough to observe that in the induced game  $\Gamma(\Pi)$ , the history  $\theta(x) = x$  for each  $x \in X = \Xi(\Pi)$ . ■

We have the correspondence between a basic extensive game and a basic protocol. Observe that we have the equivalences  $\Xi(\Pi(\Gamma)) = \Theta(\Gamma)$  in the first statement and  $\Theta(\Gamma(\Pi)) = \Xi(\Pi)$  in the second statement. These equivalences hold in the following three correspondence theorems. The reversibilities  $\Pi(\Gamma(\Pi))$  to  $\Pi$  and  $\Gamma(\Pi(\Gamma))$  to  $\Gamma$  will be considered after the correspondence theorems.

The next theorem simply states that Axiom K13(Root) corresponds to Axiom N1(Root).

**Theorem 4.2. (Root -vs-Root)**

(1): Let  $\Gamma$  be a basic extensive game with K13(Root). Then the induced protocol  $\Pi(\Gamma)$  is a basic information protocol with N1(Root).

(2): Let  $\Pi$  be a basic information protocol with N1. Then the induced game  $\Gamma(\Pi)$  is a basic extensive game with K13.

*Proof.* (1): Let  $\langle \xi, v \rangle$  be a position in  $\Pi(\Gamma)$ . Then  $\langle \xi, v \rangle$  is an initial segment of some maximal history  $\theta(x) \in \Theta(\Gamma)$ . By K13,  $\theta(x)$  starts with  $\lambda(x^o) \in W$ , where  $x^o$  is root in  $X$ . Hence,  $\langle \lambda(x^o) \rangle$  is an initial segment of every position in  $\Pi(\Gamma)$ .

(2): Since  $\Pi$  satisfies N1, every position  $\langle \xi, v \rangle$  starts with the same  $w^0$ . By G1,  $\langle w^0 \rangle$  is the smallest node in  $\Gamma(\Pi)$ . ■

The following theorem states the correspondence between K33i and N2.

**Theorem 4.3. (Injective -vs- Determination)**

(1): Let  $\Gamma$  be a basic extensive game with K33i(Injective). Then the induced protocol  $\Pi(\Gamma)$  is a basic information protocol with N2(Determination).

(2): Let  $\Pi$  be a basic information protocol with N2. Then the induced game  $\Gamma(\Pi)$  is a basic extensive game with K33i.

*Proof.* (1): Let  $\langle \xi, u \rangle$  and  $\langle \eta, v \rangle$  be positions in  $\Pi(\Gamma)$  and suppose  $\xi = \eta$ . Then  $\langle \xi, u \rangle = \theta(x)$  and  $\langle \eta, v \rangle = \theta(y)$  for some  $x$  and  $y$  in  $X$ . Let  $x_1, \dots, x_k = x$  be the path to  $x$  and  $y_1, \dots, y_k = y$  be the path to  $y$ . Since  $\xi = \eta$ , it follows by K13<sup>0</sup>, that  $x_1 = y_1$ . Then by K33i (Injection),  $x_j = y_j$  for  $j = 2, \dots, k$ . Hence,  $x = y$  and  $u = \lambda(x) = \lambda(y) = v$ .

(2): Let  $x \in X^D$ , and consider any two distinct immediate successors  $y$  and  $y'$  of  $x$ . We show  $\varphi_x(y) \neq \varphi_x(y')$ . By G1,  $x = \langle \xi, v \rangle \in \Xi$ , and  $y$  and  $y'$  are positions of the form  $y = \langle \xi, (v, a), w \rangle$  and  $y' = \langle \xi, (v, b), u \rangle$ . Since  $y$  and  $y'$  are distinct, we have  $a \neq b$  by N2. Hence, by G3b,  $\varphi_x(y) = a \neq b = \varphi_x(y')$ , which implies that  $\varphi_x$  is an injection, i.e., K33i. ■

Next, we can obtain the correspondence between K33s and N3.

**Theorem 4.4. (Surjective -vs- Independent Extension)**

(1): Let  $\Gamma$  be a basic extensive game with K33s(Surjective). Then the induced protocol  $\Pi(\Gamma)$  is a basic information protocol with N3(Independence).

(2): Let  $\Pi$  be a basic information protocol with N3. Then the induced game  $\Gamma(\Pi)$  is a basic extensive game with K33s.

*Proof.* (1): Let  $\langle \xi, w \rangle$  be a position in  $\Pi(\Gamma)$ , and  $a \in A_w$ . Then,  $\theta(x) = \langle \xi, w \rangle$  for some  $x \in X^D$ . We find an immediate successor  $y$  of  $x$  with  $\varphi_x(y) = a$  by K33s. Then,  $\theta(y) = \langle \xi, (w, a), \lambda(y) \rangle$  is a history, which is an initial segment of a maximal history by K4. Hence, by P2  $\langle \xi, (w, a), \lambda(y) \rangle$  is a position in  $\Pi(\Gamma)$ .

(2): Let  $x \in X^D$ , and consider any  $a \in A_x$ . We need to show that  $\varphi_x(y) = a$  for some immediate successor  $y$  of  $x$ . By G1,  $x = \langle \xi, v \rangle \in \Xi$ , and by G3a,  $[(v, a)] \prec w$  for some  $w \in W$ . Hence, by N3, there is a position  $\langle \xi, (v, a), u \rangle$  for some  $u \in W$ . By G1,  $y = \langle \xi, (v, a), u \rangle$  is an immediate successor of  $x$ , and by G3b,  $\varphi_x(y) = a$ . ■

Finally, we consider the reversibility in the sense of (9) and (10). We already discussed that an information protocol may have superfluous actions. Similarly, an extensive game may have superfluous actions when K33(Bijective) is replaced by K33i or K33f. Thus, we have the following definition: a basic extensive game  $\Gamma$  has no *superfluous actions* iff for any  $w \in W$  and any  $a \in A_w$ , there is some  $x, y \in X$  such that  $\lambda(x) = w$  and  $\varphi_x(y) = a$ . That is, any action  $a \in A_w$  is used at some node assigned information piece  $w$ . This does not require that  $a$  is used at every node with  $w$  even when  $a \in A_w$ .

We can have the following theorem.

**Theorem 4.5. (Reversibilities)**

(1): Let  $\Pi$  be a basic protocol with no superfluous actions. Then  $\Pi(\Gamma(\Pi)) = \Pi$ .

(2): Let  $\Gamma$  be a basic extensive game satisfying K33i(Injective).

(a): Then there is a bijection  $\psi$  from the set of nodes  $X$  of  $\Gamma$  to the set of nodes of  $\Gamma(\Pi(\Gamma))$  such that  $\psi(x) = \theta(x)$  for all  $x \in X$ , where  $\theta(x)$  is the history up to  $x$  in  $\Gamma$ .

(b): Suppose that  $\Gamma = ((X, <), (\lambda, W), \{(\varphi_x, A_x)\}_{x \in X})$  has no superfluous actions. Then  $\psi$  is structure-preserving from  $\Gamma$  to  $\Gamma(\Pi(\Gamma)) = ((X', <'), (\lambda', W'), \{(\varphi'_x, A'_x)\}_{x \in X'})$ , i.e.,  $x < y \Leftrightarrow \psi(x) <' \psi(y)$ , and for all  $x \in X$ ,  $A_x = A'_{\psi(x)}$ ,  $\lambda(x) = \lambda'(\psi(x))$  and  $\varphi_x(y) = \varphi'_{\psi(x)}(\psi(y))$  for any immediate successor  $y$  of  $x$ .

The first assertion is the same as Theorem 3.3, and can be proved in the same manner. For (2b), we need to start with a basic extensive game satisfying K33i and with no superfluous actions. Then, actually, we can write

$$\Gamma \longrightarrow \Pi(\Gamma) \longrightarrow \Gamma(\Pi(\Gamma)) \cong \Gamma.$$

The last part that  $\Gamma(\Pi(\Gamma)) \cong \Gamma$  is the assertion (2b) that the structure of  $\Gamma$  is preserved in  $\Gamma(\Pi(\Gamma))$ . In the case of no superfluous actions for  $\Gamma$  and K33i, the two structures are equivalent up to the names of nodes. If, on the other hand,  $\Gamma(\Pi(\Gamma))$  does not satisfy K33i, then the set of nodes in  $\Gamma(\Pi(\Gamma))$  may be smaller than the set in  $\Gamma$ , but we could use the notion of g-morphism given in Kaneko-Kline [4] to show a sense in which the two structures may be regarded as equivalent. Finally, with some superfluous action  $a$  in  $\Gamma$ , the structure  $\Pi(\Gamma)$  would contain the superfluous action  $a$ , but the structure  $\Gamma(\Pi(\Gamma))$  would not include  $a$ . Nevertheless, the two structures could be regarded as equivalent up to superfluous actions.

**Lemma 4.1. (Bijection between  $X$  and  $\Theta(\Gamma)$ )** Let  $\Gamma$  be a basic extensive game satisfying K33i(Injection). Then  $\theta$  is a bijection from  $X$  to  $\Theta(\Gamma)$ .



*Proof.* First, we note that  $(X, <)$  is a finite forest, that is, each connected part in  $(X, <)$  is a tree. This is guaranteed by K11 and K12. Also, the immediate successors of each node are well defined whenever successors exist. We show by induction on a tree in  $(X, <)$  from its root that  $\theta(x) = \theta(x')$  implies  $x = x'$ . When  $\theta(x) = \theta(x')$  is of length 1, we have  $x = x'$  by K13<sup>0</sup>. Now, make the inductive hypothesis that the assertion holds up to length  $k$ . Let  $\theta(x) = \langle (\lambda(x_1), a_1), \dots, (\lambda(x_k), a_k), \lambda(x) \rangle = \theta(x') = \langle (\lambda(x'_1), a'_1), \dots, (\lambda(x'_k), a'_k), \lambda(x') \rangle$ . Hence,  $a_k = a'_k$ , and

$$\langle (\lambda(x_1), a_1), \dots, (\lambda(x_{k-1}), a_{k-1}), \lambda(x_k) \rangle = \langle (\lambda(x'_1), a'_1), \dots, (\lambda(x'_{k-1}), a'_{k-1}), \lambda(x'_k) \rangle.$$

By the induction hypothesis, we have  $x_k = x'_k$ . Then, since  $\varphi_{x_k} = \varphi_{x'_k}$  is an injection by K33i,  $\varphi_{x_k}(x) = a_k = a_{k'} = \varphi_{x_k}(x')$  implies  $x = x'$ . ■

*Proof of Theorem 4.5 (2)* The set of nodes in  $\Gamma(\Pi(\Gamma))$  is the set of positions  $\Xi(\Pi(\Gamma))$  by G1. By P2, we have  $\Xi(\Pi(\Gamma)) = \Theta(\Gamma)$ . By Lemma 4.1,  $\theta : \Gamma \rightarrow \Theta(\Gamma)$  is a bijection. Thus (a) holds.

Consider (b). Since  $\Gamma$  has no superfluous actions, we have  $A_x = A_{\lambda(x)}$  for any  $x \in X$ . The other assertions can be verified. ■

## 5. Inductive Game Theory using Information Protocols

In Sections 3 and 4, we discussed equivalence and correspondence results between extensive games and information protocols. Although these constructs can be regarded as equivalent, they have certain theoretical differences, e.g., an extensive game has nodes as a base concept but an information protocol doesn't. In this section, we give a brief discourse of inductive game theory using information protocols. We will see how this differs from the inductive game theory based on extensive games in Kaneko-Kline [4] and [5], as well as how the equivalence results can be used to translate one theory into the other.

Inductive game theory has three steps: (1) playing and accumulating memories; (2) inductive derivations of personal views from accumulated memories; (3) decision making using a personal view. The first two steps are relevant for this paper. For (1), we need to define a memory function for a player and a domain of accumulation of memories. These are the sources of the inductively derived view of (2). Then, we analyze the correspondences between the axioms the inductively derived view satisfies and the choice of a domain.

### 5.1. Players, Memories and Behaviors

First, we need to add a *player-assignment*  $\pi$  and *payoff functions*  $h = (h_1, \dots, h_n)$  to an information protocol  $(W, A, \prec)$ . We denote a player set by  $N = \{1, \dots, n\}$ . A complete description of a situation including the player assignment and payoffs is written as  $\Pi = ((W, A, \prec), \pi, h)$ , where  $\pi$  and  $h = (h_1, \dots, h_n)$  are given as:

**IP4:**  $\pi : W \rightarrow 2^N$ , where  $|\pi(w)| = 1$  for all  $w \in W^D$  and  $\pi(w) = N$  for all  $w \in W^E$ ;

**IP5:**  $h_i : W^E \rightarrow R$  for all  $i \in N$ .

Definition IP4 means that each decision piece is received by only one player and each end-piece is received by all the players. Definition IP5 means that the players receive payoffs at the endpieces. The above definitions do not need any axioms, since the separation between  $W^D$  and  $W^E$  suffices for these definitions.

The information protocol of Fig.5.1 has two players with the player-assignment  $\pi(w^0) = \{1\}$  and  $\pi(u_1) = \pi(u_2) = \{2\}$ . This means that player 1 moves at the root and player 2 moves at the other decision pieces. The endpieces belong to both players by condition IP4, e.g.,  $\pi(e_1) = \{1, 2\}$ . An example of payoffs could be given as  $h_1(e_t) = t$  and  $h_2(e_t) = -t$  for  $t = 1, 2, 3, 4$ .

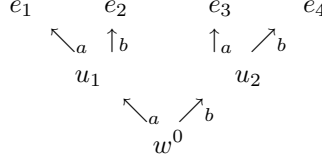


Figure 5.1

Next, we describe the memory capability of each player by a *memory function*  $\mathbf{m}_i$ . A player's memories described by his memory function will be the source for his inductively derived view. These memories will take the form of sequences of past information pieces received and actions taken. To simplify the description of a player's memory, we assume throughout the following that each information piece  $w$  contains:

**M1:** the set  $A_w$  of available actions in the sense of (5);

**M2:** the value  $\pi(w)$  of the player assignment  $\pi$  if  $w$  is a decision piece;

**M3:** player  $i$ 's own payoff  $h_i(w)$  (as a numerical value) if  $w$  is an endpiece<sup>8</sup>.

We can interpret the functions  $\pi$  and  $h_i$  as devices to extract the information about the available actions, player assignment, and payoff from each piece. Assumptions M1-M3 simplify the description of memory by providing relevant information for a personal view. Recall that we interpret information pieces as symbolic expressions that could easily contain the information expressed in M1-M3.

In this paper, we use a specific memory function called the *exact-perfect recall* (EPR-) memory function.<sup>9</sup> We take two steps to define this function. First, we define the set of positions belonging to player  $i$  in  $\Pi$ :

$$\Xi_i = \{\langle \xi, w \rangle \in \Xi : i \in \pi(w)\}. \quad (11)$$

Then, let  $\langle \xi, w \rangle_i = \langle (v_1, b_1), \dots, (v_l, b_l), v_{l+1} \rangle$  be the maximal subsequence of  $\langle \xi, w \rangle \in \Xi_i$  so that each  $v_k$  in  $\langle \xi, w \rangle_i$  is a piece of player  $i$ , i.e.,  $i \in \pi(v_k)$  for  $k = 1, \dots, l+1$ . In the 2-person protocol of Fig.5.1,  $\langle (w^0, a), (u_1, a), e_1 \rangle \in \Xi_1$  and  $\langle (w^0, a), (u_1, a), e_1 \rangle_1 = \langle (w^0, a), e_1 \rangle$ , since player 2 moves at  $u_1$ .

The EPR-memory function  $\mathbf{m}_i$  is defined by

$$\mathbf{m}_i \langle \xi, w \rangle = \{\langle \xi, w \rangle_i\} \text{ for all } \langle \xi, w \rangle \in \Xi_i. \quad (12)$$

That is, it gives player  $i$  a local memory of his past actions and information pieces received. In [4], a more general definition of a memory function is given in the context of an extensive game.

<sup>8</sup> It would be more consistent to assume that each player receives only his part of each endpiece. For simplicity, however, we do not put subscript "i".

<sup>9</sup> In Kaneko-Kline [4], a general definition of a memory function is given in an extensive game. It can be converted into the present paper for information protocols.

Our analysis of inductive game theory starts with the pair of an information protocol  $\Pi$  with payoffs and players, and an  $n$ -tuple of memory functions  $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_n)$ . Then, each player's behavior depends on his memory as described by his memory function. A *behavior pattern*  $\sigma_i$  of player  $i$  in  $(\Pi, \mathbf{m})$  is a function defined on  $\Xi_i$  with

$$\sigma_i \langle \xi, w \rangle \in \{a : \langle \xi, (w, a), u \rangle \in \Xi \text{ for some } u\}; \quad (13)$$

$$\mathbf{m}_i \langle \xi, w \rangle = \mathbf{m}_i \langle \eta, v \rangle \text{ implies } \sigma_i \langle \xi, w \rangle = \sigma_i \langle \eta, v \rangle. \quad (14)$$

This means that  $\sigma_i$  assigns an action compatible with the information protocol  $\Pi$  and the memory function  $\mathbf{m}_i$  of player  $i$ .

The concept of a behavior pattern is the same as the standard concept of a strategy. However, it should be interpreted as describing his behavior pattern regularly taken. His *decision making* is made only after his personal view is inductively derived. This paper does not touch the problem of decision making. See Kaneko-Kline [5] for decision making based on personal views.

Before moving on to our study of inductive game theory, we return to the absent-minded driver situation modelled as a 1-player protocol in Figure 1.2. First, we observe that if we apply the EPR-memory function to this example, then it does not involve any forgetfulness. A player with the EPR-memory function can distinguish between the two exits by his memories. At the first exit, he will have the memory value  $\{\langle E \rangle\}$ , while at the second exit his memory value is  $\{\langle (E, c), E \rangle\}$ . Thus his behavior pattern may differ at the two exits. The EPR-memory function removes the absent-minded driver "paradox" by separating the problem of memory from that of information transmission.

If, however, we try to capture forgetfulness in the story of the absent-minded driver, we would need a different memory function. Here we consider one alternative memory function  $\mathbf{m}_1^F$  defined by:

$$\mathbf{m}_1^F \langle \xi, w \rangle = \begin{cases} \{\langle E \rangle\} & \text{if } \langle \xi, w \rangle = \langle E \rangle \text{ or } \langle (E, c), E \rangle \\ \{\langle (E, e), 0 \rangle\} & \text{if } \langle \xi, w \rangle = \langle (E, e), 0 \rangle \\ \{\langle (E, e), 2 \rangle\} & \text{if } \langle \xi, w \rangle = \langle (E, c), (E, e), 2 \rangle \\ \{\langle (E, c), 1 \rangle\} & \text{if } \langle \xi, w \rangle = \langle (E, c), (E, c), 1 \rangle. \end{cases} \quad (15)$$

That is, player 1 forgets, at the second exit, that he was previously at an exit. At any endposition he recalls only the last piece of information he received and the last action he took. There are other types of forgetfulness that a player might have, which can be expressed by different memory functions. We have simply chosen one consistent with the absent-minded driver story. More importantly, the view derived by a player with  $\mathbf{m}_1^F$  differs from that by a player with the EPR-memory function, which will be considered in Section 5.2.

## 5.2. The Objective Situation and Experienced Domains

For inductive game theory, the choice of the memory function and the domain of accumulation of memories are paramount to determining the inductively derived view of a player. In this paper we stick to the EPR-memory function, but we consider several domains of accumulation. To describe these, we start with an objective situation  $(\Pi^o, \mathbf{m}^o)$  consisting of an information protocol  $\Pi^o = ((W^o, A^o, \prec^o), \pi^o, h^o)$  with B1-B2, N1-N3 and a profile of EPR-memory functions  $\mathbf{m}^o = (\mathbf{m}_1^o, \dots, \mathbf{m}_n^o)$ , where the player set is given as  $N^o = \{1, \dots, n\}$ . The set of positions in  $\Pi^o$  is denoted by  $\Xi^o$ , and that for player  $i$  is  $\Xi_i^o$ .

We assume that the players typically follow a specific profile of behavior patterns  $\sigma^o = (\sigma_1^o, \dots, \sigma_n^o)$ , but occasionally they make deviations from this given behavior profile. Each player learns about the situation from these trials and the memories he recalls after some repeated plays of the objective situation.

$$\cdot \cdot \cdot \rightarrow (\Pi^o, \mathbf{m}^o) \rightarrow (\Pi^o, \mathbf{m}^o) \rightarrow (\Pi^o, \mathbf{m}^o) \rightarrow \cdot \cdot \cdot$$

Figure 5.2

Let us consider the repeated situation of  $(\Pi^o, \mathbf{m}^o)$  described in Fig.5.2, where each player makes a trial once in a while independently with a small frequency (probability). It is a rare event that two or more players make simultaneous trial deviations from their behavior patterns. Memories from very rare events might disappear (be forgotten) in the mind of a player, but memories from some other forms of experiences could remain as *long-term* memories. In this paper, other than the full domain of a player's positions  $\Xi_i^o$ , we consider two other domains of accumulation: one consisting of his *active* experiences, and the other consisting of both active and *passive* experiences. The active experiences come from a player's own trials, and the passive ones come from the trials of some other players. We will see how the choice of these domains affect his inductively derived view and which axioms are satisfied by the derived view.

To describe these three domains, we need a few more definitions. Given a profile of behavior patterns  $\sigma = (\sigma_1, \dots, \sigma_n)$ , we define the *induced* endposition  $\langle \xi, w \rangle = \langle (w_1, a_1), \dots, (w_m, a_m), w_{m+1} \rangle$  by: for any  $k = 1, \dots, m$ ,

$$\text{if } j \in \pi^o(w_k), \text{ then } \sigma_j \langle (w_1, a_1), \dots, (w_{k-1}, a_{k-1}), w_k \rangle = a_k. \quad (16)$$

We say that any initial segment  $\langle (w_1, a_1), \dots, (w_k, a_k), w_{k+1} \rangle$  of  $\langle \xi, w \rangle$  is *reachable by*  $\sigma = (\sigma_1, \dots, \sigma_n)$ . That is, a position  $\langle \eta, v \rangle$  is reachable by  $\sigma$  if and only if it is an initial segment of the endposition  $\langle \xi, w \rangle$  induced by  $\sigma$ .

Let us return to the regular behavior patterns  $\sigma^o = (\sigma_1^o, \dots, \sigma_n^o)$  mentioned above. For a player  $i \in N^o$ , we define the *unilateral domain of active experiences* by:

$$D_i^A(\sigma^o) = \bigcup_{\sigma_i} \{ \langle \xi, w \rangle \in \Xi_i^o : \langle \xi, w \rangle \text{ is reachable by } (\sigma_{-i}^o, \sigma_i) \}, \quad (17)$$

where  $\sigma_i$  under the union symbol varies over the set of all possible behavior patterns for player  $i$ . We also define the *unilateral domain of passive and active experiences* by

$$D_i^U(\sigma^o) = \bigcup_{j \in N^o} \bigcup_{\sigma_j} \{ \langle \xi, w \rangle \in \Xi_i^o : \langle \xi, w \rangle \text{ is reachable by } (\sigma_{-j}^o, \sigma_j) \}. \quad (18)$$

The domain  $D_i^A(\sigma^o)$  for player  $i$  consists of positions induced by his own trials (active experiences), and  $D_i^U(\sigma^o)$  consists of positions induced either by his own trials or trials of some other players (active and passive experiences). Since simultaneous trials are excluded from either domain, we typically have the relationship:

$$D_i^A(\sigma^o) \subsetneq D_i^U(\sigma^o) \subsetneq \Xi_i^o. \quad (19)$$

For example, let  $\Pi^o$  be the 2-person protocol given as Fig.5.1, and let  $\sigma^o$  assign the choice of  $a$  to each position of each player. Then the active domain  $D_1^A(\sigma^o)$  for player 1 consists of three positions ending with  $w^0, e_1, e_3$  in Fig.5.3, i.e.,  $D_1^A(\sigma^o) = \{ \langle w^0 \rangle, \langle (w^0, a), (u_1, a), e_1 \rangle, \langle (w^0, b), (u_2, a), e_3 \rangle \}$ . The unilateral domain  $D_1^U(\sigma^o)$ , depicted in Fig.4, contains another

endposition ending with  $e_2$ , which is induced by a unilateral deviation of player 2. The position ending with  $e_4$  requires simultaneous deviations by players 1 and 2 and occurs in  $\Xi_1^o$ .

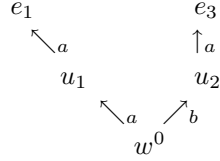


Figure 5.3

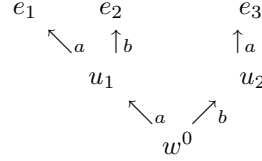


Figure 5.4

The domains  $D_i^A(\sigma^o)$  and  $D_i^U(\sigma^o)$  are natural candidates for the domain of experiences when the game has various players. When the game is very small, it may be also natural to have the entire domain  $\Xi_i^o$  for player  $i$ . In this section, we consider these three domains.

Let  $D_i$  be either  $D_i^A(\sigma^o)$ ,  $D_i^U(\sigma^o)$  or  $\Xi_i^o$ . Then we define the *memory kit* by

$$T_{D_i} = \bigcup_{\langle \xi, w \rangle \in D_i} m_i^o \langle \xi, w \rangle. \quad (20)$$

The memory kit  $T_{D_i}$  is interpreted as the set of accumulated long-term memories for player  $i$ . In the case of  $D_i^A(\sigma^o)$ , the regular experiences along the play induced by  $\sigma^o$  become long-term memories, and so do the memories induced by his own deviations. Player  $i$  constructs his personal view based on his memory kit  $T_{D_i}$ .

When  $m_i^o$  is the EPR-memory function and  $D_i$  is the domain of accumulation, the memory kit is described as

$$T_{D_i} = \{ \langle \xi, w \rangle_i : \langle \xi, w \rangle \in D_i \}.$$

The domain  $D_i$  is the set of experiences from the objective point of view, but the memory kit  $T_{D_i}$  consists of the subjective memories of player  $i$ . In Fig.5.3, the active domain  $D_1 = D_1^A(\sigma^o)$  was given above, and then the memory kit is given as  $T_{D_1} = \{ \langle w^0 \rangle, \langle (w^0, a), e_1 \rangle, \langle (w^0, b), e_3 \rangle \}$  by excluding the decision pieces of player 2.

Now, we define the *inductively derived view* (i.d.view)  $\Pi^i = ((W^i, A^i, \prec^i), \pi^i, h^i)$  from the memory kit  $T_{D_i}$  by

- ID1:**  $W^i = \{ w : w \text{ is included in some sequence in } T_{D_i} \}$ ;
- ID2:**  $A^i = \bigcup_{w \in W^i} A_w^o$ ;
- ID3:**  $\prec^i = \{ \langle \xi, w \rangle : \langle \xi, w \rangle \text{ is a subsequence of some sequence in } T_{D_i} \}$ ;
- ID4:**  $\pi^i(w) = \{ i \}$  for all  $w \in W^i$ ;
- ID5:**  $h^i(w) = h_i^o(w)$  for all  $w \in W^{iE}$ .

Since the set of available actions at  $w \in W^i$  is written on  $w$  by Assumption M1, we require as ID2 that the set of all actions in  $A_w^o$  appear in  $A^i$ . Nevertheless, the set of available actions  $A_w^i$  in the sense of (5) in  $\Pi^i$  may differ from  $A_w^o$ . Condition M2 is consistent with ID4, which states that  $\Pi^i$  is a 1-person protocol. Condition M3 about objective payoffs is used in ID5.

A salient feature of the above definition is the unique determination of the i.d.view without imposing any of B1-B2 and N1-N3. The question is which axioms the i.d.view

satisfies. This contrasts with our findings for extensive games. Although in [4] and [5] an i.d.view was defined in a parallel manner using an extensive game, we needed to make an appropriate choice of some conditions from K13<sup>o</sup>, K13, K33f, K33i, K33s and K33. In [5], we restricted our attention to an i.d.view satisfying K13 and K33 to obtain the uniqueness of it, but the choices of names of nodes remained and the uniqueness was obtained up to isomorphisms. In [4], K13<sup>o</sup> and K33f were chosen, and we needed some procedure to make comparisons of multiple i.d.views. By our direct approach with information protocols, we can avoid these types of problems.

### 5.3. Analysis of Axioms for I.d.views

Here, we analyze which axioms are satisfied by the i.d.view for each of the three domains  $D_i^A(\sigma^o)$ ,  $D_i^U(\sigma^o)$ , and  $\Xi_i^o$ . We also consider some sufficient conditions for the axioms when those are not generally satisfied by the i.d.view. We assume throughout the following, except for discussions on the absent-minded driver game in the very end of this section, that player  $i$ 's  $m_i^o$  is the EPR-memory function.

First, we show that the i.d.view  $\Pi^i = ((W^i, A^i, \prec^i), \pi^i, h^i)$  is a basic information protocol for each domain in question.

**Lemma 5.1. (I.d.views are basic)** *Let  $\Pi^i = ((W^i, A^i, \prec^i), \pi^i, h^i)$  be the i.d.view from the memory kit  $T_{D_i}$ , and let  $D_i$  be  $D_i^A(\sigma^o)$ ,  $D_i^U(\sigma^o)$  or  $\Xi_i^o$ . Then  $(W^i, A^i, \prec^i)$  satisfies B1 and B2.*

*Proof.* Condition B1 is satisfied by ID3. Consider B2. Let  $\langle \zeta, w \rangle$  be a feasible sequence in  $\prec^i$  and  $a \in A_w^i$ . Then  $w \in W^{oD}$ , and we can find an endposition  $\langle \xi, v \rangle$  in  $D_i$  such that  $\langle \xi, v \rangle_i$  includes  $\langle \zeta, (w, b), v \rangle$  as a subsequence for some  $b$ . Since  $m_i^o \langle \xi, v \rangle = \{\langle \xi, v \rangle_i\}$ , we have  $\langle \xi, v \rangle_i \in T_{D_i}$ . Hence, by ID3,  $\langle \zeta, (w, b), v \rangle$  is a feasible sequence in  $\prec^i$ . ■

This lemma is partial in the sense that it does not state whether or not the non-basic axioms are satisfied. The next result is that the non-basic axioms are all satisfied for the active domain  $D_i^A(\sigma^o)$ .

**Theorem 5.1. (I.d.views for  $D_i^A(\sigma^o)$  are full)** *Let  $\Pi^i = ((W^i, A^i, \prec^i), \pi^i, h^i)$  be the i.d.view from the memory kit  $T_{D_i}$ , and  $D_i = D_i^A(\sigma^o)$ . Then  $(W^i, A^i, \prec^i)$  satisfies N1-N3.*

*Proof.* Consider N1. Let  $\langle \xi, w \rangle$  be the endposition induced by  $\sigma^o$ . Then  $\langle \xi, w \rangle_i$  is an endposition in  $\Xi^i$ . Let  $\langle v \rangle$  denote the initial segment of  $\langle \xi, w \rangle_i$  of length 1. Since  $D_i = D_i^A(\sigma^o)$ , all players other than  $i$  follow  $\sigma^o$ , so every player  $j$  before  $v$  follows  $\sigma_j^o$ . Hence,  $\langle \eta, v \rangle_i$  has  $\langle v \rangle$  as an initial segment for any  $\langle \eta, v \rangle \in D_i = D_i^A(\sigma^o)$ .

Consider N2. Let  $\langle \xi, w \rangle$  and  $\langle \eta, v \rangle$  be two positions in  $\Xi^i$ . Then, we have two positions  $\langle \xi', w \rangle$  and  $\langle \eta', v \rangle$  in  $D_i^A(\sigma^o)$  with  $\langle \xi', w \rangle_i = \langle \xi, w \rangle$  and  $\langle \eta', v \rangle_i = \langle \eta, v \rangle$ . Since  $D_i = D_i^A(\sigma^o)$ , all players other than  $i$  follow  $\sigma^o$ . Hence,  $\xi = \eta$  if and only if  $\xi' = \eta'$ . Hence, by N2 on  $\Pi^o$ , we have  $w = v$ .

Consider N3. Let  $\langle \xi, w \rangle$  be a position in  $\Pi^i$  and  $a \in A_w^i$ . Then  $\langle \xi, w \rangle = \langle \xi', w \rangle_i$  for some position  $\langle \xi', w \rangle$  in  $D_i^A(\sigma^o)$ . This position  $\langle \xi', w \rangle$  is reachable by some profile  $(\sigma_{-i}^o, \sigma_i)$  having the additional property  $\sigma_i \langle \xi', w \rangle = a$ . This profile will induce some endposition  $\langle \eta, u \rangle$  in  $\Pi^o$ . This has an initial segment  $\langle \eta', u' \rangle$  so that  $\langle \eta', u' \rangle_i = \langle \xi, (w, a), u' \rangle$  is a position in  $\Pi^i$ . ■

The above result can be obtained alternatively from a result for extensive games in Kaneko-Kline [4], using Theorems 3.1 and 3.2.

For the other two domains,  $D_i^U$  and  $\Xi_i^o$ , Axioms N1 and N2 may not be satisfied by the i.d.view. Counter examples will be given presently. When the domain is  $\Xi_i^o$ , however, Axiom N3 will be satisfied.

**Theorem 5.2.(N3 for  $\Xi_i^o$ )** *Let  $\Pi^i = ((W^i, A^i, \prec^i), \pi^i, h^i)$  be the i.d.view from the memory kit  $T_{D_i}$ , and  $D_i = \Xi_i^o$ . Then  $(W^i, A^i, \prec^i)$  satisfies N3.*

*Proof.* Let  $\langle \xi, w \rangle$  be a position in  $\Pi^i$  and  $a \in A_w$ . Then, there is a position  $\langle \eta, w \rangle$  in  $\Xi_i^o$  such that  $\langle \eta, w \rangle_i = \langle \xi, w \rangle$ . By N3 for  $\Pi^o$ , there is a position  $\langle \eta, (w, a), v \rangle$  in  $\Xi^o$ . There is an endposition  $\langle \eta', v' \rangle$  in  $\Xi_i^o$  such that  $\langle \eta, (w, a), v \rangle$  is an initial segment of  $\langle \eta', v' \rangle$ . Hence, for some  $u$ ,  $\langle \xi, (w, a), u \rangle$  is an initial segment of  $\langle \eta', v' \rangle_i$ . This means that we have a position  $\langle \xi, (w, a), u \rangle$  is a position in  $\Pi^i$ . ■

To see a possible violation of N1, we consider the i.d.view derived from  $\Xi_2^o$  in Fig.5.1. Recall  $\sigma^o$  that assigns action  $a$  to every decision piece. The i.d.view for player 2 has two starts  $u_1, u_2$  such as depicted in Fig.5.5 and violates N1.



Figure 5.5

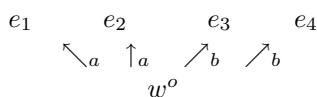


Figure 5.6

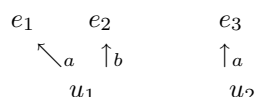


Figure 5.7

Next, consider the i.d.view derived from  $\Xi_1^o$  for player 1 in the same objective protocol. It is depicted in Fig.5.6 and violates N2. In this example, since 1 moves at the root, his i.d.view could never violate N1. This observation is verified in the next theorem. Before it, we consider the problems when  $D_i = D_i^U(\sigma^o)$ .

When  $D_2 = D_2^U(\sigma^o)$ , the i.d.view for player 2 becomes Fig.5.7, and N1 is violated. A violation of N2 is obtained for the i.d.view for  $D_1 = D_1^U(\sigma^o)$ . The i.d.view of player 2 in Fig.5.7 might appear to be a violation of N3, since  $u_2$  has no immediate successor for the action  $b$ . However, careful inspection shows that N3 is not actually violated in this example since there is no feasible sequence of the form  $\langle (u_2, b), w \rangle$  for  $\Pi^2$ . Indeed, we can construct a counterexample against N3. One example comes from the 2-player protocol  $\Pi^o$  of Fig.5.8 where player 1 moves only at the root  $w^0$  and player 2 moves at  $u$  and  $v$ . Let  $\sigma^o$  assign the

action  $a$  everywhere. Then player 2's i.d.view for  $D_2^U(\sigma^o)$  violates N3.

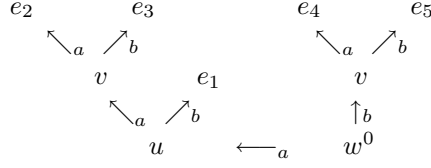


Figure 5.8

In sum, for the full domain  $\Xi_i^o$ , Axiom N3 is always satisfied, but Axioms N1, N2 might not hold. For the domain  $D_i^U(\sigma^o)$ , none of the non-basic axioms N1-N3 are guaranteed. Nevertheless, we can find some sufficient conditions for these axioms.

**Theorem 5.3. (Sufficient Conditions for N1 and N2 on  $D_i^U(\sigma^o)$  or  $\Xi_i^o$ )** Let  $\Pi^i = ((W^i, A^i, \prec^i), \pi^i, h^i)$  be the i.d.view from the memory kit  $T_{D_i}$  and let  $D_i$  be  $D_i^U(\sigma^o)$  or  $\Xi_i^o$ .

(1): Suppose that  $i \in \pi^o(w^0)$ . Then  $(W^i, A^i, \prec^i)$  satisfies N1.

(2): Suppose that for any  $\langle \xi, w \rangle, \langle \eta, v \rangle \in \Xi_i^o$ , if  $\langle \xi \rangle_i = \langle \eta \rangle_i$  and it is a nonempty sequence, then  $w = v$ . Then  $(W^i, A^i, \prec^i)$  satisfies N2.

*Proof.* (1): If  $i \in \pi^o(w^0)$ , then  $\langle w_0 \rangle$  is the root of  $\Pi^i$ .

(2): Let  $\langle \xi, w \rangle$  and  $\langle \eta, v \rangle$  be two positions in  $\Xi^i$  with  $\xi = \eta$ . Then, we have two positions  $\langle \xi', w \rangle$  and  $\langle \eta', v \rangle$  in  $\Xi_i^o$  so that  $\langle \xi', w \rangle_i = \langle \xi, w \rangle$  and  $\langle \eta', v \rangle_i = \langle \eta, v \rangle$ . Thus,  $\langle \xi' \rangle_i = \xi = \eta = \langle \eta' \rangle_i$ , which implies  $w = v$  by the supposition. Thus, we have N2 for  $\Pi^i$ . ■

We note that if the sequences  $\langle \xi \rangle_i = \langle \eta \rangle_i$  in (2) were allowed to be empty, we would obtain N1, too.

Finally, we consider a sufficient condition for N3 when  $D_i = D_i^U(\sigma^o)$ . It requires that player  $i$ 's only decision piece in  $W^o$  be the root piece  $w^o$ . For player 1 in the protocol of Fig.5.1, this condition is met, and thus his i.d.view will satisfy N3.

**Theorem 5.4. (Sufficient Condition for N3 on  $D_i^U(\sigma^o)$ )** Let  $\Pi^i = ((W^i, A^i, \prec^i), \pi^i, h^i)$  be the i.d.view from the memory kit  $T_{D_i}$ , and  $D_i = D_i^U(\sigma^o)$ . Assume that  $W_i^{oD} = \{w^o\}$ . Then  $(W^i, A^i, \prec^i)$  satisfies N3.

*Proof.* Let  $\langle \xi, w \rangle$  be a position in  $\Pi^i$  and  $a \in A_w$ . Then it follows from  $W_i^{oD} = \{w^o\}$  that  $\langle \xi, w \rangle = \langle w^o \rangle$ . Since  $\langle w^o \rangle$  is also a position in  $\Pi^o$ , by N3 for  $\Pi^o$ , there is a position  $\langle (w^o, a), v \rangle$  in  $\Pi^o$ . This  $\langle (w^o, a), v \rangle$  is an initial segment of an endposition in  $\langle \xi, u \rangle$  in  $D_i^U(\sigma^o)$  because  $W_i^{oD} = \{w^o\}$ . Also, since only  $w^o$  and  $u$  belong to player  $i$  by the assumption,  $\langle (w^o, a), u \rangle$  is a position in  $\Xi^i$ . ■

Theorems 5.2 - 5.4 and the counterexamples given above show that the non-basic axioms would be satisfied by the i.d.view for certain circumstances, but not typically. Thus, these results mean that the correspondence theorems given in Sections 3 and 4 have substantive contents from the viewpoint of inductive game theory.

Finally, let us return to the absentminded-driver game. When player 1's  $m_1^o$  is the EPR-memory function and the objective protocol is  $\Pi^o$  given in Fig.1.2 the domain of experiences  $D_1^A, D_1^U$  and  $\Xi_1^o$  turn out to be identical, and the i.d.view  $\Pi^1$  coincides with  $\Pi^o$ . Since he



does not forget anything, with the help of his memory function, he can reconstruct the true protocol.

Consider the case where his memory function is the forgetful one given as  $\mathbf{m}_1^F$  of (15). Then, we should be careful about the choice of domain. In particular, the domain  $\Xi_1^o$  differs now from  $D_1^A(\sigma^o) = D_1^U(\sigma^o)$  even though it is a 1-player game. The reason for this difference is that the player cannot get to the endpiece with 2 by any of his strategies. However, one possibility is that he makes mistakes, in which case endpiece 2 could be experienced sometimes. With the domain  $\Xi_1^o$  and the memory function  $\mathbf{m}_1^F$ , player 1 would get the i.d.view of Fig.1.3 violating N2.

To obtain the i.d.view of Fig.1.4, we need to have a narrower domain of experiences. In Kaneko-Kline [5] such a domain was given by introducing some limitations on actions. Suppose that the driver has never tried the action to exit  $e$  so that the domain  $D_1$  is a single path described by the set of positions  $\prec^1 = \{\langle E \rangle, \langle (E, c), E \rangle, \langle (E, c), (E, c), 1 \rangle\}$ . In this case, with the memory function  $\mathbf{m}_1^F$  he would derive the i.d.view of Fig.1.4. But M1 still allows player 1 to find  $e$  is available at the exit  $E$ . In this case,  $A_E^o$  written on  $E$  is  $\{c, e\}$ , while  $A_E^1$  defined by (5) from  $\prec^1$  consists of only one action  $c$ . This apparent difference causes no confusion by the presumption that player 1 is aware of the action  $e$ , but having never tried this action, he excludes it from consideration in his i.d.view.

## 6. Conclusions

We have developed the theory of information protocols in order to have simpler representations of extensive game situations for inductive game theory than those provided by the theory of extensive games. An information protocol is defined based on information pieces and actions as its primitives as well as on a causality relation of histories to new pieces. As in Kaneko-Kline [4] and [5], we formulated individual memory separately using a memory function. We showed the equivalence and correspondence results between the theory of information protocols and the theory of extensive games.

In particular, we showed that an information protocol with all the axioms is equivalent to an extensive games of Kuhn [7] with the replacement of information sets by information pieces. We use such a protocol to describe the objective description of the game situation in question. Then, we showed the correspondences between the axioms for protocols and the axioms for extensive games. The correspondence results help us consider choices of weak forms of extensive games for the subjective descriptions of the players.

Although the two theories are connected in a clear-cut manner by these correspondences, we may have quite different theoretical practices in using them for inductive game theory. First, the theory of information protocols is far simpler than the theory of extensive games. As briefly mentioned in the very end of Section 5.2, one consideration in terms of isomorphisms between extensive games disappears if we use information protocols. Some other considerations are also simplified by adopting information protocols. For example, the inductive process of a player becomes clear-cut and straightforward.

The power of the theory of information protocols may manifest itself more when we consider social situations with relatively large numbers of players and with coarse information pieces. In such a situation, if we follow the inductive game theory presented in Section 5, we would find a large gap between the objective social situation and subjective personal views: When information is coarse to a player, a personal view given by accumulated experiences is extremely simple relative to the objective situation. In this case, a player may be satisfied

by his simplistic view on society, or he may try to collect more information about society. Communication and/or education may function for the latter purpose.

A problem of the above sort was discussed by Kaneko-Matsui [6]. It is about discrimination and prejudices involved in the interactions of several ethnic groups. Each player can observe only the ethnic differences of people. This is very coarse relative to the objective situation. If a person constructs his view based on his observations of people's responses to such ethnic differences, the view is simple and understandable but far from reality. The view contains a lot of prejudicial aspects. The theory of information protocols gives clear-cut results on the gap between a personal view and the objective social situation.

A final comment is: in spite of the advantages of information protocols, we do not claim that the theory of extensive game should be eliminated from the future development of inductive game theory. It suggests an alternative method to organize the thoughts of people, which may not be so efficient. People sometimes follow inefficient methods using hypothetical concepts to complete their descriptions. Without the help of such hypothetical concepts, it might be difficult to think about social situations. The use of hypothetical nodes may give a player a different perspective and/or may suggest a hint of a more complicated society or environment. For these reasons we would like to keep the theory of extensive games for the future research of inductive game theory.

#### REFERENCES

1. P. Dubey, M. Kaneko, *Information Patterns and Nash Equilibria in Extensive Games I*, *Mathematical Social Sciences* **8** (1984), 111–139.
2. J. Isbell, *Finitary Games*, *Contributions to the Theory of Games III* (1957), 79–96, Princeton University Press.
3. M. Kaneko, *Epistemic Logics and their Game Theoretical Applications: Introduction*, *Economic Theory* **19** (2002), 7–62.
4. M. Kaneko, J. J. Kline, *Inductive Game Theory: A Basic Scenario*, SSM.DP.1143, University of Tsukuba, 2006, <http://www.sk.tsukuba.ac.jp/SSM/libraries/pdf1126/1143.pdf>.
5. M. Kaneko, J. J. Kline, *Small and Partial Views derived from Limited Experiences*, SSM.DP.1166, University of Tsukuba, 2007, <http://www.sk.tsukuba.ac.jp/SSM/libraries/pdf1151/1166.pdf>.
6. M. Kaneko, A. Matsui, *Inductive Game Theory: Discrimination and Prejudices*, *Journal of Public Economic Theory* **1** (1999), 101–137. Errata: the same journal **3** (2001), 347.
7. H.W. Kuhn, *Extensive Games and the Problem of Information*, *Contributions to the Theory of Games II* (1953), Kuhn, H. W. and A. W. Tucker, eds. 193–216. Princeton University Press.
8. R.D. Luce, H. Raiffa, *Games and Decisions*, John Wiley and Sons Inc., Boston, 1957.
9. E. Mendelson, *Introduction to Mathematical Logic*, 3rd ed. Wadsworth & Brooks, Monterey, 1987.

10. M. Piccione, A. Rubinstein, *On the Interpretation of Decision Problems with Imperfect Recall*, *Games and Economic Behavior* **20** (1997), 3–24.
11. M. Osborne, A. Rubinstein, *A Course in Game Theory*, MIT Press, Cambridge, 1994.
12. R. Selten, *Reexamination of the Perfectness Concept of Equilibrium Points in Extensive Games*, *International Journal of Game Theory* **4** (1975), 25–55.
13. J. von Neuman, O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, 1944.