19 UTILITY THEORIES IN COOPERATIVE GAMES

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1 Introduction

Cooperative game theory begins with descriptions of coalitional behavior. For every permissible coalition, a subset of the players of the game, there is a given set of feasible outcomes for its members. Each outcome is presupposed to arise from cooperative behavior by the members of the coalition; specific individual actions are secondary.[1] Cooperative games take several forms—games with side payments, games without side payments, partition function form games, and others, including, for example, bargaining games. In this paper we focus on games with and without side payments.

Cooperative game theory has two parts. One part is the description of game situations, the form or model of the game, and the other part is the description of expected outcomes. The second part is called *solution theory*. Utility theory is foundational to both parts. Utility theory for a solution theory, however, may involve additional assumptions, sometimes hidden. Therefore the utility theory behind the description of a game situation may not be the same as that behind a solution concept applied to the game. In this chapter, in addition to exploring various models of games, we will consider the assumptions behind various solution concepts.

The predominant forms of cooperative games are *games with side payments* and *games without side payments*. A game with side payments summarizes the possible outcomes to a coalition by one real number, the total payoff achievable by the coalition. In contrast, a game without side payments describes the possibilities open to a coalition by a set of outcomes, where each outcome states the payoff to each player in the coalition. The concepts of games with and without side payments are not disjoint; a game with side payments can be described as a game without side payments. Because of the simplicity of a game with side payments, cooperative game theory has been more extensively developed for games with side payments than for games without side payments. Because of this simplicity, however, games with side payments require special consideration of the underlying utility theory.

We will first discuss games with side payments. These require the assumptions of side payments (SP) and of "transferable utility" (TU). Together these

[1] Recently there has been considerable interest in "Nash's Program", that is, the study of cooperative games in terms of noncooperative game theory through formulating cooperative behavior as moves in an extensive form game. This is more faithful to "the ontological version of methodological individualism"—the individual is the actor—than cooperative game theory and is capable of analyzing the postulate of cooperation. Some limitations of Nash's program, however, are that the extensive form description of complex social interactions may be too complicated to analyze and the results may be highly dependent on specific details of the extensive form game. Nash's Program complements cooperative game theory in understanding cooperative aspects of societies.

assumptions appear to imply that utility can be transferred between players at a one-to-one rate. This conclusion is sometimes misunderstood to imply that utility itself can be transferred between players. A specific form of this misunderstanding is that a game with side payments necessarily involves interpersonal utility comparisons. To clarify the roles of the assumptions SP and TU, we will illustrate the derivation of games with side payments from several models of social and economic situations. As noted earlier, some solution concepts may deviate from the intentions of SP and TU. We consider to what extent such deviations exist and to what extent they may be justified.

A *game without side payments* requires neither the assumption SP nor the assumption TU. For these games, standard economic utility theory suffices. Since neither transferable utility nor side payments are required for games without side payments, solution concepts developed for games without side payments might avoid subtle difficulties in utility theories. Many solution concepts have been developed first, however, for games with side payments and then extended to games without side payments. Thus, the difficulties present in interpretation of solution concepts for games with side payments also may arise in extensions of these solution concepts to games without side payments. Games without side payments, together with some solution concepts, are discussed in the later part of this chapter.

In Section 2, we review the concept of a game with side payments and several examples from the literature. In Section 3, we consider the assumptions of transferable utility and of side payments. We give axiomatic characterizations of the transferable utility assumption in the cases of no uncertainty and of uncertainty. In Section 4, we discuss some solution concepts for games with side payments, specifically, the core, the von Neumann-Morgenstern stable set, the Shapley value and the nucleolus. In Section 5, we discuss games without side payments and see how those solution concepts depend upon the assumptions of transferable utility and of side payments.

2 Games with Side Payments

In the literature of game theory, a game with side payments is often given as an abstract mathematical construct, but this construct is typically derived from a model of a social or an economic situation. The consideration of utility theory behind a game with side payments is relevant to this derivation and especially to the question of how faithfully the derived game describes the underlying situation. Hence our method of evaluating the utility theory behind a game with side payments begins with the derivation of games from underlying social and economic situations.

Such a derivation relies upon the following two assumptions, mentioned in Section 1:

(SP): side payments;

(TU): transferable utility.

The assumption of transferable utility is a requirement on the utility function of each player while the assumption of side payments is a part of the rules of the game. The term "a game with side payments" is a slight misnomer in that both assumptions SP and TU are required for the theory. If at least one of them is violated, the derived game does not faithfully describe the underlying situation, and then the general theory of games without side payments is needed. The assumptions SP and TU are logically independent, but are often related in specific situations. We will illustrate that in some situations, side payments are regarded as a part of the situation, and in other situations, TU is assumed but side payments are prohibited.

Before the discussions on the assumptions SP and TU, we give some basic definitions. A *game with side payments* is given as a pair (N, v), where $N = \{1, ..., n\}$ is the *player set* and $v : 2^N \to \mathbb{R}$ is the *characteristic function* satisfying $v(\emptyset) = 0$ and $v(S) + v(T) \leq v(S \cup T)$ for disjoint subsets S, T of N. The second condition is called *superadditivity*. The function v assigns to each coalition S in 2^N the "maximum total payoff" that can be obtained by collective activities of the players in S. A game (N, v) describes a social situation in terms of the payoffs achievable by the collective activities of groups of players.

Typically a characteristic function v is used to describe what can be obtained by each coalition of players in a game. The total payoff $v(S)$ is interpreted as available to the players in S. This suggests the following definition: a payoff vector $(a_i)_{i \in S}$ is said to be *feasible* for a coalition S iff

$$\sum_{i \in S} a_i \leq v(S). \tag{2.1}$$

The characteristic function v is also used to restrict the payoff possibilities to players. An example is individual rationality: a payoff vector $(a_i)_{i \in S}$ is said to be *individually rational* iff

$$a_i \geq v(\{i\}) \quad \text{for all } i \in S. \tag{2.2}$$

This states that cooperation gives each player at least what he can independently guarantee for himself. Those are basic concepts for solution theory, discussed further in Section 4.

The above definition of feasibility is based on the assumptions TU and SP. Without these assumptions, we may lose the intended meaning of (2.1). In the following subsections, we discuss the roles of the assumptions of transferable utility and side payments in three examples.

2.1 *Transferable Utility and Side Payments*

Consider an individual player i with utility function $U_i : X \times \mathbb{R} \to \mathbb{R}$. The domain $X \times \mathbb{R}$ is called the *outcome space for player i*. The set X may represent a commodity space, a set of social alternatives or the outcome space of a noncooperative game. The set $\mathbb{R}$ of real numbers is typically interpreted as representing the set of increments (and decrements) of a perfectly divisible composite commodity called "money". This commodity represents purchasing power for other commodities outside the model. The value $U_i(x, \xi)$ represents the utility from the outcome x and the increment (or decrement) ξ of money from a given initial level.

The interpretation of the unbounded domain of money is that, without meeting any boundary conditions, any individually rational outcome can be achieved. That is, relative to individually rational payoffs that might arise from the game, incomes are sufficiently large to avoid the need for boundary conditions. This is also related to the assumption SP and will be clarified further in a subsequent example.

The *transferable utility assumption* TU, also called "quasi-linearity" in the economics literature, is that U_i is linearly separable with respect to ξ, that is, there is a function $u_i : X \to \mathbb{R}$ such that

$$U_i(x, \xi) = u_i(x) + \xi \text{ for all } (x, \xi) \in X \times \mathbb{R}. \tag{2.3}$$

The utility function in (2.3) is interpreted as uniquely determined up to a parallel transformation. That is, as will be clarified in Section 3.1, if $V_i(x, \xi) = v_i(x) + \xi$ and $v_i(x) = u_i(x) + c$ for some constant c, then $V_i(x, \xi)$ can be regarded as equivalent to $U_i(x, \xi)$.

The term "transferable utility" is motivated by the following observation. When two players have utility functions of form (2.3), since the utility level of each player changes by the amount of a transfer, a transfer of money between the players appears to be a transfer of utility.

Let $x_0 \in X$ be an arbitrarily chosen outcome, interpreted as an initial situation or the "status quo". For a utility function $U_i(x, \xi)$ of form (2.3), it holds that for any $x \in X$

$$U_i(x, \xi) = U_i(x_0, u_i(x) - u_i(x_0) + \xi).$$

This formula implies that $u_i(x) - u_i(x_0)$ represents the monetary equivalent of the change in utility brought about by the change from x_0 to x. In other words,

$u_i(x) - u_i(x_0)$ is the amount of "willingness to pay" for the transition of the outcome from x_0 to x.

In the terminology of economics, (2.3) implies that there are no income effects on the choice behavior of player i. "No income effects" means that preferences over X are independent of money holdings, that is, $u_i(x) - u_i(x_0)$ does not depend on ξ. TU is also a sufficient condition for the well-definedness of consumer surplus. When player i compares paying the amounts of money p and p_0 for x and x_0 respectively, his "surplus" due to the change from x_0 to x is $(u_i(x) - p) - (u_i(x_0) - p_0)$. When $u_i(x_0) - p_0$ is normalized to equal zero by a parallel transformation of u_i, $u_i(x) - p$ is defined as the player's *consumer surplus* due to the change [cf. Hicks (1956)].

The no income effects condition can be regarded as a local approximation to a situation where the initial income of the consumer is large relative to any money transfers that may arise in the game. This also provides a justification for the assumption of the unbounded domain of money, which will be further clarified in the next subsection. We should always keep these justifications in mind: some applications or extensions of games with side payments are not consistent with these justifications.

"Side payments" simply means that transfers of money are allowed, in addition to any sort of transfer embodied in the outcome x. The assumption of side payments is independent of the assumption of transferable utility. In the following, we consider the role of side payments in the contexts of market games, majority voting games and games derived from strategic games.

2.2 A Market Game

Consider an exchange economy with players $1, ..., n$ and commodities $1, ..., m$, $m+1$. The set X, called the *consumption space* of the first m commodities, is taken as the non-negative orthant $\mathbb{R}^m_+$ of $\mathbb{R}^m$. Each player i has an *endowment* of commodities $\omega_i \in X$, describing his initial holding of the first m commodities. Each player also has an endowment of the $(m+1)^{th}$ commodity but this is assumed to be sufficiently large so that it is not binding. Thus we do not need to specify the endowment of money: only increments or decrements from the initial level are considered. The value $U_i(x, \xi)$ represents the utility from consuming commodity bundle x and the initial money holdings plus the increment or decrement in ξ.

Under the transferable utility assumption (2.3) for all players, a game with side payments is defined as follows: for each coalition S,[2]

$$v(S) = \max \sum_{i \in S} u_i(x_i) \quad \text{subject to}$$
$$\sum_{i \in S} x_i = \sum_{i \in S} \omega_i \text{ and } x_i \in X \quad \text{for all } i \in S. \tag{2.4}$$

The characteristic function assigns to each coalition the maximum total payoff achievable by exchanges of commodities among the members of the coalition.

The characteristic function (2.4) may *appear* to suggest that the players in a coalition maximize total utility and players make interpersonal comparisons of utilities. As we discuss below, some solution concepts based on the characteristic function may indeed make interpersonal comparisons of utilities. These deviate from the interpretation of the value $v(S)$ as simply a description of the Pareto frontier and the feasible payoffs for S, as intended by von-Neumann-Morgenstern (1944). Definition (2.4) itself is a mathematical one and does not involve any behavioral assumptions or interpersonal utility comparisons.

To describe the Pareto frontier, define an *allocation* $(x_i, \xi_i)_{i \in S}$ *for* S by

$$(x_i, \xi_i) \in X \times \mathbb{R} \text{ for all } i \in S \text{ and } \sum_{i \in S} (x_i, \xi_i) = \sum_{i \in S} (\omega_i, 0). \tag{2.5}$$

An allocation $(x_i, \xi_i)_{i \in S}$ is said to be *Pareto-optimal for* S iff there is no other allocation $(y_i, \eta_i)_{i \in S}$ for S such that

$$U_i(y_i, \eta_i) \ \geq U_i(x_i, \xi_i) \text{ for all } i \in S; \text{ and}$$
$$U_i(y_i, \eta_i) \ > U_i(x_i, \xi_i) \text{ for some } i \in S. \tag{2.6}$$

The value $v(S)$ describes Pareto-optimal allocations in the following sense:

PROPOSITION 2.1 An allocation $(x_i, \xi_i)_{i \in S}$ is Pareto-optimal for S if and only if $v(S) = \sum_{i \in S} u_i(x_i)$.

PROOF If $(x_i, \xi_i)_{i \in S}$ is not Pareto-optimal for S, then (2.6) holds for some allocation $(y_i, \eta_i)_{i \in S}$ for S. This, together with $\sum_{i \in S} \eta_i = \sum_{i \in S} \xi_i = 0$, implies that $\sum_{i \in S} u_i(x_i) < \sum_{i \in S} u_i(y_i) \leq v(S)$. Conversely, if $\sum_{i \in S} u_i(x_i) < v(S)$, there is a feasible vector $(y_i)_{i \in S}$ with $\sum_{i \in S} u_i(x_i) < \sum_{i \in S} u_i(y_i)$. This implies that (2.6) holds for an appropriate choice of $(\eta_i)_{i \in S}$. ∎

[2]When $u_i(x_i)$ is continuous, the following maximization problem is well defined. In the sequel, when we use "max", we assume that the maximum is well defined.

The characteristic function v delineates feasible payoffs. In the market, the *feasibility* of payoffs $(a_i)_{i\in S}$ for coalition S is given by:

$$\text{for some allocation } (x_i, \xi_i)_{i\in S},\ a_i \leq u_i(x_i) + \xi_i \text{ for all } i \in S. \tag{2.7}$$

This feasibility is summarized by the characteristic function v, since $(a_i)_{i\in S}$ is feasible for S if and only if

$$\sum_{i\in S} a_i \leq v(S). \tag{2.8}$$

Thus, we obtain (2.1). In the terminology of economics, the value $v(S)$ is the maximum sum of the consumer surpluses over the players in S. From the no income effects condition, this sum $v(S)$ is independent of the distribution of the money holdings among the members of S.

The definition of Pareto-optimality (2.6) is unaffected by monotone increasing transformations of utility functions. That is, the Pareto-optimality of an allocation for a coalition S is unaffected by such transformations of the utility functions of its members. On the other hand, the definition of the characteristic function requires particular (transformations of) utility functions. Nevertheless, Proposition 2.1 guarantees that the value $v(S)$ determines the Pareto frontier for coalition S.

In the context of markets, the side payments assumption simply means that transfers of the last commodity are possible. In other contexts such as voting games, discussed in the next subsection, side payments have a nontrivial meaning.

One criticism of the above formulation is that players' allocations of money are unbounded below. Even with budget constraints, if incomes are sufficiently large relative to the value of the activity of the game, then individual rationality will ensure that the budget constraint is not binding. This can be formulated without any difficulty; the budget constraint is ignored in the above formulation for simplicity. Moreover, the TU assumption implies that there are no income effects. This suggests that amounts of payments and receipts of transfers must be small relative to incomes. Thus, to ignore the budget constraint is consistent with the interpretation of TU as the absence of income effects. If income effects are not negligible, the concept of a game without side payments is more appropriate.

An implication of the above paragraph is that a game with side payments and certain solution concepts may be inappropriate if the game is large and large coalitions are required to achieve the solution of the game. That is, these payoffs may require transfers from individual players to other players in excess of the players' (hidden) budget constraints. This may be the case, for example, for the von-Neumann-Morgenstern stable set concept, to be discussed in Section 4.

2.3 A Majority Voting Game with Side Payments

In the above example, side payments arise naturally. In voting games, this may not be the case. In fact, side payments may well be prohibited.

Consider a voting situation with n players where one alternative is chosen by majority voting from a set X of social alternatives. Suppose that the utility function of each player i, U_i: $X \times \mathbb{R} \to \mathbb{R}$, is of form (2.3). Define a characteristic function $v : 2^N \to \mathbb{R}$ by

$$v(S) = \begin{cases} \max_{x \in X} \sum_{i \in S} u_i(x) \text{ if } |S| > \frac{n}{2} \\ \\ \min_{x \in X} \sum_{i \in S} u_i(x) \text{ otherwise,} \end{cases} \tag{2.9}$$

where $|S|$ is the number of members in S. A majority coalition S, $|S| > \frac{n}{2}$, can choose any social alternative x from X. Therefore the members of a majority coalition can maximize the total payoff $\sum_{i \in S} u_i(x)$. A minority coalition S, $|S| \leq \frac{n}{2}$, cannot make an effective choice. Thus $v(S)$ is defined as the value the members of S can certainly guarantee for themselves.

The main issue of the majority voting game is the choice of a social alternative $x \in X$. Besides choosing x, the players are allowed to make transfers of money, that is, side payments. This allows the possibility of obtaining the consent of other players to a particular outcome by purchasing their votes.

In a voting game, as in a market game, for a majority coalition S the value $v(S)$ determines the Pareto frontier for S. For a minority coalition S, $v(S)$ also determines the Pareto frontier among all feasible outcomes that the members of S can guarantee for themselves.

In the market game of the above subsection, side payments have only a trivial meaning in the sense that transfers of money are essential to the definition of a market; if such transfers are prohibited, the situation is no longer a market. On the other hand, in voting situations, such side payments are sometimes difficult or regarded as impossible. In such a case, the formulation (2.9) is inappropriate: we need the formulation of a game without side payments.

The following example illustrates a difference between the cases with and without side payments.

EXAMPLE 2.1 Consider a three-person voting games with total player set $N = \{1, 2, 3\}$ and $X = \{x, y\}$. The utility functions of the players are given by

$$u_1(x) = u_2(x) = 10, \; u_3(x) = 0,$$

$$u_1(y) = u_2(y) = 0, \;\; u_3(y) = 15.$$

The characteristic function, calculated according to (2.9), is:

$$v(N) = 20, \quad v(\{1,2\}) = 20,$$

$$v(\{2,3\}) = v(\{1,3\}) = 15, \text{ and } v(\{1\}) = v(\{2\}) = v(\{3\}) = 0.$$

An individually rational and Pareto optimal payoff is given by a vector $a = (a_1, a_2, a_3)$ where $a_1 + a_2 + a_3 = 20$ and $a_i \geq 0$ for all $i = 1, 2, 3$. Suppose $a = (5, 5, 10)$. To achieve the payoff vector a, it is necessary that each of players 1 and 2 make a transfer of 5 to player 3. When side payments are prohibited, only two payoff vectors, $(10, 10, 0)$ and $(0, 0, 15)$, are possible. Thus the voting situations with and without side payments are dramatically different.

Our next example illustrates a voting game where the description of the set of alternatives X effectively allows side payments.

EXAMPLE 2.2 Consider again a three-person voting game with total player set $N = \{1, 2, 3\}$ but $X = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1 \text{ and } x_i \geq 0$ for all $i = 1, 2, 3\}$; one dollar is to be distributed by majority voting. The utility functions of the players are given by

$$U_i(x, \xi) = x_i + \xi \quad \text{ for } (x, \xi) \in X \times \mathbb{R} \text{ and } i = 1, 2, 3.$$

Then the derived characteristic function is

$$v(S) = \begin{cases} 1 \text{ if } |S| \geq 2 \\ 0 \text{ otherwise.} \end{cases}$$

In the game theory literature this is often called a *simple majority game*. An individually rational and Pareto-optimal payoff vector for the game is given by (a_1, a_2, a_3) with $a_1 + a_2 + a_3 = 1$ and $a_i \geq 0$ for all $i = 1, 2, 3$. In fact, this payoff vector can be realized without any transfers. Therefore side payments are effectively built into the description of the political situation underlying the game with side payments.

Example 2.1 illustrates a situation where allowing side payments dramatically changes the set of feasible outcomes. In Example 2.2, side payments are an intrinsic part of the game and whether side payments are allowed does not affect the set of possible outcomes. In general, "direct" payments may be included in the description of the game. The assumption of SP further facilitates the possibility of payments between players.

2.4 A Cooperative Game Derived from a Strategic Form Game

Let $G = (N, \{\Sigma_i\}_{i\in N}, \{h_i\}_{i\in N})$ be an n-person finite strategic form game, that is, $N = \{1, ..., n\}$ is the player set, Σ_i is a finite strategy space for player $i \in N$, and $h_i : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{R}$ is the payoff function of player i. The *space of mixed strategies* of player i is the set of all probability distributions over Σ_i, denoted by $M(\Sigma_i)$. Note that $M(\Sigma_i)$ is the $\mid \Sigma_i \mid -1$ dimensional unit simplex. When the players in a coalition S cooperate, they can coordinate their strategies to play a *joint mixed strategy*, a probability distribution over $\Sigma_S = \Pi_{i\in S}\ \Sigma_i$. We denote the set of all joint strategies for S by $M(\Sigma_S)$, which is also a unit simplex. The payoff function $h_i(\cdot)$ is extended to $M(\Sigma_N)$ as the expectation of $h_i(s)$ over Σ_N. In fact, we substitute $h_i(\cdot)$ for u_i in the expression (2.3) so that $U_i(s, \xi) = h_i(s)+\xi$. Thus the whole utility function U_i is defined on $M(\Sigma_N)\times\mathbb{R}$, where the set X of Section 2.1 is now $M(\Sigma_N)$.

In the derivation of a cooperative game with side payments from a strategic game, transfers of money between the players in a coalition are permitted, that is, SP is assumed. When transferable utility in the sense of Subsection 2.1 is assumed, the total utility

$$\sum_{i\in S} h_i(\sigma_S, \sigma_{-S}), \quad \text{where } \sigma_S \in M(\Sigma_S) \text{ and } \sigma_{-S} \in M(\Sigma_{-S}),$$

is independent of the monetary transfers. This means that the total utility $\sum_{i\in S}\ h_i(\sigma_S, \sigma_{-S})$ can be freely distributed among the players in S by the players via side payments $(\xi_i)_{i\in S}$ with $\sum_{i\in S} \xi_i = 0$. Each player evaluates an outcome (σ_S, σ_{-S}) by the expected value of $h_i(\cdot)$ and may make transfers to other players in return for the agreements to play the joint mixed strategy.

Von Neumann and Morgenstern (1944) defined the characteristic function v,

$$v(S) = \max_{\sigma_S\in M(\Sigma_S)}\ \min_{\sigma_{-S}\in M(\Sigma_{N-S})} \sum_{i\in S} h_i(\sigma_S\ , \sigma_{-S}) \text{ for all } S \in 2^N. \tag{2.10}$$

That is, the value $v(S)$ is defined by regarding the game situation as a two-person zero-sum game with one player taken as S and the other as the complementary coalition $N-S$.

The situation discussed in this subsection differs from the previous situations in that uncertainty is involved; players can choose joint mixed strategies.[3] This raises the question of the rationales for TU and SP. We will discuss this further in Section 3. There, we assume that when side payments are permitted, even

[3]By uncertainty we mean the Knightian risk in the sense that probabilities are well-defined and objectively gernerated.

though they might play mixed strategies, players can make monetary transfers without uncertainty.[4]

In the standard treatment of a strategic game $G = (N, \{\Sigma_i\}_{i \in N}, \{h_i\}_{i \in N})$, the game G is a closed world in the sense that no additional structure is assumed. In the treatment here, side payments can be made by making transfers of money. Money represents purchasing power in the world outside the game. In this sense, here the game is not a closed world.

3 Axiomatic Characterization of Transferable Utility

It may be helpful in understanding the assumption of transferable utility to look at an axiomatic characterization of preferences having transferable utility representations. We will discuss axioms for both preferences over outcomes with and without uncertainty. The derivation with no uncertainty is close to the classical utility theory [cf., Debreu (1959)]. With uncertainty, the derivation is a special case of the von Neumann-Morgenstern utility theory.

3.1 Transferable Utility with no Uncertainty

In the absence of uncertainty, a preference relation $\succsim_i$ is defined on $X \times \mathbb{R}$. Consider the following four conditions on $\succsim_i$:

(T1): $\succsim_i$ is a complete preordering on $X \times \mathbb{R}$;

(T2): $\succsim_i$ is strictly monotone on $\mathbb{R}$;

(T3): for any $(x, \xi), (y, \eta) \in X \times \mathbb{R}$ with $(x, \xi) \succsim_i (y, \eta)$, there is an $\epsilon \in \mathbb{R}$ such that $(x, \xi) \sim_i (y, \eta + \epsilon)$;

(T4): $(x, \xi) \sim_i (y, \eta)$ and $\epsilon \in \mathbb{R}$ imply $(x, \xi + \epsilon) \sim_i (y, \eta + \epsilon)$,

where $\sim_i$ is the indifference part of the relation $\succsim_i$. Conditions T1 and T2 are standard. Condition T3 means that some amount of money substitutes for a change in outcome. Under T2 the ϵ in T3 is nonnegative. Condition T4, the most essential for TU, means that the player's choice behavior on X does not depend on his money holdings.

The following result[5] holds [cf. Kaneko (1976)]:

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[4]This is a simplifying assumption. Since we assume risk neutrality, uncertain monetary transfers are equivalent to transfers without uncertainty.

[5]Applying T4 to the expression $U_i(x, \xi)$, Aumann (1960) obtained $u_i(x) + \xi$.

PROPOSITION 3.1 A preference relation $\succsim_i$ satisfies T1-T4 if and only if there is a function $u_i : X \to \mathbb{R}$ such that $(x, \xi) \succsim_i (y, \eta) \Leftrightarrow u_i(x) + \xi \geq u_i(y) + \eta$.

A utility function $U_i(x, \xi) = u_i(x) + \xi$ is one representation of a preference relation $\succsim_i$ satisfying T1-T4. Note that any monotone transformation $\varphi(u_i(x) + \xi)$ of U_i is also a representation of the preference relation $\succsim_i$. Nevertheless, as already seen in Section 2, the representation $u_i(x) + \xi$ has a special status in defining a game with side payments.

Without assuming T4 and with a specified total income I_i, we would not have a utility function of the form $u_i(x) + \xi$. When we would like to consider a game situation with income effects, T4 may be modified to express income effects. Specifically: for any $(x, \xi), (y, \eta) \in X \times \{\zeta \in \mathbb{R} : \zeta \geq -I_i\}$,

(T4′): $(x, \xi) \sim_i (y, \eta)$ and $\xi < \eta$ imply $(x, \xi + \varepsilon) \succsim_i (y, \eta + \varepsilon)$ for any $\varepsilon > 0$.

In economics terminology, the decision variable x is assumed to be "normal". This means that when income increases, player i chooses (or demands) a "better" x than y.[6] Under T4′ together with T1-T3 and some additional condition, e.g., X is a finite set, we can obtain a utility function representation of a preference, but not necessarily of the form $u_i(x) + \xi$. In this case, a game without side payments is required.

PROOF OF PROPOSITION 3.1 If there is a utility function U_i of form (2.3), then $\succsim_i$ determined by U_i satisfies T1-T4. Suppose, conversely, that $\succsim_i$ satisfies T1-T4. Choose an arbitrary x_0 in X. For each x in X, define $u_i(x)$ by

$$u_i(x) = \eta - \xi, \text{ where } (x, \xi) \sim_i (x_0, \eta). \tag{3.1}$$

The existence of such numbers ξ and η is ensured by T3 and the difference $\eta - \xi$ is uniquely determined by T2 and T4. Note that (3.1) and T2 imply $u_i(x_0) = 0$ and $(x, \xi) \sim_i (x_0, u_i(x) + \xi)$, i.e., $u_i(x)$ is the amount of willingness-to-pay for the transition from x_0 to x. The function $u_i(x)$ represents the preference relation $\succsim_i$. Indeed, $(x, \xi) \succsim_i (y, \eta) \Longleftrightarrow (x_0, u_i(x) + \xi) \sim_i (x, \xi) \succsim_i (y, \eta) \sim_i (x_0, u_i(y) + \eta) \Longleftrightarrow u_i(x) + \xi \geq u_i(y) + \eta$. ∎

The following facts hold [Kaneko (1976)]:

$$U_i(x, \xi) = u_i(x) + \xi \text{ is quasi-concave iff } u_i(x) \text{ is concave;} \tag{3.2}$$

$$U_i(x, \xi) = u_i(x) + \xi \text{ is continuous iff } u_i(x) \text{ is continuous.} \tag{3.3}$$

[6] This is useful, especially for comparative statics analysis [cf. Kaneko (1983)].

In (3.2) and (3.3) some algebraic and topological structures on X are assumed. It follows from (3.2) and (3.3) that a condition for $\succsim_i$ to be convex or to be continuous is the concavity or continuity of u_i respectively.

3.2 Transferable Utility with Uncertainty

When the game situation involves uncertainty, as in Section 2.4, $U_i(x, \xi) = u_i(x) + \xi$ is a von Neumann-Morgenstern utility representation. In this case, the domain of a preference relation $\succsim_i$ is the set of probability distributions on $X \times \mathbb{R}$. Now we describe conditions on a preference relation $\succsim_i$ with this domain implying that there exists a utility function representation of form (2.3).

A *probability distribution on* $X \times \mathbb{R}$ *with finite support* is a function $p : X \times \mathbb{R} \to [0, 1]$ satisfying the property that for some finite subset S of $X \times \mathbb{R}$, $\sum_{t \in S} p(x, \xi) = 1$ and $p(x, \xi) > 0$ implies $(x, \xi) \in S$. We extend $X \times \mathbb{R}$ to the set $M(X \times \mathbb{R})$ of all probability distributions on $X \times \mathbb{R}$ with finite supports. Regarding a one-point distribution $f_{(x,\xi)}$ (i.e., $f_{(x,\xi)}(x, \xi) = 1$) as (x, ξ) itself, the space $X \times \mathbb{R}$ becomes a subset of $M(X \times \mathbb{R})$. Also, $M(X) \times \mathbb{R}$ is a subset of $M(X \times \mathbb{R})$; this is relevant in Section 2.4 (where we take X as Σ_N). For $p, q \in M(X \times \mathbb{R})$ and $\lambda \in [0, 1]$, we define a convex combination $\lambda p * (1 - \lambda)q$ by

$$(\lambda p * (1 - \lambda)q)(x, \xi) = \lambda p(x, \xi) + (1 - \lambda)q(x, \xi) \text{ for all } (x, \xi) \in X \times \mathbb{R}. \quad (3.4)$$

With this operation, $M(X \times \mathbb{R})$ is a convex set. Usually, $\lambda p * (1 - \lambda)q$ is regarded as a compound lottery in the sense that p and q occur with probabilities λ and $(1 - \lambda)$ respectively and then the random choice according to p or q is made. Condition (3.4) requires that the compound lottery be reducible into one lottery.

We impose the following three axioms on $\succsim_i$;

(NM1): $\succsim_i$ is a complete preordering on $M(X \times \mathbb{R})$;

(NM2): $p \succsim_i q \succsim_i r$ implies $\alpha p * (1 - \alpha)r \sim_i q$ for some $\alpha \in [0, 1]$;

(NM3): for any $\alpha \in (0, 1)$, (1) $p \succ_i q$ implies $\alpha p * (1 - \alpha)r \sim\succ_i \alpha q * (1 - \alpha)r$; and (2) $p \sim_i q$ implies $\alpha p * (1 - \alpha)r \sim_i \alpha q * (1 - \alpha)r$, where $\succ_i$ is the nonsymetric part of $\succsim_i$.

Condition NM1 is the same as T1 except that condition NM1 is applied to the larger domain $M(X \times \mathbb{R})$; thus NM1 implies T1. Condition NM2 states that for any lottery q between two other lotteries p and r, there is a compound lottery $\alpha p * (1 - \alpha)r$ indifferent to q. Condition NM2, as condition T3, is a continuity property. Condition NM3 is called the independence axiom, a sort of "Sure-Thing" Principle. This condition means that the comparison of compound lotteries is based on the (surer) outcomes of these lotteries, which implies that the evaluation of a lottery depends eventually upon the sure outcomes of the lottery, as is shown in (3.7) below.

The following is known as the Expected Utility Theorem [cf. von Neumann-Morgenstern (1944), Herstein-Milnor (1953)]:

PROPOSITION 3.2 A preference relation $\succsim_i$ satisfies NM1-NM3 if and only if there is a function $V_i : M(X \times \mathbb{R}) \to \mathbb{R}$ such that for any $p, q \in M(X \times \mathbb{R})$ and $\lambda \in [0, 1]$,

$$p \succsim_i q \Longleftrightarrow V_i(p) \geq V_i(q); \tag{3.5}$$

$$V_i(\lambda p * (1 - \lambda)q) = \lambda V_i(p) + (1 - \lambda)V_i(q). \tag{3.6}$$

The function V_i is called a *von Neumann-Morgenstern utility function*. In contrast to the representability in Section 3.1, $V_i(x, \xi)$ allows only a positive linear transformation, not necessarily an arbitrary monotonic transformation, i.e., if U_i also satisfies (3.5) and (3.6), there are real numbers $a > 0$ and b such that $U_i(p) = aV_i(p) + b$ for all $p \in M(X \times \mathbb{R})$.

Since $X \times \mathbb{R}$ is a subset of $M(X \times \mathbb{R})$, V_i assigns a value $V_i(x, \xi)$ to each (x, ξ) in $X \times \mathbb{R}$. For each $p \in M(X \times \mathbb{R})$, the value $V_i(p)$ is represented as the expected value of $V_i(x, \xi)$ with $p(x, \xi) > 0$. Indeed, since each $p \in M(X \times \mathbb{R})$ has finite support S, by repeating application of (3.6) a finite number of times, we obtain

$$V_i(p) = \sum_{(x,\xi) \in S} p(x, \xi)V_i(x, \xi). \tag{3.7}$$

That is, the utility from the probability distribution p is given as the expected utility value with respect to the distribution p. This fact motivates the term "expected utility theory".[7]

Proposition 3.2, the Expected Utility Theorem, is more fully discussed in Hammond (1998). Here we give a sketch of the proof.

PROOF OF PROPOSITION 3.2 The "if" part is straightforward. Consider the "only-if" part. First of all, we note that it follows from NM1 and NM3.(1) that

$$p \succ_i q \text{ and } \alpha > \beta \text{ imply } \alpha p * (1 - \alpha)q \succ_i \beta p + (1 - \beta)q. \tag{3.8}$$

[7]The space $M(X \times \mathbb{R})$ of probability distributions with finite supports is not big enough to treat some interesting examples such as the St. Petersburg game. For this purpose $M(X \times \mathbb{R})$ can be extended to the space of probability distributions with countable supports, for which Proposition 3.2 holds. To obtain (3.7) for a distribution with countable support, however, an additional condition is required.

Indeed, since $p = \frac{\beta}{\alpha}p * (1-\frac{\beta}{\alpha})p \succ_i \frac{\beta}{\alpha}p * (1-\frac{\beta}{\alpha})q$, we have $\alpha p * (1-\alpha)q \succ_i \alpha(\frac{\beta}{\alpha}p * (1-\frac{\beta}{\alpha})q) * (1-\alpha)q = \beta p * (1-\beta)q$.

Suppose that $a \succ_i b$ for some $a, b \in M(X \times \mathbb{R})$. If such distributions a and b do not exist, the claim is shown by assigning zero to every p. Now we define $V_{ab}(p)$ for any p with $a \succsim_i p \succsim_i b$ by

$$V_{ab}(p) = \lambda, \text{ where } p \sim_i \lambda a * (1-\lambda)b. \tag{3.9}$$

The unique existence of such λ is ensured by NM2 and (3.8). Then it follows from NM1 and (3.8) that $V_{ab}(p) \geq V_{ab}(q) \Leftrightarrow p \succsim_i q$, which is (3.5). Finally, $\mu := V_{ab}(\lambda p * (1-\lambda)q)$ satisfies

$$\begin{aligned} \mu a * (1-\mu)b &\sim_i \lambda p * (1-\lambda)q \quad \text{(by (3.9))} \\ &\sim_i \lambda[V_{ab}(p)a * (1-V_{ab}(p))b] * (1-\lambda)[V_{ab}(q)a * (1-V_{ab}(q)b] \quad \text{(by NM3.(2))} \\ &\sim_i [\lambda V_{ab}(p) + (1-\lambda)V_{ab}(q)]a * (1-[\lambda V_{ab}(p) + (1-\lambda)V_{ab}(q)])b. \end{aligned}$$

The coefficients for a in the first and last terms must be the same by NM1 and (3.8), that is, $\mu = V_{ab}(\lambda p * (1-\lambda)q) = \lambda V_{ab}(p) + (1-\lambda)V_{ab}(q)$. Thus (3.6) holds.

It remains to extend the function V_{ab} to the entire space $M(X \times \mathbb{R})$. We give a sketch of how this extension is made [cf., Herstein-Milnor (1953) for a more detailed proof]. Let c, d, e, f be arbitrary elements in $M(X \times \mathbb{R})$ with $e \succsim_i c \succsim_i a$ and $b \succsim_i d \succsim_i f$. Applying the above proof, we obtain utility functions V_{cd} and V_{ef} satisfying (3.5) and (3.6) with domains $\{p : c \succsim_i p \succsim_i d\}$ and $\{p : e \succsim_i p \succsim_i f\}$. Then $V_{cd}(c) = V_{ef}(e) = 1$ and $V_{cd}(d) = V_{ef}(f) = 0$. We define new utility functions U_{cd} and U_{ef} by the following positive linear transformations:

$$U_{cd}(p) = (V_{cd}(p) - V_{cd}(a))/(V_{cd}(b) - V_{cd}(a)) \text{ for all } p \text{ with } c \succsim_i p \succsim_i d;$$

$$U_{ef}(p) = (V_{ef}(p) - V_{ef}(a))/(V_{ef}(b) - V_{ef}(a)) \text{ for all } p \text{ with } e \succsim_i p \succsim_i f.$$

Then it can be shown that these functions U_{cd} and U_{ef} coincide on $\{p : c \succsim_i p \succsim_i d\}$. This fact ensures that we can define $V_i(p) = U_{cd}(p)$ for any $p \in M(X \times \mathbb{R})$, where c, d are chosen so that $c \succsim_i p \succsim_i d$ and $c \succsim_i a \succ_i b \succsim_i d$. Since $U_{cd}(p)$ satisfies (3.5) and (3.6), so does the function V_i. ∎

When $\succsim_i$ satisfies T2-T4 on the domain $X \times \mathbb{R}$ in addition to NM1-NM3 on $M(X \times \mathbb{R})$, it holds that there is a monotone function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying

$$V_i(x, \xi) = \varphi(u_i(x) + \xi) \text{ for all } (x, \xi) \in X \times \mathbb{R}.$$

Indeed, since the preference $\succsim_i$ over $X \times \mathbb{R}$ is represented by $u_i(x) + \xi$ and is also represented by the restriction of V_i to $X \times \mathbb{R}$, the functions $u_i(x) + \xi$ and

$V_i(x, \xi)$ are related by a monotone transformation φ. The function φ expresses the risk attitude of player i.

For $u_i(x) + \xi$ to be a von Neumann-Morgenstern utility function, we need one more assumption:

(RN): $\frac{1}{2}(x, \xi) * \frac{1}{2}(x, \eta) \sim_i (x, \frac{1}{2}\xi + \frac{1}{2}\eta)$ for all $(x, \xi), (x, \eta) \in X \times \mathbb{R}$.

This assumption describes risk neutrality with respect to money; given x, player i is indifferent between ξ and η with equal probabilities and the average of ξ and η.

From RN and (3.6) it follows that

$$\frac{1}{2}\varphi(u_i(x) + \xi) + \frac{1}{2}\varphi(u_i(x) + \eta) = \varphi(u_i(x) + \frac{1}{2}\xi + \frac{1}{2}h). \tag{3.10}$$

Indeed since ξ, η are arbitrary elements of $\mathbb{R}$, $u_i(x) + \xi$ and $u_i(x) + \eta$ can take arbitrary values. Thus (3.10) can be regarded as a functional equation: for each α and β in $\mathbb{R}$,

$$\frac{1}{2}\varphi(\alpha) + \frac{1}{2}\varphi(\beta) = \varphi(\frac{1}{2}\alpha + \frac{1}{2}\beta).$$

This, together with the monotonicity of φ, implies that φ can be represented as $\varphi(\alpha) = a\alpha + b$ for all α, where $a > 0$ and b are given constants. We can normalize a and b to be $a = 1$ and $b = 0$. Thus we have the following:

PROPOSITION 3.3 A preference relation $\succsim_i$ satisfies NM1-NM3, T2-T4 and RN if and only if $\succsim_i$ is represented by a utility function of the form $V_i(x, \xi) = u_i(x) + \xi$ in the sense of (3.5), (3.6) and (3.7).

The assumption RN of risk neutrality coheres to the assumption TU for the following reason. We can regard the theory of games we are considering here as static but describing a recurrent sitation behind the theory. Each game is a representative of the recurrent and stationary game situation, and the solution represents a stationary state. In this interpretation, TU means that incomes are large enough relative to money transfers occurring in each period. Suppose that financial institutions are well developed in the sense that each individual can borrow or lend money freely if borrowings are small relative to his income. Then an individual player can borrow and lend money to make his consumption level constant over periods. This is preferable if his utility function is concave over consumption. In this case, it suffices to calculate the average gains or loses in such a recurrent situation. Hence his utility function can be best approximated by risk-neutral utility function.

4 Solution Concepts for Games with Side Payments

A game with (or without) side payments describes the payoff each coalition S can obtain by the cooperation of the members of S. Solution theory addresses the question of how payoffs are distributed. Each solution concept explicitly or implicitly describes the behavior of coalitions and makes some prediction on the occurrence of distributions of payoffs. Some solution concepts are faithful to the basic objective of the definition of the characteristic function discussed in Section 2, but some depend critically upon the numerical expression of the characteristic function. In this section, we discuss four solution concepts, namely, the core, the von Neumann-Morgenstern stable set, the nucleolus, and the Shapley value.

Let a game (N, v) with side payments be given. An *imputation* is an individually rational Pareto optimal payoff vector $a = (a_1, ..., a_n)$, that is, a satisfies $a_i \geq v(\{i\})$ for all $i \in N$ and $\sum_{i \in N} a_i = v(N)$. We denote the set of imputations by $I(N, v)$. For imputations a and b in $I(N, v)$, we say that *a dominates b via a coalition S*, denoted by a *dom*$_S$ b, iff

$$a_i > b_i \text{ for all } i \in S \tag{4.1}$$

and

$$v(S) \geq \sum_{i \in S} a_i. \tag{4.2}$$

Condition (4.1) means that every player in S prefers a to b and (4.2), called *effectiveness* by von Neumann-Morgenstern (1944), means that the imputation a is feasible for coalition S [cf. (2.7) and (2.8)]. We denote a *dom*$_S$ b for some S by a *dom* b.

4.1 The Core

The *core* is defined to be the set of all undominated imputations, that is,

$$\{a \in I(N, v) : \text{not } b \textit{ dom } a \text{ for any } b \in I(N, v)\}.$$

Although the core is defined to be a set, the stability property of the core is an attribute of each imputation in the core. The core can alternatively be defined to be the set of all imputations satisfying *coalitional rationality:*

$$\sum_{i \in S} a_i \geq v(S) \text{ for all } S \in 2^N. \tag{4.3}$$

In the market game of Section 2.2, if $v(S) > \sum_{i \in S} a_i$ for some coalition S then there is an allocation $(x_i, \xi_i)_{i \in S}$ for S such that $u_i(x_i) + \xi_i > a_i$ for all

$i \in S$, that is, the players in S can be better off by their own exchanges of commodities. The coalitional rationality of the core rules out such possibilities. This definition simply depends upon individual preferences and the feasibility described by the characteristic function. No interpersonal comparisons are involved in the definition of the core.[8,9]

For two-player games, the core is simply the imputation space $I(N, v)$. For more than two players, games may have empty cores. In the following we consider the role of side payments in some examples of games with empty cores and some with nonempty cores.

The core of the game (N, v) of Example 2.1 is empty. Indeed, consider the regular triangle with height 20, as in Figure 4.1.

Each point in the triangle in Figure 4.1 corresponds to a vector (a_1, a_2, a_3), where a_i is the height of the perpendicular to the base i. The inequalities

$$a_1 + a_2 \geq 20 = v(\{1,2\}),\ a_2 + a_3 \geq 15 = v(\{2,3\}),$$

$$\text{and } a_1 + a_3 \geq 15 = v(\{1,3\})$$

determine the areas that the corresponding coalitions can guarantee. The core is the intersection of those three areas. In this example, the core is empty.

Example 3.1 Consider another three-person voting games with total player set $N = \{1, 2, 3\}$ and $X = \{x, y\}$. The utility functions of the players are given by

$$\overline{u}_1(x) = \overline{u}_2(x) = 10,\ \overline{u}_3(x) = 0$$

$$\overline{u}_1(y) = \overline{u}_2(y) = 0,\ \ \overline{u}_3(y) = 0.$$

---

[8] In the literature on market games, the nonemptiness of the core and the relationship between the core and the competitive equilibria has been extensively studied. The reader can find a comprehensive list of references in Shubik (1984).

[9] Since side payments permit unbounded transfers of the commodity "money," the competitive equilibrium concept requires some modification. A *competitive equilibrium* is a pair $(p, (x^i, p(\omega^i - x^i))_{i \in N})$ consisting of a *price vector* p and an *allocation* $(x^i, p(\omega^i - x^i)))_{i \in N}$ with the following properties:

$$u^i(x^i) + p(\omega^i - x^i) \geq u^i(y^i) + p(\omega^i - y) \text{ for all } y \in X.$$

Since money can be traded in any amount, positive or negative, the budget constraint is non-binding. Under the assumptions of concavity and continuity on the utility functions and the assumption that $\Sigma_{i \in N}\ \omega^i > 0$, the existence of a competitive equilibrium is proven by using the Kuhn-Tucker Theorem [cf. Uzawa (1958) and Negishi (1960)].

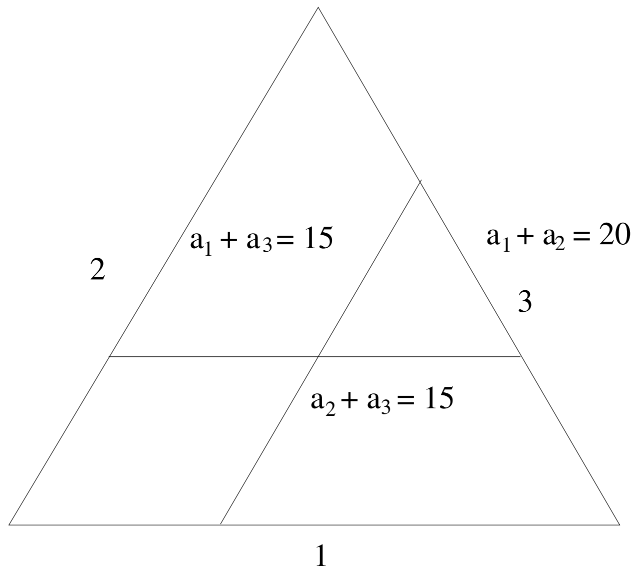

Figure 4.1 (N, v)

The characteristic functions, defined by (2.9), are given as follows:

$$\overline{v}(N) = 20, \qquad \overline{v}(\{1,2\}) = 20,$$

$$\overline{v}(\{2,3\}) = \overline{v}(\{1,3\}) = 10, \text{ and } \quad \overline{v}(\{1\}) = \overline{v}(\{2\}) = \overline{v}(\{3\}) = 0.$$

The core of the game $(N, \overline{v})$ consists of the single imputation, (10,10,0), designated by A in Figure 4.2. Since $(10{,}10{,}0) = (\overline{u}_1(x), \overline{u}_2(x), \overline{u}_3(x))$ this imputation is obtained by choosing alternative x and making no side payments. Any other imputation is dominated. For domination, side payments may be required. For example, the imputation (14,6,0) is dominated by (10,8,2). Players 2 and 3 can choose x and make a side payment of 2 units of money from player 2 to player 3, so as to ensure the payoffs of 8 and 2 for themselves.

A necessary and sufficient condition for the nonemptiness of the core of the voting game in Section 2.3 was given in Kaneko (1975). This condition states that every majority coalition has the same most preferred social alternative x^*, i.e., $v(S) = \sum_{i \in S} u_i(x^*)$ for all $S \in 2^N$ with $|S| > \frac{n}{2}$. In this case the core consists of the unique payoff vector $(u_1(x^*), ..., u_n(x^*))$; the common alternative x^* is chosen and no side payments are made.

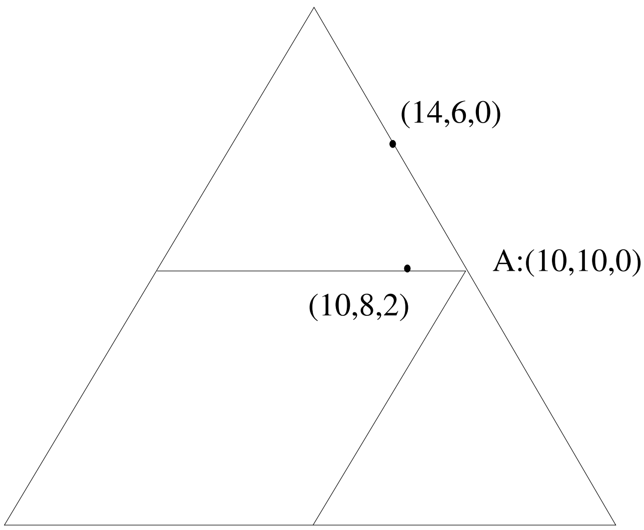

Figure 4.2 $(N, \bar{v})$

Now we will see how much side payments are required for the core. For this purpose, we consider briefly the Shapley-Shubik (1971) assignment game and its core. In the assignment game model, only pairs of players from two groups L and M ($L \cup M = N$ and $L \cap M = \emptyset$) play essential roles, i.e., an essential coalition T is $T = \{i, j\}$, $i \in L$ and $j \in M$. We denote the set of all such essential pairs by $\mathcal{P}$. Now $\Pi(S)$ denotes the set of all partitions of S into essential pairs or singleton coalitions. The value $v(S)$ of an arbitrary coalition S is obtained by partitioning coalition S into pairs and singletons, that is, a game (N, v) with side payments is called an *assignment game* iff

$$v(S) = \max_{\pi \in \Pi(S)} \sum_{T \in \pi} v(T) \quad \text{for all } S \in 2^N. \tag{4.4}$$

The assignment game has interesting applications to markets with indivisible goods [cf. Shapley-Shubik (1971)].

For the core of the assignment game (N, v), side payments are effectively required only for essential pairs. Indeed, define a *pairwise feasible payoff vector* $a = (a_1, ..., a_n)$ by $a_i \geq v(\{i\})$ for all $i \in N$, where for some partition $\pi \in \Pi(N)$, $a_i + a_j \leq v(\{i, j\})$ if $\{i, j\} \in \pi$, and $a_i = v(\{i\})$ if $\{i\} \in \pi$.

That is, a pairwise feasible payoff vector is obtained by cooperation of essential pairs in some partition π. We denote the set of all pairwise feasible payoff vectors by $P(N, v)$. This set is typically much smaller than the entire imputation space $I(N, v)$. One can prove that the core of the assignment game (N, v) coincides with the set $\{a \in P(N, v) : a_i + a_j \geq v(\{i, j\})$ for all $\{i, j\} \in \mathcal{P}\}$. In the definition of a pairwise feasible payoff vector and in coalitional rationality for essential pairs $\{i, j\} \in \mathcal{P}$, side payments are allowed only between two players in each essential coalition. Thus, for the consideration of the core of an assignment game, side payments are only required within essential coalitions. In different game models, we may not be able to make exactly the same assertion, but often a similar tendency can be found.

4.2 The von Neumann-Morgenstern Stable Set

Now consider the von Neumann-Morgenstern stable set. Let (N, v) be a game with side payments. A subset K of $I(N, v)$ is called a *stable set* iff it satisfies the following two properties:

INTERNAL STABILITY: for any $a, b \in K$, neither "*a dom b*" nor "*b dom a*";

EXTERNAL STABILITY: for any $a \in I(N, v) - K$, there is $b \in K$ such that "*b dom a*."

Von Neumann-Morgenstern described the stability property of a stable set as follows: each stable set is a candidate for a stable standard of behavior in recurrent situations of the game. Once a stable set has become socially acceptable, each imputation in the stable set is a possible stable (stationary) outcome. The stability of each outcome in the stable set is supported by the entire structure of the stable set. In general each game also has a great multiplicity of stable sets. Two of these stable sets for the above three person game examples are depicted in Figures 4.3 and 4.4.

In Figures 4.3 and 4.4, the stable sets consist of the points in the bold lines. Which outcome in a stable set and which stable set arises is determined by the history of the society. For a full explanation, see von Neumann-Morgenstern (1944).

The definition of a stable set is based on dominance relations. Thus, like the core, the definition depends only upon individual preferences and the feasibility described by the characteristic function. Nevertheless, the definition of a stable set depends crucially upon the entire imputation space $I(N, v)$, in contrast to the core. Some imputations in $I(N, v)$ need large transfers among all the players. For example, the point $B = (15, 0, 5)$ in Figure 4.4 is in the stable set and is obtained by choosing alternative x and making the transfer of 5 each to players 1 and 3 from player 2. The point $C = (0, 20, 0)$ is not in the stable set but needs to be taken into account for a stable set.

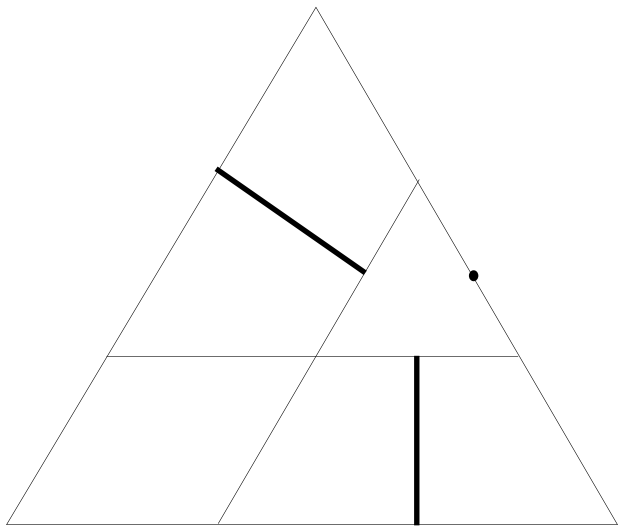

Figure 4.3 (N, v)

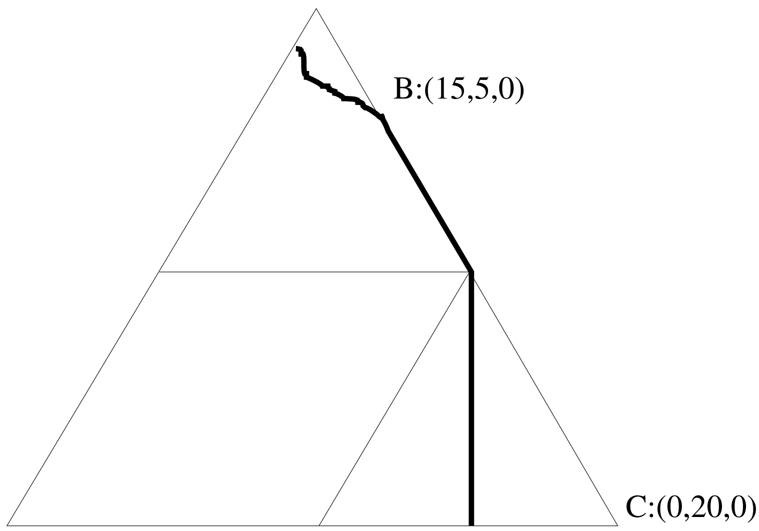

Figure 4.4 $(N, \bar{v})$

When the game involves a large number of players, the dependence of a stable set upon the entire imputation space becomes problematic. Imputations where a few players get all the surplus and the others only receive their individually rational payoffs cannot be ignored. It may require cooperation and agreement among a large number of players to make large amounts of side payments to obtain such imputations. In this case, the justification for the framework of games with side payments and with no boundary conditions for money, discussed in Section 2, becomes problematic.

In the Shapley-Shubik assignment game described above, for example, the core can be defined by coalitional rationality for essential pairs in $\mathcal{P}$ and the pairwise feasible payoff space $P(N, v)$; it does not need the entire imputation space $I(N, v)$. In contrast to the core, a stable set crucially depends upon the specification of the entire feasible payoff set. If we adopt a different set of feasible payoff vectors, a stable set would change drastically.

Here we do not intend to suggest the superiority of the core to the stable set. The stable set has a richer underlying interpretation than the core, and may give some good hints for applications of game theory to new and different models of social problems. Our intent is to suggest that simplistic applications or extensions of the stable set may violate the original justification and motivation for the framework of games with side payments.

4.3 The Nucleolus

Some solution concepts appear to make intrinsic use of the monetary representation of $v(S)$. In this and the following subsections we discuss two such solution concepts, the nucleolus and the Shapley value. It is often claimed that these concepts involve interpersonal utility comparisons. We consider how we might interpret these interpersonal comparisons.

Let (N, v) be a game and let a be an imputation. Define the "dissatisfaction" of coalition $S \in 2^N$ by

$$e(a, S) = v(S) - \sum_{i \in S} a_{i.} \tag{4.5}$$

Let $\theta(a)$ be the 2^n-vectors whose components are $e(a, S), S \in 2^N$ and are ordered in a descending way, i.e., $\theta_t(a) \geq \theta_s(a)$ for all s and t from 1 to 2^n with $t \leq s$. The lexicographic ordering $\succ_\ell$ is defined as follows:

$$\begin{gathered} a \succ_\ell b \text{ iff there is an } s \ (s = 1, \cdots, 2^n) \text{ such that} \\ \theta_t(a) = \theta_t(b) \text{ for all } t = 1, ..., s-1 \text{ and } \theta_s(a) > \theta_s(b). \end{gathered} \tag{4.6}$$

The relation $\succ_\ell$ is a complete ordering on $I(N,v)$. The *nucleolus* is defined to be the minimal element in $I(N,v)$ with respect to the ordering $\succ_\ell$. Schmeidler (1969) showed that the nucleolus exists and is unique.

The nucleolus has various technical merits. One merit is the unique existence; this facilitates comparative statics, for example. Also, when the core is nonempty, the nucleolus belongs to the core and, for any $\epsilon \geq 0$, when the ϵ-core is nonempty, the nucleolus belongs to the ϵ-core. In the examples of Section 4.1, the nucleoli are $(\frac{25}{3}, \frac{25}{3}, \frac{10}{3})$ and (10,10,0) respectively. In the first case the nucleolus is in the $\epsilon-$core, and in the second case the nucleolus coincides with the core. The nucleolus is related to other solution concepts—the bargaining set M_1^i of Aumann-Maschler (1964) and the kernel of Davis-Maschler (1965).

The nucleolus is frequently regarded as a possible candidate for a normative outcome of a game, meaning that the nucleolus expresses some equity or fairness.[10] Sometimes, it is regarded as a descriptive concept since it always belongs to the core or the ϵ-core. Either interpretation, normative or descriptive, presents, however, some difficulties related to the treatments of TU and SP. The first difficulty is in the question of how to interpret comparisons of dissatisfactions $v(S) - \sum_{i\in S} a_i$ and $v(T) - \sum_{i\in T} b_i$ for different coalitions S, T and different imputations a, b. If the dissatisfactions are compared for a single coalition, the minimization of dissatisfaction is equivalent to the original role of $v(S)$ described by (2.7) and (2.8), but comparisons are required over different coalitions. The second difficulty is the lack of motivation for the criterion of lexicographic minimization of dissatisfactions.

The first difficulty consists of two parts: (a) individual utilities (gains or losses) are compared over players; and (b) sums of utilities (gains or losses) for some players are compared for different coalitions. In either case, making such comparisons already deviates from the initial intention of the characteristic function discussed in Section 2.

Since a normative observer may be motivated to minimize dissatisfactions, the second difficulty is less problematic if the nucleolus is regarded as normative rather than as descriptive. The question here is the basis for the criterion of minimizing dissatisfactions in the lexicographic manner. Thus the first question is more relevant form the normative viewpoint.

The intuitive appeal of the nucleolus to some researchers may be based on the feature that dissatisfactions are compared using monetary units, perhaps because monetary comparisons are familiar from our everyday life. This may be the basis for interpersonal utility comparisons inherent in the nucleolus. However, this does not clarify the meaning of comparisons of dissatisfaction

[10] The normative aspect attributed to the nucleolus is derived chiefly by its similarity to Rawles' (1970) minmax principle or the leximin welfare function as the interpretation of the maximin principle given by economists.

over different coalitions. Moreover, as discussed in Section 2, the assumption of transferable utility prohibits income effects and for distributional normative issues, income effects are central.

4.4 The Shapley Value

The value, introduced by Shapley (1953), resembles the nucleolus as a game theoretical concept; it exists uniquely for any game (N, v). From the viewpoint of utility theory, the Shapley value also needs the intrinsic use of the particular definition of a characteristic function. Nevertheless, it is less problematic than the nucleolus. First, we give a brief review of the Shapley value.

Shapley (1953) derived his value originally from four axioms on a solution function. A *solution function* ψ is a function on the set Γ of all n-person superadditive characteristic function games (N, v), with fixed player set N, which assigns a payoff vector to each game. Since the player set N is fixed, the game is identified with a characteristic function v. Thus, a value function $\psi : \Gamma \to \mathbb{R}^n$ is denoted by $\psi(v) = (\psi_1(v), ..., \psi_n(v))$. Shapley gave the following four axioms on ψ:

(S1): Pareto Optimality: for any game $(N, v) \in \Gamma$, $\sum_{i\in N} \psi_i(v) = v(N)$;

(S2): Symmetry: for any permutation π of $N, \psi(\pi v) = (\psi_{\pi(1)}(v), ..., \psi_{\pi(n)}(v))$, where πv is defined by $\pi v(S) = v(\{\pi(i) : i \in S\})$ for all $S \in 2^N$;

(S3): Additivity: for any two games, $v, w \in \Gamma$, $\psi(v + w) = \psi(v) + \psi(w)$, where $v + w$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \in 2^N$;

(S4): Dummy Axiom: for any game $v \in \Gamma$ and $i \in N$, if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \in 2^N$ with $i \notin S$, then $\psi_i(v) = v(\{i\})$.

In general, the solution function ψ depends upon the game described by a characteristic function, but condition S2 means that ψ should not depend on the names of players given by the index numbers $1, 2, ..., n$.

Shapley (1953) proved the following: if a solution function ψ satisfies conditions S1 through S4, then ψ is uniquely determined as

$$\psi_i(v) = \sum_{S\subseteq N-\{i\}} \frac{|S|!(n - |S| - 1)!}{n!}[v(S \cup \{i\}) - v(S)] \text{ for all } i \in N. \tag{4.7}$$

Although each of the above axioms and Shapley's result are mathematically clear, they do not indicate the utility theory underlying the concept of the Shapley value. Formula (4.7), however, does provide some utility theoretic interpretation of the Shapley value.

Suppose that the players come to participate in the game in random order and that each player i is paid his marginal contribution $v(S \cup \{i\}) - v(S)$ when the players S are already in the game and then player i enters the game. Before the game is played, it is equally probable for player i that he comes to the game at any place in the ordering of $1, 2, ..., n$. The probability that player i follows the players in S is given by the coefficients in formula (4.7). Thus player i's expected utility from the random entry process is given as formula (4.7).

In the above interpretation, the utility theory underlying the Shapley value is relatively clear. The marginal contribution $v(S \cup \{i\}) - v(S)$ is the monetary payoff to player i and the expectation of these marginal payoffs is taken: the risk neutral von Neumann-Morgenstern utility function suffices. In this interpretation, however, the game is assumed to be played in a different manner than that intended by the motivation initially given for a game in characteristic function form.

Similarly to the nucleolus, the Shapley value is also interpreted as a normative (fair or equitable) outcome, mainly because of the symmetry condition. As already mentioned, Symmetry simply states that a solution function does not depend upon the names of the players, a necessary but not sufficient condition for an equitable outcome, since the game itself may be inequitable.

5 Games Without Side Payments and Some Solution Concepts

Although a game with side payments is a convenient tool, it requires SP and TU. The transferable utility assumption may be inappropriate for some situations in that it ignores income effects. Side payments may be prohibited or impossible. When either SP or TU does not hold, games without side payments are required. In this section we discuss a game without side payments together with some solution concepts from the viewpoint of utility theories.

The term "game without side payments" is slightly misleading since the game may satisfy SP and TU. However, we follow the standard terminology.

5.1 Games Without Side Payments

A game without side payments is given as a pair (N, V) consisting of the player set N and a characteristic function V on 2^N. For each coalition S, the set $V(S)$ is a subset of $\mathbb{R}^S$, where $\mathbb{R}^S$ is $|S|$ −dimensional Euclidean space with coordinates labelled by the members of S.[11] The set $V(S)$ describes the set of all payoff vector for coalition S that are attainable by the members in S themselves. We assume the following technical conditions: for all $S \in 2^N$,

$$V(S) \text{ is a closed subset of } \mathbb{R}^S; \tag{5.1}$$

[11] $\mathbb{R}^\emptyset = \{0\}$.

$$a^S \in V(S) \text{ and } b^S \leq a^S \text{ imply } b^S \in V(S); \tag{5.2}$$

$$\{a^S \in V(S) : a_i^S \geq \max V(\{i\}) \text{ for all } i \in S\} \text{ is nonempty and bounded.} \tag{5.3}$$

Within the framework of games without side payments, a game with side payments is described as

$$V(S) = \{a^S : \sum_{i \in S} a_i^S \leq v(S)\} \tag{5.4}$$

for all $S \in 2^N$. Thus $V(S)$ describes directly the set of attainable payoffs for S. The three examples of games with side payments in Section 2 are directly described by (5.4). It will be seen below that using the framework without side payments, the assumptions of transferable utility and side payments are not needed.

A game without side payments is a heavy mathematical tool. It is suitable to discuss general problems such as the nonemptiness of the core [cf. Scarf (1967)] but when a specific game situation is given, it is often more convenient to work on the situation directly instead of describing it as a game without side payments. Nevertheless, in order to see general principles underlying cooperative games, it is useful to formulate game situations in terms of games without side payments. In the following, we will see the descriptions of the examples given in Section 2 in terms of games without side payments.

5.2 Examples

Market Games

Consider a market game with n players and $m+1$ commodities. In contrast to the previous formulation of market games with quasi-linear utility functions, we now assume that the continuous utility function U_i is defined on $\mathbb{R}_+^{m+1}$ and the endowment of player i is given as a vector in $\mathbb{R}_+^{m+1}$. The $m+1^{th}$ commodity is treated in the same way as the first m commodities. An *S-allocation* $(x_i)_{i \in S}$ is defined by $\sum_{i \in S} x^i = \sum_{i \in S} \omega_i$ and $x_i \in \mathbb{R}_+^{m+1}$ for all $i \in S$. The characteristic function V is defined by

$$V(S) = \{a^S \in \mathbb{R}^S : a_i^S \leq U_i(x_i) \text{ for some } S\text{-allocation } (x_i)_{i \in S}\} \tag{5.5}$$

for all $S \in 2^N$. Then this characteristic function V satisfies conditions (5.1)-(5.3). For this definition, only the existence of a utility function U_i representing a preference relation $\succsim_i$ is required [see Debreu (1959) for conditions ensuring the existence of a continuous utility function].

For the definition (5.5), we can assume that the utility function U_i satisfies the transferable utility assumption, i.e., linear separability. If, however, the endowments of $\omega_{i,m+1}$ of the $m+1^{th}$ commodity are small, then side payments may not be freely permitted. If the endowments $\omega_{i,m+1}$ are sufficiently large to avoid the relevant constraints, then side payments are effectively unbounded. This is the case of a market game in Section 2.2. Nevertheless, side payments are still part of the problem.

Voting Games

Consider a voting game where the assumption TU is satisfied but no side payments are permitted. In such a case, the characteristic function is given by

$$V(S) = \begin{cases} \{a^S \in \mathbb{R}^S : \text{for some } x \in X,\ a_i \leq u_i(x) \text{ for all } i \in S\} \\ \qquad \text{if } |S| > \frac{n}{2}, \\ \\ \{a^S \in \mathbb{R}^S : \text{for all } x \in X,\ a_i \leq u_i(x) \text{ for all } i \in S\} \\ \qquad \text{if } |S| \leq \frac{n}{2}. \end{cases} \tag{5.6}$$

This majority voting game has been extensively discussed in the social choice literature [Nakamura (1975), Moulin (1988), for example].

The above formulation of $V(S)$ illustrates that the assumptions SP and TU are independent. (Recall also the discussion of the relationships between SP and TU in Subsection 2.3.) Nevertheless, unless side payments are totally prohibited, it may be better to take side payments into account when building a model, since, as discussed in Section 4, they may affect solutions significantly.

Cooperative Games Derived from Strategic Games

Suppose that side payments are not allowed in the normal form game $G = (N, \{\Sigma_i\}_{i\in N}, \{h_i\}_{i\in N})$. This means that either the economy including the game G has money but money transfers are prohibited, or that G is a full description of the game in question and nothing other than in the game is available in playing the game. In either case, the relevant utility functions of players, given by $\{h_i\}_{i\in N}$, are von Neumann-Morgenstern utility functions over the domain $M(\Sigma_N)$.

Corresponding to definition (2.10), the characteristic function V_α is defined by: for all $S \in 2^N$,

$$\begin{aligned} V_\alpha(S) = \{a \in \mathbb{R}^S :\ & \text{there is some } \sigma \in M(\Sigma_S) \text{ such that} \\ & \text{for any } \sigma_{-S} \in M(\Sigma_{N-S}), a_i \leq h_i(\sigma_S, \sigma_{-S}) \text{ for all } i \in S\}. \end{aligned} \tag{5.7}$$

The value $V_\alpha(S)$ of the characteristic function V_α is the set of all payoff vectors for the members of the coalition S that can be obtained by the cooperation of the members of S. This is a faithful extension of definition (2.10) in the absence of side payments.

In (2.10), in fact, the min-max value, which is obtained by changing the order of the max and min operators, coincides with the value of (2.10) because of the von Neumann Mini-Max Theorem. This suggests another definition of a characteristic function; for all $S \in 2^N$,

$$\begin{aligned} V_\beta(S) = \{a \in \mathbb{R}^S :\ & \text{for any } \sigma_{-S} \in M(\Sigma_{N-S}) \text{ there is} \\ & \text{a } \sigma_S \in M(\Sigma_S) \text{ such that } a_i \leq h_i(\sigma_S, \sigma_{-S}) \text{ for all } i \in S\}. \end{aligned} \tag{5.8}$$

Unlike games with side payments, these two definitions may give different sets [cf. Aumann (1961)]. The first and second are often called the α- and β-*characteristic functions*. A general nonemptiness result for the α-core, defined using the α-characteristic function, is obtained in Scarf (1971). The β-core, defined by the β-characteristic function, is closely related to the Folk Theorem for repeated games [cf. Aumann (1959, 1981)].

5.3 Solution Concepts

The characteristic function V of a game without side payments describes, for each coalition S, the set of payoff vectors attainable by the members of S. Once V is given, the imputation space and dominance relations are extended to a game without side payments in a straightforward manner. The imputation space $I(N, V)$ is simply the set

$$\{a \in V(N) : a_i \geq \max V(\{i\}) \text{ for all } i \in N\}.$$

The dominance relation *a dom b* is defined by:

$$\text{for some } S \in 2^N,\ a_i > b_i \text{ for all } i \in S \text{ and and } (a_i)_{i \in S} \in V(S).$$

The core is defined to be the set of all undominated imputations. The von Neumann-Morgenstern stable set is also defined via internal and external stability requirements in the same way as in a game with side payments.

Consider the core and stable set for a voting game without side payments for Example 3.1. Since no transfer of money is allowed, the problem is which alternative x or y to choose. In both examples, players 1 and 2 prefer x to y and thus x is chosen. Actually, x constitutes the core and also the unique stable set. In the first example, when side payments are involved, Player 3 can compensate for Player 1 or 2 to obtain his cooperation for the alternative y. This causes the core to be empty. Our point is that the possibility of side payments may drastically change the nature of the game. But this is almost independent of the assumption TU.

The nucleolus and Shapley value are based intrinsically on the numerical expression of the characteristic function with side payments. Nevertheless, some authors modify the definitions of these concepts for games without side payments. Here we discuss only one example—the λ-transfer value introduced by Shapley (1969).

Shapley transformed a game (N, V) without side payments into a game (N, v_λ) with side payments by using "utility transfer weights" $\lambda = (\lambda_1, ..., \lambda_n) > 0$ by defining

$$v_\lambda(S) = \max\{\sum_{i \in S} \lambda_i a_i : a \in V(S)\} \quad \text{for all } S \in 2^N. \tag{5.9}$$

The λ-transfer value is defined as follows: a payoff vector $a = (a_1, ..., a_n)$ is a *λ-transfer value* iff there are transfer weights $\lambda = (\lambda_1, ..., \lambda_n) > 0$ such that a is the Shapley value of the game (N, v_λ) and a is feasible in (N, V) i.e., $a \in V(N)$.

Shapley (1969) proved the existence of a λ-transfer value for a game without side payments, but uniqueness does not hold. Aumann (1985) provided an axiomatization of the $\lambda-$transfer value.

From the viewpoint of utility theory, it is difficult to interpret the transformation from (N, V) to (N, v_λ) and the λ-transfer value. Some authors claim that utility units are compared with the help of the weights. In fact, the matter of the interpretation of the NTU value has been the subject of lively debate; see Roth (1980, 1987), Aumann (1985b, 1986, 1987) and Scafuri and Yannelis (1984).

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