Combinatorial Structure of Constructible Complexes

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Abstract

This thesis is a study of the combinatorial structure of a certain kind of complexes: constructible complexes and strongly constructible complexes. The notion of constructible complexes is known as a weaker notion than that of shellable complexes, and this thesis is aimed to be a foundation of the study of shellable complexes. The theme of this thesis is a characterization of constructible complexes in terms of their face posets, and the relation between constructibility and partitionability.

After reviewing some preliminaries on posets and complexes in Chapter 2, the property of constructible complexes is studied in Chapter 3. In this chapter, a new notion named recursively dividable posets is defined and it is shown that a complex is constructible if and only if its face poset is recursively dividable. Usually constructibility is defined only for simplicial complexes, but corresponding to the notion of recursively dividable posets, constructibility is generalized for non-simplicial complexes.

In Chapter 4, a notion of strongly constructible complexes is defined by strengthening the condition of constructible complexes. For this notion, as like as Chapter 3, a notion of strongly dividable posets is defined and it is shown that a complex is strongly constructible if and only if its face poset is strongly dividable. Moreover, strongly dividable posets are shown to be signable. Since it is known that a simplicial complex is partitionable if and only if its face poset is signable, this result means that strongly constructible simplicial complexes are partitionable.

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Contents

Chapter 1

Introduction

After Brugesser and Mani [2] introduced the notion, shellability has come to be of fundamental importance in the study of complexes and has been extensively studied by many researchers. (See [8], [19, Chapter 8], [20].) One of the most famous applications is the proof of the Upper Bound Theorem for convex polytopes by McMullen [12] which used the result of Brugesser and Mani [2] that convex polytopes are shellable. Shellability is indeed a very useful notion, but it is sometimes so strong that it is hard to show whether the complex in interest is shellable or not. Spherical fans and oriented matroid polytopes are the examples that are not known whether they are shellable or not. (c.f. [10].) The difficulty comes from the fact that shellability is not a topological property: all shellable pseudomanifolds (i.e., d-dimensional complexes in which every $(d-1)$-dimensional face belongs to at most 2 facets) are homeomorphic to balls or spheres, but there exist non-shellable balls and non-shellable spheres. For example, the extension of the Upper Bound Theorem for simplicial spheres could not be solved until Stanley [16] finally proved it by using the notion of Cohen-Macaulayness. (The definition of Cohen-Macaulayness is in Appendix A.) Cohen-Macaulayness is a weaker notion than shellability. Other than Cohen-Macaulayness, there are also important notions weaker than shellability such as partitionability and constructibility. Among these, we treat constructibility in this thesis. Constructibility is a very important notion in the study of shellability as Chapter 5 indicates, but there is so little known about constructibility. So this thesis is aimed to be a foundation of the study of constructibility. The theme of this thesis is a characterization of constructibility in terms of their face posets, and the relation with partitionability.

The combinatorial properties of complexes are reflected in their face posets, so the notions of complexes must be translated into the terms of face posets. For example, Björner and Wachs [6] showed that a complex is shellable if and only if its face poset admits a recursive coatom ordering. Similarly, Kleinschmidt and Onn [10] showed that a simplicial complex is partitionable if and only if its face poset is signable. These translations make the notions tractable, so we must first seek the condition of face posets equivalent to constructibility. For this sake, in Chapter 3 we introduce a notion of recursively dividable posets and show that a complex is constructible if and only if its face poset is recursively dividable. Usually constructibility is defined only for simplicial complexes, but corresponding to the notion of recursively dividable posets, we generalize constructibility for non-simplicial complexes in Section 3.2.

Next we study the partitionability of constructible complexes. In Chapter 4, we define a notion of strongly constructible complexes by strengthening the condition of constructible complexes. For this notion, as like as Chapter 3, we define a notion of strongly dividable posets and show that a complex is strongly constructible if and only if its face poset is strongly dividable. Then we show that strongly dividable posets are signable. This result means that strongly constructible simplicial complexes are partitionable.

Before proceeding to the main subjects, we review some preliminaries on posets and complexes in Chapter 2.

Chapter 2

Preliminaries

In this chapter, we review the terminology on posets, complexes, and the notion of shellability and partitionability.

2.1 Posets

A poset P is a finite partially ordered set, i.e., a finite set with an *order relation* $\leq$ which satisfies the conditions below.

(i) $x \leq x$. (reflexive law)

(ii) $x \leq y$ and $y \leq z$ imply $x \leq z$. (transitive law)

(iii) $x \leq y$ and $y \leq x$ imply $x = y$. (antisymmetric law)

Generally, the order relation is not defined for all pairs of two elements in a poset. If the order relation is defined for all pairs in a poset P, P is called a *totally ordered set* or a *linearly ordered set.*

Any subset of a poset P is again a poset with the induced order relation. A *chain* in a poset P is a totally ordered subset of P. The *length* of a chain is the number of its elements minus 1.

For $x \leq y$ in P, $[x, y]_P = \{z \in P \,|\, x \leq z \leq y\}$ is called an *interval.* If there is no confusion, we will omit the subscript P.

A poset is called *bounded* if it has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. A bounded poset is called *graded* if every maximal chain has the same length. Every interval of a graded poset is again a graded poset. The *rank* of an element x in a graded poset P is the length of a maximal chain of $[\hat{0}, x]_P$. The *rank* of P is the rank of $\hat{1}$, i.e., the length of a maximal chain of P. The rank of P is denoted by $\mathrm{rank}(P)$.

The *covering relation* is denoted by $x \prec y$ which means that $x \leq y$ and there is no $z \in P$ such that $x \leq z \leq y$. If $x \prec y$, y is called a *cover* of x, and x is called a *cocover* of y. The elements of P which covers $\hat{0}$ are called *atoms* and those that are covered by $\hat{1}$ are called *coatoms*.

2.2 Complexes

A *(convex) polytope* P means the convex hull of a finite set of points in a Euclidean space. A hyperplane defines two closed halfspaces in this space. If one of these closed halfspaces contains the whole polytope P, the intersection of P and this hyperplane is called a *face* of P. The *dimension* of a polytope or a face is the dimension of the affine hull of it. The 0-dimensional faces are called *vertices* and the 1-dimensional faces are called *edges*. The empty set ϕ is defined to be a face of dimension -1. A d-dimensional polytope which has exactly $d+1$ vertices is called a *simplex*.

A *polyhedral complex* Δ is defined to be a finite set of polytopes P in some Euclidean space such that

(i) if $P \in \Delta$, then all the faces of P (including the empty set) is contained in Δ, and

(ii) if $P, Q \in \Delta$, then $P \cap Q$ is a face of both P and Q.

The *underlying space* $\|\Delta\|$ of a polyhedral complex Δ is the set $\cup_{P\in\Delta} P$. The maximal faces (concerning the inclusion relation) of a polyhedral complex are called *facets*. The *dimension* of a polyhedral complex is the maximum dimension of its facets. If all the facets have the same dimension, the polyhedral complex is called *pure*. If all the faces of a polyhedral complex are simplices, it is called a *simplicial complex*. (See Figure 2.1.)

The notion of polyhedral complexes is a special case of a more general notion. Next we review this notion called regular cell complexes. (See [5, Section 4.7]. They are called

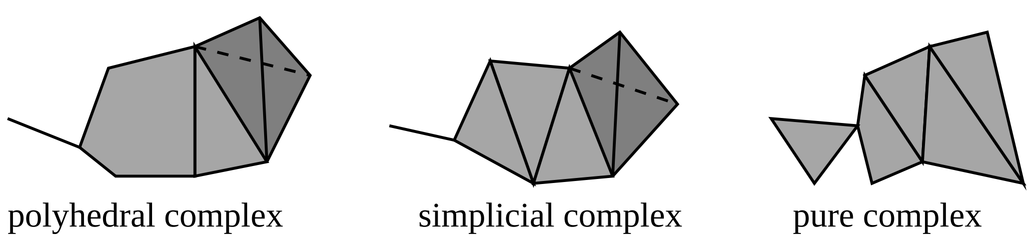

Figure 2.1: Examples.

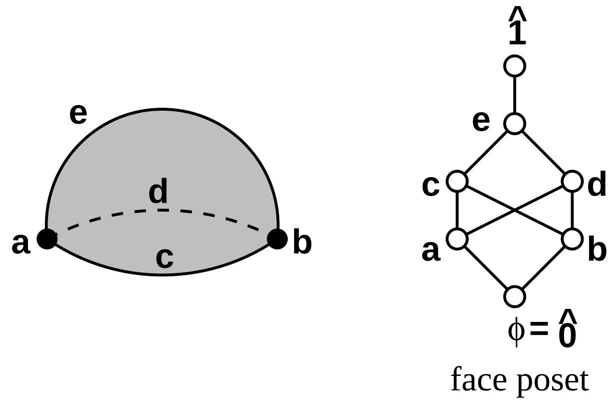

Figure 2.2: A regular cell complex and its face poset.

regular CW complexes in [3].)

A *ball* in a topological space T is a subspace $\sigma \in T$ which is homeomorphic to the standard d-dimensional ball, for some d. The *relative interior* of σ is denoted by $\mathring{\sigma}$. The *boundary* $\partial\sigma$ of a ball σ is $\sigma - \mathring{\sigma}$. A *regular cell complex* Δ is a finite collection of balls σ in a Hausdorff space $\|\Delta\| = \cup_{\sigma\in\Delta}\sigma$ such that

(i) the interiors $\mathring{\sigma}$ partition $\|\Delta\|$ (i.e., every $x \in \|\Delta\|$ lies in exactly one $\mathring{\sigma}$), and

(ii) the boundary $\partial\sigma$ is a union of some members of Δ, for all $\sigma \in \Delta$.

The space $\|\Delta\|$ is called the *underlying space* of Δ. The balls σ are called *faces* and the maximal faces are called *facets*. The *dimension* of a regular cell complex is the maximum dimension of the facets. If every facets has the same dimension, the complex is called *pure*. The *boundary complex* $\partial\sigma$ of σ is the complex made of all the faces in the boundary $\partial\sigma$.

The *face poset* $F(\Delta)$ of a regular cell complex Δ is a poset consisting of all the faces of Δ ordered by inclusion and adjoining a greatest element $\hat{1}$. Note that the face poset of a pure complex is always a graded poset with a unique maximal element $\hat{1}$ and a unique minimal element $\hat{0} = \phi$. (See Figure 2.2. In this figure, a poset is represented by a Hasse diagram, i.e., a graph in which vertices denote the elements of the poset and a pair of vertices are joined with an edge if and only if the upper vertex in the figure covers the lower vertex.)

In this thesis, we are interested in the notions of complexes such as shellability, partitionability and constructibility, but all of these are defined only for pure complexes. Therefore the posets we are concerning are always graded posets. So in the rest of this thesis, by a *poset* we mean a graded poset.

A d-dimensional regular cell complex Δ is called a *pseudomanifold* if every $(d-1)$-dimensional face of Δ belongs to at most two facets. If the underlying space $\|\Delta\|$ of a pseudomanifold Δ is homeomorphic to a ball or a sphere, Δ is called a *ball* or a *sphere* respectively.

2.3 Shellable complexes

Definition 2.1. An ordering $F_1, F_2, \ldots, F_n$ of facets of a pure d-dimensional regular cell complex is called a *shelling* if $d = 0$, or if

(i) the boundary complex ∂F_1 of F_1 has a shelling, and

(ii) for $2 \leq j \leq n$, $F_j \cap \left(\cup_{i=1}^{j-1} F_i\right)$ is a $(d-1)$-dimensional regular cell complex having a shelling which can extend to a shelling of the boundary complex ∂F_j of F_j,

and a regular cell complex is called *shellable* if it admits a shelling.

An example of a shellable complex is the boundary complex of a polytope. The shellability of the boundary complex of a polytope is proved by Brugesser and Mani [2] by using the method called "line shelling". In fact, they proved a weaker version of shellability than Definition 2.1, (They did not require the condition that the shelling of $F_j \cap \left(\cup_{i=1}^{j-1} F_i\right)$ can be extended to a shelling of ∂F_j.) but their proof is also valid for our definition. (See [19, Chapter 8].)

It is clear that a shellable pseudomanifold is a ball or a sphere, but the converse is not true. It is easy to see that all 2-dimensional balls and spheres are shellable as the following proposition shows, but in 3 and higher dimensional cases, some examples of non-shellable balls and non-shellable spheres are known. (See [1], [11], [13], [19, Chapter 8], [20].) Later, we show an example of a non-shellable ball made by Ziegler [20] in Section 3.1 and another one made by Walker [18] in Section 3.2.

Proposition 2.2. *2-dimensional balls and spheres are shellable.*

Proof. A facet of a ball Δ is called *free* if its intersection with the boundary of Δ is a $(d-1)$-dimensional ball. If Δ has no free facets, it is called *strongly non-shellable*. If Δ is not shellable, it has a strongly non-shellable ball as its subcomplex. In fact, if there is no strongly non-shellable ball, then we can remove free facets one by one until there is no

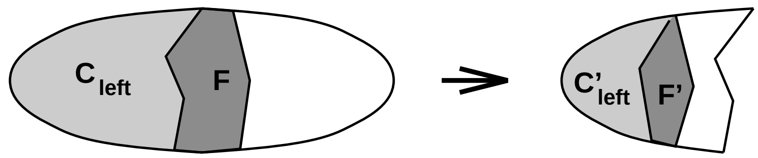

Figure 2.3: Proof of Proposition 2.2.

facet left. But if this is possible, we get a shelling of Δ by following the removing process reversively. This is a contradiction. (See also [20].)

Now we show that there is no 2-dimensional strongly non-shellable ball at all. Let Δ' be a 2-dimensional strongly non-shellable ball. Take one facet F of Δ'. The boundary of F has a path which divides Δ' into two balls C_{left} and C_{right} such that C_{left} does not contain F as Figure 2.3. (If not, F must be a free facet.) Then C_{left} also has a non-free facet of Δ and we can similarly divide C_{left} into two balls. Proceeding this process, we get a free facet in the leftmost side. This is a contradiction. So there is no strongly non-shellable ball, and hence every 2-dimensional ball is shellable.

Let Δ be a 2-dimensional sphere and F be one of its facet. Then if we remove the facet F from Δ, the remainder is a 2-dimensional ball and it is shellable. So Δ is also shellable. $\square$

The notion of shellability is translated into the terms of face posets as below.

Definition 2.3. (Björner and Wachs [6])
An ordering $a_1, a_2 \ldots, a_n$ of coatoms of a poset P is called a *recursive coatom ordering* if $\text{rank}(P) = 1$, or if the following conditions hold.

(i) If $z \leq a_i, a_j$ in P and $i < j$, then there exist an index $k < j$ and an element $v \in P$ such that $z \leq v \prec a_k, a_j$.

(ii) For every j, there is a recursive coatom ordering of $[\hat{0}, a_j]$ in which the coatoms of $[\hat{0}, a_j]$ that come first are those that are covered by some a_i for $i < j$.

Theorem 2.4. (Björner [3])

A regular cell complex Δ *is shellable if and only if its face poset* $F(\Delta)$ *admits a recursive coatom ordering.*

2.4 Partitionable complexes

The notion of partitionable complexes is defined only for simplicial complexes.

Definition 2.5. Let Δ be a pure simplicial complex and $F(\Delta)$ be its face poset. Then Δ is called *partitionable* if $F(\Delta) - \{\hat{1}\}$ can be partitioned into intervals of the form $[\psi(G), G]_{F(\Delta)}$, where G is a facet and $\psi(G)$ is a face of G. (See Figure 2.4.)

Partitionability is known to be weaker than shellability, i.e., all shellable simplicial complexes are partitionable. To see this fact, let Δ be a shellable simplicial complex and $F_1, F_2, \ldots, F_n$ be a shelling of Δ. If we define $\psi(F_i)$ as the face consists of the vertices $v \in F_i$ such that $F_i - v$ is contained in one of the previous facets, where $F_i - v$ means the face consists of the vertices of F_i except v. (See Figure 2.5.) Then the faces in $[\psi(F_i), F_i]_{F(\Delta)}$ are exactly the faces of F_i that are not contained in some previous facets. In fact, if a face G of F_i is not in $[\psi(F_i), F_i]_{F(\Delta)}$, i.e., G does not contain $\psi(F_i)$, then G is contained in $F_i - v$ for some $v \in \psi(F_i)$ and it is contained in some previous facets. Conversely, if a face G of F_i is in $[\psi(F_i), F_i]_{F(\Delta)}$, i.e., G contains $\psi(F_i)$, the previous facets cannot contain G from the construction of $\psi(F_i)$. Hence we can easily see that the intervals $[\psi(F_i), F_i]_{F(\Delta)}$ partition $F(\Delta)$ and Δ is partitionable. (See [19].)

The notion of partitionable complexes is translated into the notion of signable posets by Kleinschmidt and Onn [10].

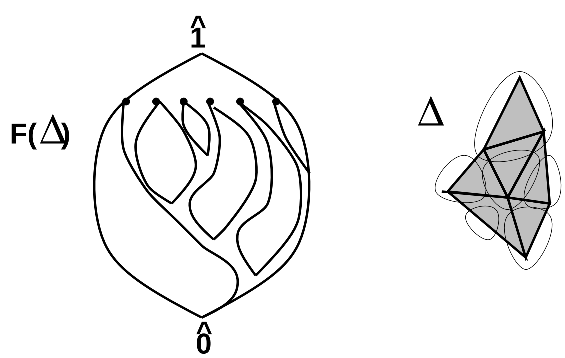

Figure 2.4: Partitionable complex.

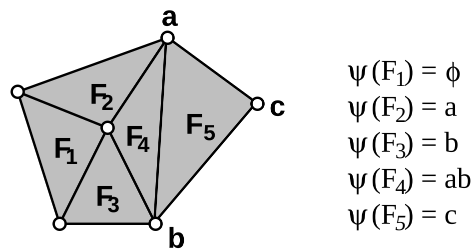

Figure 2.5: Example of a shelling and $\psi(F_i)$.

Definition 2.6. (Kleinschmidt and Onn [10])

A *signing* of a poset P is an assignment $\chi(x,y) \in \{-,+\}$ of signs to each pair of elements $x, y, \in P$ such that $x \prec y \prec \hat{1}$. A coatom y is called *positive* under χ if $\chi(x,y) = +$ for all its cocovers x.

Every upper interval $[w, \hat{1}]$ for each $w \in P$ inherits a signing from that of P by restriction. A signing χ is called *exact* if there is exactly one positive coatom in the interval $[w, \hat{1}]$ (under the restricted signing) for every $w \in P$, and a poset P is called *signable* if it admits an exact signing. (See Figure 2.6.)

Theorem 2.7. (Kleinschmidt and Onn [10])

A simplicial complex Δ is partitionable if and only if its face poset $F(\Delta)$ is signable.

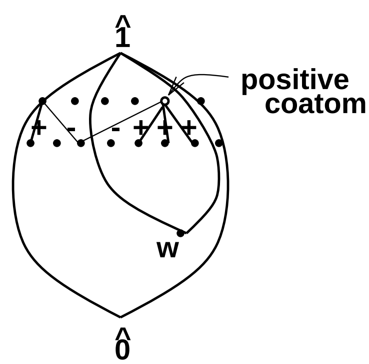

Figure 2.6: Signing and positive coatom.

Chapter 3

Constructible complexes

3.1 Constructible complexes

The notion of constructible complexes appears in [15]. (See also [4].) This notion is defined for simplicial complexes as below.

Definition 3.1. A pure d-dimensional simplicial complex Δ is said to be *constructible* if

(i) Δ is a simplex, or

(ii) there exist d-dimensional constructible subcomplexes C_1 and C_2 such that $\Delta = C_1 \cup C_2$ and that $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible complex.

Intuitively, a shellable complex is made by adding facets one by one, but in the case of a constructible complex, we can add a lump of complexes at one time. (See Figure 3.1.) So constructibility is weaker than shellability, i.e., all shellable simplicial complexes are constructible. More precisely, if $F_1, F_2, \ldots, F_n$ is a shelling of a shellable simplicial complex Δ, then $C_1 = F_1 \cup F_2 \cup \cdots \cup F_{n-1}$ and $C_2 = F_n$ satisfy the condition above.

Example 3.2. Ziegler [20] made an example of a 3-dimensional non-shellable simplicial ball which has 10 vertices and 21 facets. This complex Δ has vertices $\{1, 2, \ldots, 9, 0\}$ and facets:

a: 1234	b: 1256	f: 1569	k: 2560	p: 3678	s: 4578
	c: 2367	g: 1629	l: 2670	q: 3248	t: 4137
	d: 3478	h: 1249	m: 2730	r: 3268	u: 4157
	e: 4185	i: 1489	n: 2310		
		j: 1859	o: 2150		

This complex is indeed non-shellable, (In fact, this is strongly non-shellable.) but if we divide this complex into two 3-dimensional balls $C_1 = \mathrm{d} \cup \mathrm{p} \cup \mathrm{q} \cup \mathrm{r} \cup \mathrm{s} \cup \mathrm{t} \cup \mathrm{u}$ and $C_2 = \cup\{\text{other facets}\}$, both C_1 and C_2 are shellable. Because every 2-dimensional ball is shellable, $C_1 \cap C_2$ is also shellable. So this complex is an example of a non-shellable but constructible complex.

As like as the case of shellable pseudomanifolds, constructible pseudomanifolds are balls or spheres. Whether there are non-constructible simplicial balls or not is not known yet. (It is very unlikely that all balls and spheres are constructible.)

Open Problem. Are there non-constructible simplicial balls?

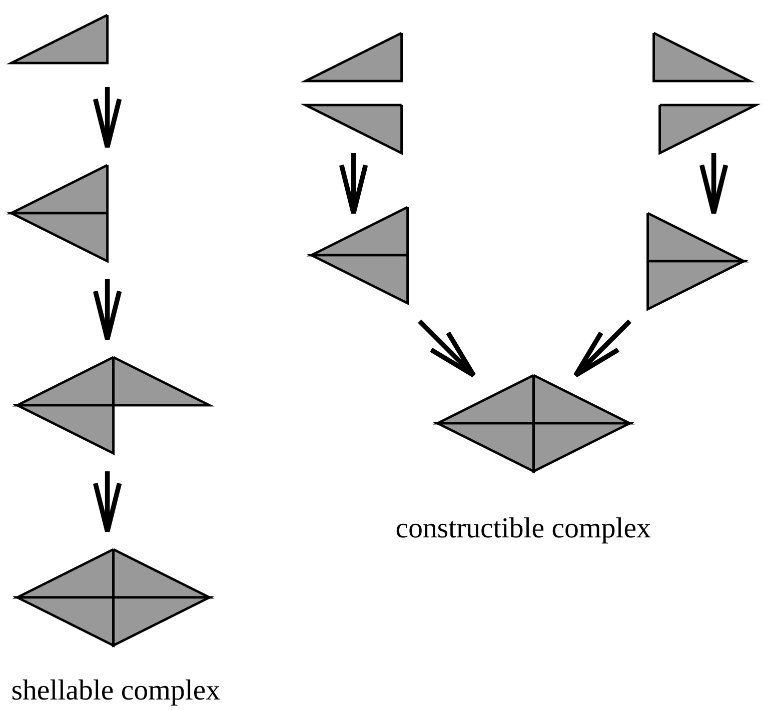

Figure 3.1: Shellable complex and constructible complex.

3.2 Recursively dividable posets

In this section, we define the notion of recursively dividable posets, which is related to the notion of constructible complexes. In what follows, we denote

$$\hat{I}(X) = \{y \in P \mid {}^{\exists}x \in X,\ y \leq x\} \cup \{\hat{1}\},$$

for a subset X of P, i.e., $\hat{I}(X)$ is the order ideal generated by X and adjoining the maximal element.

Definition 3.3. We call a poset P *recursively dividable* if $\mathrm{rank}(P) = 1$, or if the following conditions hold.

(R1) For each coatom x of P, $[\hat{0}, x]$ is recursively dividable.

(R2) If there exists more than one coatom in P, the set of coatoms can be divided into two disjoint non-empty sets X and Y satisfying the following.

(a) $\hat{I}(X)$ and $\hat{I}(Y)$ are recursively dividable.

(b) Let $Z = \{z \in P \mid {}^{\exists}x \in X, {}^{\exists}y \in Y,\ z \prec x, y\}$. Then $\hat{I}(Z)$ is recursively dividable.

(c) $\hat{I}(X) \cap \hat{I}(Y) = \hat{I}(Z)$.

(See Figure 3.2.) We call such a division a *recursive division*.

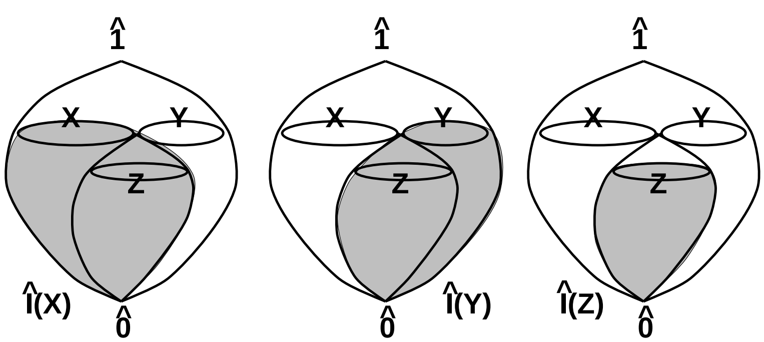

Figure 3.2: $\hat{I}(X)$, $\hat{I}(Y)$ and $\hat{I}(Z)$.

It turns out that a poset admitting a recursive coatom ordering is recursively dividable.

Proposition 3.4. *A poset which admits a recursive coatom ordering is recursively dividable.*

Proof. Let $x_1, x_2, \ldots, x_n$ be a recursive coatom ordering of a poset P of rank d, and let the coatoms be divided into $\{x_1, x_2, \ldots, x_{n-1}\}$ and $\{x_n\}$. Then this division is a recursive division of P. To see this, we use an induction on the rank of P and the number of the coatoms. When the rank is 1, the statement is clear.

(R1) $[\hat{0}, x_i]$ admits a recursive coatom ordering and is rank $d-1$, so is recursively dividable by the induction hypothesis.

(R2)(a) $\hat{I}(\{x_1, x_2, \ldots, x_{n-1}\})$ has a recursive coatom ordering $x_1, x_2, \cdots, x_{n-1}$ and has only $n-1$ coatoms, so is recursively dividable by the induction hypothesis. And $\hat{I}(\{x_n\})$ is recursively dividable because $[\hat{0}, x_n]$ is recursively dividable as above.

(R2)(b) Let $Z = \{z \in P \mid z \prec x_i, x_n \text{ , for } 1 \leq {}^{\exists}i \leq n-1\}$. Then Z is the first part of a recursive coatom ordering of $[\hat{0}, x_n]$ from the condition (ii) of Definition 2.3. So it is clear that $\hat{I}(Z)$ has a recursive coatom ordering induced by that of $[\hat{0}, x_n]$. Since $\hat{I}(Z)$ is rank $d-1$, $\hat{I}(Z)$ is recursively dividable by the induction hypothesis.

(R2)(c) Let $w \leq x_i, x_n$ $(1 \leq {}^{\exists}i < n)$. Then from the condition (i) of Definition 2.3, $w \leq {}^{\exists}z \prec x_j,\ x_n$, for $1 \leq {}^{\exists}j < n$. So we have $\hat{I}(\{x_1, x_2, \ldots, x_{n-1}\}) \cap \hat{I}(\{x_n\}) \subset \hat{I}(Z)$. On the other hand, we have $\hat{I}(Z) \subset \hat{I}(\{x_1, x_2, \ldots, x_{n-1}\})$ and $\hat{I}(Z) \subset \hat{I}(\{x_n\})$, so we conclude that $\hat{I}(\{x_1, x_2, \ldots, x_{n-1}\}) \cap \hat{I}(\{x_n\}) = \hat{I}(Z)$.

□

Now we show the relation between the notion of constructible complexes and that of recursively dividable posets.

Theorem 3.5. *A simplicial complex Δ is constructible if and only if its face poset $F(\Delta)$ is recursively dividable.*

Proof. We use an induction on the number of the facets and the dimension. If Δ has only one facet, then Δ is a simplex, so it is constructible. And because a simplex is shellable, its face poset admits a recursive coatom ordering, so $F(\Delta)$ is recursively dividable. Hence the statement holds.

Let Δ be a d-dimensional constructible complex and Δ has more than one facet. Then there exist constructible subcomplexes C_1 and C_2 of Δ such that $C_1 \cup C_2 = \Delta$ and that $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible complex. If we divide the coatoms of $F(\Delta)$ into $\{\text{the facets of } C_1\} = X$ and $\{\text{the facets of } C_2\} = Y$, this division will be a recursive division of $F(\Delta)$. Here, the face posets of C_1 and C_2 are $\hat{I}(X)$ and $\hat{I}(Y)$ respectively. First, the condition (R1) of Definition 3.3 holds because each facet is a simplex. Next, by the induction hypothesis, $\hat{I}(X)$ and $\hat{I}(Y)$ are recursively dividable because C_1 and C_2 are constructible, so the condition (R2)(a) holds. On the other hand, the face poset $F(C_1 \cap C_2)$ of $C_1 \cap C_2$ consists of the elements of $\{w \in P \mid {}^{\exists}x \in X,\ {}^{\exists}y \in Y,\ w \leq x, y\}$. But since $C_1 \cap C_2$ is a pure complex, any $w \in F(C_1 \cap C_2)$ has a coatom z of $F(C_1 \cap C_2)$ such that $w \leq z$, so $F(C_1 \cap C_2) = \hat{I}(Z)$ where $Z = \{z \in P \mid {}^{\exists}x \in X,\ {}^{\exists}y \in Y,\ z \prec x, y\}$. Using the induction hypothesis, $\hat{I}(Z)$ becomes recursively dividable because $C_1 \cap C_2$ is constructible and its dimension is smaller than Δ. Hence the condition (R2)(b) is satisfied. This observation also implies that the condition (R2)(c) holds.

Conversely, if $F(\Delta)$ is recursively dividable and it has more than one coatom, the coatoms of $F(\Delta)$ can be divided into X and Y according to a recursive division of $F(\Delta)$. Let C_1 and C_2 be the subcomplexes of Δ whose facets are X and Y respectively. Then clearly the face posets of C_1 and C_2 are $\hat{I}(X)$ and $\hat{I}(Y)$ respectively, and $C_1 \cup C_2 = \Delta$. Because $\hat{I}(Z) = \hat{I}(X) \cap \hat{I}(Y)$, where $Z = \{z \in P \mid {}^{\exists}x \in X,\ {}^{\exists}y \in Y,\ z \prec x, y\}$, by the condition (R2)(c) of Definition 3.3, we have $\hat{I}(Z) = F(C_1 \cap C_2)$. Now $\hat{I}(X)$, $\hat{I}(Y)$ and $\hat{I}(Z)$

are recursively dividable, C_1, C_2 and $C_1 \cap C_2$ are constructible by the induction hypothesis, which means that Δ is constructible. $\square$

The notion of constructibility is defined only for simplicial complexes as Definition 3.1, but recursive dividability is defined for face posets of more general complexes. Corresponding to this, we can naturally generalize constructibility for regular cell complexes.

Definition 3.6. A pure d-dimensional regular cell complex Δ is defined to be *constructible* if $d = 0$, or if

(i) Δ is a complex which has only one facet and $\partial\Delta$ is constructible, or

(ii) if Δ has more than one facet, then

  (a) each facet of Δ is constructible, and

  (b) there exist d-simensional constructible subcomplexes C_1 and C_2 such that $\Delta = C_1 \cup C_2$ and that $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible complex.

The same theorem as Theorem 3.5 holds for this generalized constructibility. It can be easily shown by the same way as Theorem 3.5.

Theorem 3.7. *A regular cell complex Δ is constructible if and only if its face poset $F(\Delta)$ is recursively dividable.*

Proof. The way of the proof is same as that of Theorem 3.5, but the treatment for the case when there is only one facet and the verification of the condition (R1) is different. In Theorem 3.5, the statement for these cases is satisfied because Δ is a simplex, but here we use an induction on the dimension. If Δ has only one facet, then Δ is constructible if and only if $\partial\Delta$ is constructible, and $F(\Delta)$ is recursively dividable if and only if $[\hat{0}, a]$ is recursively dividable where a is the only one coatom of $F(\Delta)$. Hence the statement holds by the induction hypothesis. When there is only one facet and the dimension is 0, the statement is clearly satisfied. $\square$

As is mentioned in Section 3.1, we do not know whether there are non-constructible simplicial balls or not. But as for non-simplicial case, we have an example of non-constructible balls.

Example 3.8. In [18], Walker made an example of a 3-dimensional non-shellable ball which have 3 facets in a 3-dimensional Euclidean space as below. (See Figure 3.3.)

$$\begin{aligned} C_1 &= ([1,3]\times[0,1]\times[1,3]) \cup ([1,3]\times[3,4]\times[1,3]) \cup \\ &\quad ([2,3]\times[1,3]\times[1,2]) \cup ([1,2]\times[2,3]\times[1,2]), \\ C_2 &= ([1,3]\times[0,4]\times[0,1]) \cup ([1,3]\times[0,4]\times[3,4]) \cup \\ &\quad ([1,2]\times[1,2]\times[1,3]) \cup ([2,3]\times[1,2]\times[2,3]), \\ C_3 &= ([0,1]\times[0,4]\times[0,4]) \cup ([3,4]\times[0,4]\times[0,4]) \cup \\ &\quad ([1,3]\times[2,3]\times[2,3]). \end{aligned}$$

In this example, every two facets form a torus, so this is clearly non-constructible.

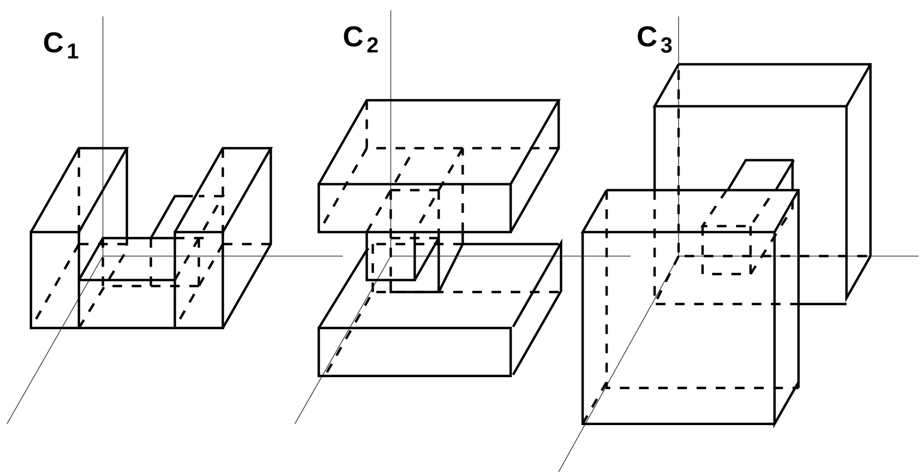

Figure 3.3: Non-constructible 3-ball.

3.3 Barycentric subdivision

For every face G (not the empty face) of a regular cell complex Δ, we assign a point $\hat{G}$ in $\mathring{G}$. Let us define a simplicial complex Δ' to be a collection of faces $\{\langle \hat{G}_1, \hat{G}_2, \ldots, \hat{G}_t \rangle \mid G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_t\}$, where $\langle \hat{G}_1, \hat{G}_2, \ldots, \hat{G}_t \rangle$ is a simplex whose vertices are $\{\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_t\}$. This simplicial complex Δ' is called a *barycentric subdivision* of Δ. (See Figure 3.4.)

It is known that the barycentric subdivision of a shellable complex is again shellable [6]. Similarly, it is easy to see from the following proposition that the barycentric subdivision of a constructible complex is again constructible. In this proposition, the *order complex* $\Delta(P)$ of a poset P means the simplicial complex whose faces are the chains of P. It is easy to see that $\Delta(P)$ is constructible if and only if $\Delta\left(P - \{\hat{0}, \hat{1}\}\right)$ is constructible, and the barycentric subdivision of a regular cell complex Δ is combinatorially equivalent to the order complex $\Delta\left(F(\Delta) - \{\hat{0}, \hat{1}\}\right)$, where $F(\Delta)$ is the face poset of Δ.

Proposition 3.9. *The order complex of a recursively dividable poset is constructible.*

Proof. We use an induction on the number of the facets and the dimension. Let us divide the coatoms of P into X and Y according to a recursive division of P, and let $Z = \{z \in P \mid {}^{\exists}x \in X,\ {}^{\exists}y \in Y,\ z \prec x, y\}$. Let C_1 be $\Delta\big(\hat{I}(X)\big)$, and C_2 be $\Delta\big(\hat{I}(Y)\big)$. It is clear that $C_1 \cup C_2 = \Delta(P)$ and that $C_1 \cap C_2 = \Delta\big(\hat{I}(Z)\big)$. We must show that $\Delta\big(\hat{I}(X)\big)$, $\Delta\big(\hat{I}(Y)\big)$ and $\Delta\big(\hat{I}(Z)\big)$ are constructible, but these conditions are satisfied by the induction hypothesis because $\hat{I}(X)$, $\hat{I}(Y)$ and $\hat{I}(Z)$ are recursively dividable. $\square$

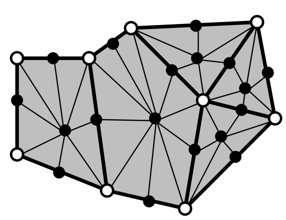

Figure 3.4: Barycentric subdivision.

Chapter 4

Strongly constructible complexes

In this chapter, we discuss the relation between constructible complexes and partitionable complexes. In the following sections, we define a notion of strongly constructible complexes which is stronger than that of constructible complexes, and show that they are partitionable.

4.1 Strongly constructible complexes

Definition 4.1. Let Δ be a pure d-dimensional regular cell complex and D be an empty set or a $(d-1)$-dimensional pure subcomplex of Δ. A pair (Δ, D) is defined to be *strongly constructible* if $d = 0$, or if

(i) Δ is a complex which has only one facet and $(\partial\Delta, \phi)$ is strongly constructible, or

(ii) there exist d-dimensional subcomplexes C_1 and C_2 such that $C_1 \cup C_2 = \Delta$ and satisfying,

  (a) the pair $(C_1, D \cap C_1)$ is strongly constructible,

  (b) the pair $\big(C_2, (D \cup C_1) \cap C_2\big)$ is strongly constructible, and

  (c) $C_1 \cap C_2$ is $(d-1)$-dimensional and the pair $(C_1 \cap C_2, \phi)$ is strongly constructible,

and Δ is said to be *strongly constructible* if (Δ, ϕ) is strongly constructible.

Intuitively, the definition of strongly constructible complexes requires that the cut end of the division must be inherited in a nice way. Figure 4.1 shows the situation of the definition. Figure 4.2 is an example of a division satisfying the condition of constructibility but not satisfying the condition of strong constructibility. The following proposition shows that if Δ is a pseudomanifold, (Δ, D) is strongly constructible only if D is connected. (Here we say a complex is *connected* if every two vertices x and y in the complex has a sequence of vertices $x = v_1, v_2, \ldots, v_n = y$ such that v_i and v_{i+1} is joined with an edge, for $1 \leq i < n$.) So we can see that the division in Figure 4.2 does not satisfy the condition of strong constructibility.

Proposition 4.2. *Let Δ be a pseudomanifold and D be its $(d-1)$-dimensional pure subcomplex. If (Δ, D) is strongly constructible, then D is connected.*

Proof. Assume that D is disconnected. Because (Δ, D) is strongly constructible, Δ can be divided into C_1 and C_2 satisfying the condition of Definition 4.1. Then one of $D \cap C_1$ and $(D \cup C_1) \cap C_2 = (D \cap C_2) \cup (C_1 \cap C_2)$ is disconnected. In fact, if $D \cap C_1$ is connected, $D \cap C_1$ has only one connected component.

(Case 1) If $D \cap C_2$ has only one connected component, then $D \cap C_2$ and $C_1 \cap C_2$ must be disconnected because if not, $D \cap C_2$ and $C_1 \cap C_2$ has a common vertex and this vertex

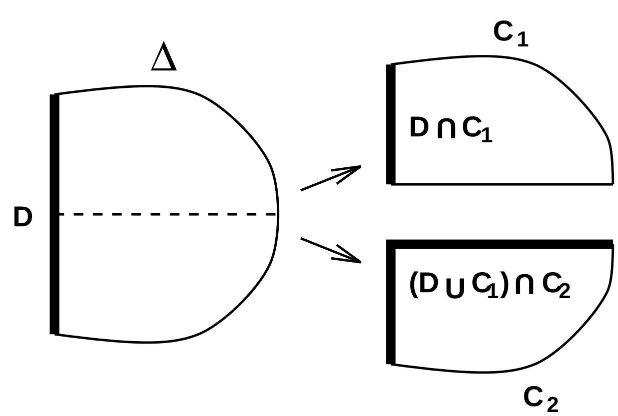

Figure 4.1: Strongly constructible complex.

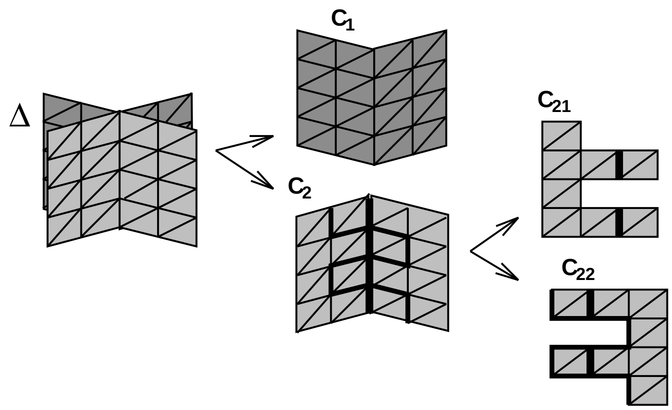

Figure 4.2: Division not satisfying the condition of strong constructibility.

must be in $D \cap C_1$. This contradicts the disconnectivity of D. So $(D \cup C_1) \cap C_2$ has two connected components $D \cap C_2$ and $C_1 \cap C_2$, and hence it is disconnected.

(Case 2) If $D \cap C_2$ has more than one connected component, $(D \cup C_1) \cap C_2$ can not be connected because $D \cap C_1$ has only one connected component.

Thus one of $D \cap C_1$ and $(D \cup C_1) \cap C_2$ is disconnected. But in the end of this division process, all the complexes are divided into simplices and any $(d-1)$-dimensional pure subcomplex of a d-dimensional simplex is connected. This is a contradiction. □

The relation between shellability, constructibility and strong constructibility is as follows.

Proposition 4.3. *Shellable complexes are strongly constructible.*

Proof. If $F_1, F_2, \ldots, F_n$ is a shelling, $C_1 = F_1 \cup F_2 \cup \cdots \cup F_{n-1}$ and $C_2 = F_n$ satisfies the condition. □

Proposition 4.4. *Strongly constructible complexes are constructible.*

Proof. This is shown easily by an induction on the number of the facets and the dimension. □

Proposition 4.5. *There are strongly constructible complexes that are not shellable.*

Proof. Example 3.2 is not shellable but strongly constructible. (Easy to see.) □

It is not known whether strong constructibility is strictly stronger than constructibility.

Open Problem. Are there constructible complexes which are not strongly constructible?

4.2 Strongly dividable posets

In this section, we introduce a notion of strongly dividable posets and show that a complex is strongly constructible if and only if its face poset is strongly dividable.

Definition 4.6. Let P be a poset and W be a subset of cocovers of coatoms of P. Then the pair (P, W) is defined to be *strongly dividable* if $\mathrm{rank}(P) = 1$, or if the following conditions hold.

(S1) For each coatom x of P, $\big([\hat{0}, x], \phi\big)$ is strongly dividable.

(S2) If there exist more than one coatom in P, the set of coatoms can be divided into two disjoint non-empty sets X and Y satisfying the conditions below. Here, we define $Z = \{z \in P \mid {}^{\exists}x \in X,\ {}^{\exists}y \in Y,\ z \prec x, y\}$.

- (a) $\hat{I}(W) \cap \hat{I}(X) = \hat{I}\big(W \cap \hat{I}(X)\big)$, and the pair $\big(\hat{I}(X), W \cap \hat{I}(X)\big)$ is strongly dividable.
- (b) $\hat{I}(W \cup Z) \cap \hat{I}(Y) = \hat{I}\big((W \cup Z) \cap \hat{I}(Y)\big)$, and the pair $\big(\hat{I}(Y), (W \cup Z) \cap \hat{I}(Y)\big)$ is strongly dividable.
- (c) $\big(\hat{I}(Z), \phi\big)$ is strongly dividable.
- (d) $\hat{I}(X) \cap \hat{I}(Y) = \hat{I}(Z)$.

A poset P is said to be *strongly dividable* if the pair (P, ϕ) is strongly dividable. A division of coatoms satisfying the conditions above is called a *strong division.*

Remark. In (S2)(b) of Definition 4.6, the condition $\hat{I}(W \cup Z) \cap \hat{I}(Y) = \hat{I}\big((W \cup Z) \cap \hat{I}(Y)\big)$ is actually redundant because this condition is always satisfied. But it is included in this definition in order to simplify the argument.

Theorem 4.7. *Let Δ be a d-dimensional pure regular cell complex and $F(\Delta)$ be its face poset. And let D be a $(d-1)$-dimensional pure subcomplex of Δ or an empty set, and W be the set of elements of $F(\Delta)$ correspondent to the facets of D. Then the pair (Δ, D) is strongly constructible if and only if the pair $\big(F(\Delta), W\big)$ is strongly dividable.*

Proof. We use an induction on the number of the facets and the dimension. When Δ has only one facet, (Δ, D) is strongly constructible if and only if $(\partial\Delta, \phi)$ is strongly constructible. And $\big(F(\Delta), W\big)$ is strongly dividable if and only if $([\hat{0}, a], \phi) = \big(F(\partial\Delta), \phi\big)$ is strongly dividable, where a is the only one coatom of $F(\Delta)$. So the statement holds by the induction on the dimension. When the dimension is 0, the statement clearly holds.

Let us assume that Δ has more than one facet and the pair (Δ, D) is strongly constructible. Then there exist d-dimensional subcomplexes C_1 and C_2 of Δ satisfying the condition of Definition 4.1. Let us divide the coatoms of $F(\Delta)$ into $\{$the facets of $C_1\} = X$ and $\{$the facets of $C_2\} = Y$. Here, the face posets of C_1 and C_2 are $\hat{I}(X)$ and $\hat{I}(Y)$ respectively. Let $Z = \{z \in P \,|\, {}^{\exists}x \in X, {}^{\exists}y \in Y,\ z \prec x, y\}$. Now we verify that all the conditions of Definition 4.6 hold. First, the condition (S1) of Definition 4.6 holds by the induction hypothesis because each facet has a smaller dimension than Δ. Next, $D \cap C_1$ is pure because $(C_1, D \cap C_1)$ is strongly constructible. This fact deduce that $\hat{I}(W) \cap \hat{I}(X) = \hat{I}(W \cap X)$. And by the induction hypothesis, the pair $\big(\hat{I}(X), W\big)$ is strongly dividable. So the condition (S2)(a) holds. Similarly, we can verify the condition (S2)(b). On the other hand, the face poset $F(C_1 \cap C_2)$ of $C_1 \cap C_2$ consists of the elements of $\{w \in P \,|\, {}^{\exists}x \in X, {}^{\exists}y \in Y,\ w \leq x, y\}$. But since $C_1 \cap C_2$ is a pure complex because $(C_1 \cap C_2, \phi)$ is strongly dividable, any $w \in F(C_1 \cap C_2)$ has a coatom z of $F(C_1 \cap C_2)$ such that $w \leq z$, so $F(C_1 \cap C_2) = \hat{I}(Z)$. Hence $(\hat{I}(Z), \phi)$ is also strongly dividable by the induction hypothesis because $(C_1 \cap C_2, \phi)$ is strongly constructible, and the condition (S2)(c) holds. This fact also imply condition (S2)(d).

Conversely, let us assume that $\big(F(\Delta), W\big)$ is strongly dividable and there is more than one coatom. Then the coatoms of $F(\Delta)$ can be divided into X and Y according to a strong division of $F(\Delta)$. Let $Z = \{z \in P \,|\, {}^{\exists}x \in X, {}^{\exists}y \in Y,\ z \prec x, y\}$. Let C_1 and C_2 be the subcomplexes of Δ whose facets are X and Y respectively. Then clearly the face posets of C_1 and C_2 are $\hat{I}(X)$ and $\hat{I}(Y)$ respectively, and $C_1 \cup C_2 = \Delta$. Because $\hat{I}(Z) = \hat{I}(X) \cap \hat{I}(Y)$, we have $\hat{I}(Z) = F(C_1 \cap C_2)$. Now the pairs $\big(\hat{I}(X), W \cap \hat{I}(X)\big)$, $\big(\hat{I}(Y), (W \cup Z) \cap \hat{I}(X)\big)$ and $(\hat{I}(Z), \phi)$ are strongly dividable, it follows that the pairs (C_1, D), $\big(C_2, (D \cup C_1) \cap C_2\big)$ and $(C_1 \cap C_2, \phi)$ are strongly constructible by the induction hypothesis, which means that Δ is strongly constructible. $\square$

Corollary 4.8. *A regular cell complex* Δ *is strongly constructible if and only if its face poset* $F(\Delta)$ *is strongly dividable.*

4.3 Signability of strongly dividable posets

Now we prove the theorem which guarantees the signability of strongly dividable posets.

Theorem 4.9. *Strongly dividable posets are signable.*

For the proof of the theorem, we need a technical lemma.

Lemma 4.10. *Let Q be a poset and W be a subset of cocovers of coatoms of Q. Then if the pair (Q, W) is strongly dividable, there is a signing χ of Q such that*

(i) there is exactly one positive coatom in $[w, \hat{1}]_Q$ if $w \notin \hat{I}(W)$, and

(ii) there is no positive coatom in $[w, \hat{1}]_Q$ if $w \in \hat{I}(W)$.

Proof. We use an induction on the number of the coatoms. When there is only one coatom, the following signing is the required signing :

$$\chi(a,b) = \begin{cases} - & \text{if } a \in W \\ + & \text{if } a \notin W. \end{cases}$$

If there is more than one coatom, let the coatoms of Q be divided into X and Y according to a strong division of Q, and let $Z = \{z \in Q \mid {}^{\exists}x \in X,\ {}^{\exists}y \in Y,\ z \prec x, y\}$. In this division, we assume that $\big(\hat{I}(X), W \cap \hat{I}(X)\big)$ is strongly dividable and $\big(\hat{I}(Y), (W \cup Z) \cap \hat{I}(Y)\big)$ is strongly dividable. Remark that the coatoms of $[w, \hat{1}]_P$ belong only to X if $w \in \hat{I}(X)$ and $w \notin \hat{I}(Z)$, and that the coatoms of $[w, \hat{1}]_P$ belong only to Y if $w \in \hat{I}(Y)$ and $w \notin \hat{I}(Z)$.

First, by using the induction hypotheses, we have a signing χ_X for $\hat{I}(X)$ satisfying that

(i) there is exactly one positive coatom in $[w, \hat{1}]_{\hat{I}(X)}$ if $w \notin \hat{I}\big(W \cap \hat{I}(X)\big)$, and

(ii) there is no positive coatom in $[w, \hat{1}]_{\hat{I}(X)}$ if $w \in \hat{I}\big(W \cap \hat{I}(X)\big)$.

Next we need a signing χ_Y of $\hat{I}(Y)$ such that

(i) there is no positive coatom in $[w, \hat{1}]_{\hat{I}(Y)}$ if $w \in \hat{I}(Z)$, and

(ii) otherwise, satisfying the following :

   (a) there is exactly one positive coatom in $[w, \hat{1}]_{\hat{I}(Y)}$ if $w \notin \hat{I}(W)$, and

   (b) there is no positive coatom in $[w, \hat{1}]_{\hat{I}(Y)}$ if $w \in \hat{I}(W)$.

The existence of such a signing χ_Y is shown by the induction hypothesis because the condition for χ_Y is equivalent to the following condition and because $\big(\hat{I}(Y), (W \cup Z) \cap \hat{I}(Y)\big)$ is strongly dividable.

(i) There is no positive coatom in $[w, \hat{1}]_{\hat{I}(Y)}$ if $w \in \big(\hat{I}(Z) \cup \hat{I}(W)\big) \cap \hat{I}(Y) = \hat{I}\big((W \cup Z) \cap \hat{I}(Y)\big)$, and

(ii) otherwise there is exactly one positive coatom in $[w, \hat{1}]_{\hat{I}(Y)}$.

Let χ be a signing such that

$$\chi(a, b) = \begin{cases} \chi_X & \text{if } b \in X \text{ and } a \text{ is its cocover} \\ \chi_Y & \text{if } b \in Y \text{ and } a \text{ is its cocover.} \end{cases}$$

Then we observe that this signing χ is the required signing of Q. In fact,

- if w is not in $\hat{I}(W)$, then
  - if w is in $\hat{I}(X)$ and not in $\hat{I}(Z)$, then all the coatoms of $[w, \hat{1}]_Q$ are in X and $[w, \hat{1}]_Q$ has exactly one coatom from the construction of χ_X,
  - if w is in $\hat{I}(Y)$ and not in $\hat{I}(Z)$, then all the coatoms of $[w, \hat{1}]_Q$ are in Y and $[w, \hat{1}]_Q$ has exactly one coatom from the construction of χ_Y, and
  - if w is in $\hat{I}(Z) = \hat{I}(X) \cap \hat{I}(Y)$, then $[w, \hat{1}]_Q$ has one positive coatom in X and no positive coatom in Y from the construction of χ_X and χ_Y, so it has exactly one positive coatom at all,

  so there is exactly one positive coatom in $[w, \hat{1}]_Q$, and

- if w is in $\hat{I}(W)$, then

– since $\hat{I}(W) \cap \hat{I}(X)$ must be equal to $\hat{I}\big(W \cap \hat{I}(X)\big)$ by the strong dividability of (Q, W), if w is in $\hat{I}(X)$, it is also in $\hat{I}\big(W \cap \hat{I}(X)\big)$ and $[w, \hat{1}]_Q$ has no positive coatom by the construction of χ_X and χ_Y, (without assuming the strong dividability, w can be outside of $\hat{I}\big(W \cap \hat{I}(X)\big)$ and $[w, \hat{1}]_Q$ can have one positive coatom in X.)

– if w is in $\hat{I}(Y)$, $[w, \hat{1}]_Q$ has no positive coatom from the construction of χ_X and χ_Y,

so there is no positive coatom in $[w, \hat{1}]_Q$.

□

Proof of Theorem 4.9. We use an induction on the number of the coatoms. If there is only one coatom, the statement is clear.

If there is more than one coatom, let the coatoms of P be divided into X and Y according to a strong division of P, and let $Z = \{z \in P \,|\, {}^{\exists}x \in X, {}^{\exists}y \in Y,\ z \prec x, y\}$. In this division, we assume that $\big(\hat{I}(X), \phi\big)$ is strongly dividable and $\big(\hat{I}(Y), Z\big)$ is strongly dividable.

First, by the induction hypothesis, we have an exact signing χ_X of $\hat{I}(X)$.

Next we need a signing χ_Y of $\hat{I}(Y)$ which satisfies that

(i) there is exactly one positive coatom in $[w, \hat{1}]_{\hat{I}(Y)}$ if $w \notin \hat{I}(Z)$, and

(ii) there is no positive coatom in $[w, \hat{1}]_{\hat{I}(Y)}$ if $w \in \hat{I}(Z)$.

The existence of such a signing χ_Y is assured by Lemma 4.10 because $\big(\hat{I}(Y), Z\big)$ is strongly dividable.

Let χ be a signing such that

$$\chi(a, b) = \begin{cases} \chi_X & \text{if } b \in X \text{ and } a \text{ is its cocover} \\ \chi_Y & \text{if } b \in Y \text{ and } a \text{ is its cocover.} \end{cases}$$

Then χ is an exact signing of P. In fact, if $w \in \hat{I}(X)$ and $w \notin \hat{I}(Z)$, all the coatoms of $[w, \hat{1}]_P$ are the members of X and there exists exactly one positive coatom in $[w, \hat{1}]_P$ by the construction of χ_X, and if $w \in \hat{I}(Y)$ and $w \notin \hat{I}(Z)$, all the coatoms of $[w, \hat{1}]_P$ are the members of Y and again there is exactly one positive coatom in $[w, \hat{1}]_P$ by the construction of χ_Y. If $w \in \hat{I}(X) \cap \hat{I}(Z) = \hat{I}(Y) \cap \hat{I}(Z) = \hat{I}(Z)$, $[w, \hat{1}]_P$ has one positive coatom in X and no positive coatom in Y, so there is exactly one positive coatom in $[w, \hat{1}]_P$. □

Corollary 4.11. *Strongly constructible simplicial complexes are partitionable.*

Chapter 5

Why we study constructible complexes?

As is mentioned in Section 3.1, a shellable complex is made by adding facets one by one. This nature of shellable complexes provides us with the way of using inductions on the number of the facets, and this is why shellable complexes are tractable and why we want to show whether the complexes in interest are shellable or not. Although shellability makes the problems easy to treat, the recognition problem of shellability is very difficult. So the weaker notions are sometimes used instead of shellability. For example, the h-vectors of a shellable simplicial complex are non-negative, but to prove the non-negativity, it is sufficient to show that the complex is Cohen-Macaulay or partitionable (See [16], [19, Chapter 8]). These notions such as partitionability or Cohen-Macaulayness is useful, but the definitions of them are too far different from that of shellability and it is unlikely that the study of these notions is of some help for the study of shellability.

Different from partitionability and Cohen-Macaulayness, constructibility resembles to shellability. This notion is defined inductively and it fits the argument of inductions on the number of facets as like as shellability. Moreover, constructibility is seemingly more tractable than shellability. For instance, it seems that the study of non-shellable balls will make some contribution to the understanding of shellability. To study non-shellable balls, the balls to be studied has many variations and we have to treat too many cases. But as

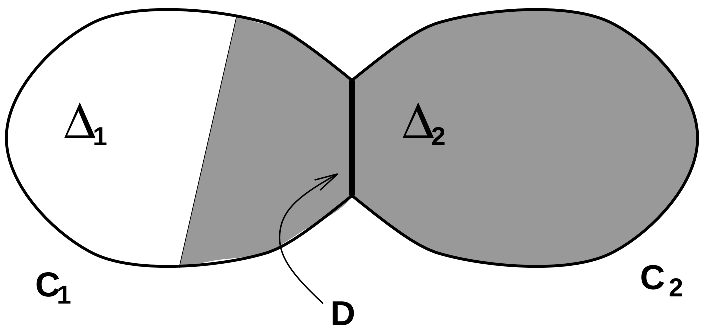

Figure 5.1: Divide Δ into Δ_1 and Δ_2.

for constructibility, we can reduce the number of the cases to be studied smaller as the proposition below indicates.

Proposition 5.1. *Let Δ be a d-dimensional ball and D be its $(d-1)$-dimensional face such that all of its faces are on the boundary of Δ except D itself. If we divide Δ by D, we have two balls C_1 and C_2 such that $C_1 \cup C_2 = \Delta$ and $C_1 \cap C_2 = D$. In this case, Δ is constructible if and only if both C_1 and C_2 are constructible.*

Proof. The first part of the statement is clear. For the latter part, let C_1 and C_2 are constructible. Then it is obvious that Δ is constructible. Conversely, let Δ be constructible. Then Δ can be divided into two constructible subcomplexes Δ_1 and Δ_2. If $\Delta_1 \cap \Delta_2$ equals to D, then $\{C_1, C_2\} = \{\Delta_1, \Delta_2\}$, hence both C_1 and C_2 are constructible. If $\Delta_1 \cap \Delta_2$ does not equal to D, then one of Δ_1 and Δ_2 contains D conserving the condition. Without loss of generality, we can assume that Δ_1 is contained in C_1. (See Figure 5.1.) Here we use an induction on the number of the facets of Δ. Because Δ_2 has smaller number of facets than Δ, both $C_1 \cap \Delta_2$ and C_2 are constructible. As for C_1, C_1 is divided into Δ_1 and $C_1 \cap \Delta_2$, and $\Delta_1 \cap (C_1 \cap \Delta_2) = \Delta_1 \cap \Delta_2$. Since all Δ_1, $C_1 \cap \Delta_2$, $\Delta_1 \cap \Delta_2$ are constructible, C_1 is also constructible. □

By this proposition, in the study of non-constructible balls, we can assume that the d-dimensional ball has no $(d-1)$-dimensional face such that all of its faces are on the boundary except itself. Thus constructibility is tractable, and is not so different from shellability, the study of this notion is expected to be of some help for the understanding of shellability.

Chapter 6

Remarks

In [17, Section 2 in Chapter III], Cohen-Macaulay simplicial complexes are conjectured to be partitionable. It is known that constructible complexes are Cohen-Macaulay, so it is very likely that constructible complexes are partitionable, so our partial result of the classification problem in Section 4.3 may further be extended in the future study. But the method of proof of Section 4.3 cannot be extended further because there may be constructible complexes that are not strongly constructible. It is thinkable that the proof of the fact that constructible complexes are partitionable will be done as a corollary of showing more generally that Cohen-Macaulay complexes are partitionable, or by showing that constructible complexes are always strongly constructible (or transforming constructible complexes into strongly constructible complexes).

Figure 6.1 shows the known relation among the classes of complexes and corresponding classes of posets. The following facts are known at present:

- Partitionability is strictly weaker than strong constructibility. (By Theorem 4.11. It is strictly weaker because there even exist partitionable complexes that are not connected.)

- Constructibility is weaker than strong constructibility. (By Proposition 4.4.)

- Strong constructibility is strictly weaker than shellability. (By Proposition 4.5.)

Yet we have some questions as below. These questions are the subjects in the future work.

Open Problems.

- Is constructibility strictly weaker than strong constructibility?
- Are all constructible complexes partitionable?
- Are there non-constructible simplicial balls?

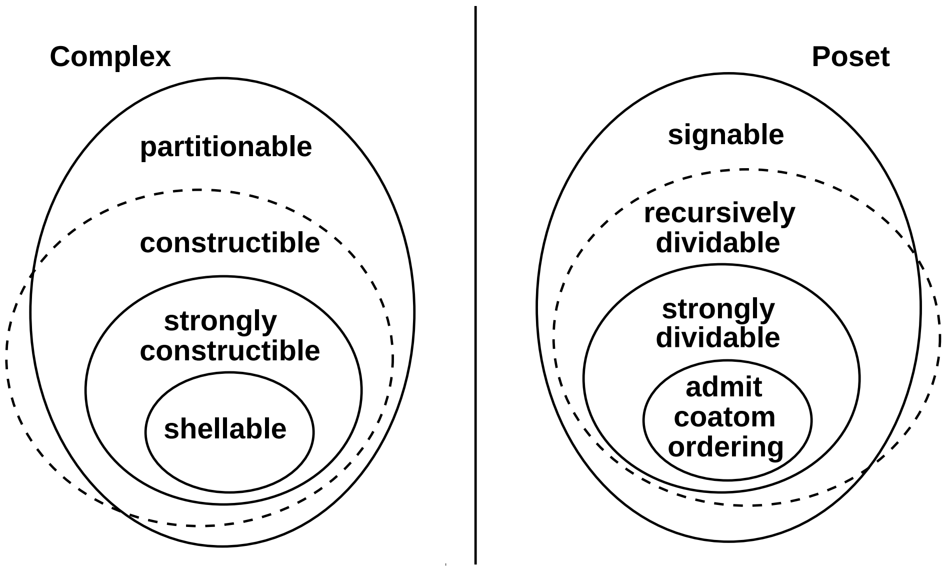

Figure 6.1: The relation among the classes of complexes and posets.

Appendix A

Cohen-Macaulay complexes

Cohen-Macaulayness is the key notion to the Stanley's solution of the Upper Bound Theorem for simplicial spheres [16]. Here we review the definition briefly. For the detail, see [7], [9], [16].

Definition A.1. Let $A = k[X_1, X_2, \ldots, X_n]$ be the polynomial ring in n-variables over a field k. For a simplicial complex Δ whose vertices are $\{v_1, v_2, \ldots, v_n\}$, let I_Δ be the ideal generated by the monomials $\{X_{i_1} X_{i_2} \cdots X_{i_t} \mid \langle v_{i_1}, v_{i_2}, \ldots, v_{i_t}\rangle \in \Delta\}$, where $\langle v_{i_1}, v_{i_2}, \ldots, v_{i_t}\rangle$ is the face whose vertices are $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$. The quotient algebra $A_\Delta = A/I_\Delta$ is called the *face ring* or the *Stanley-Reisner ring* of Δ. If A_Δ is Cohen-Macaulay ring over k, Δ is called *Cohen-Macaulay* over k.

The link $\mathrm{lk}_\Delta F$ of $F \in \Delta$ is the set of faces $\{G \in \Delta \mid G \cap F = \phi$ and $G \cup F \in \Delta\}$. Reisner [14] showed the following theorem.

Theorem A.2. *A simplicial complex Δ is Cohen-Macaulay over k if and only if the reduced homology of* $\mathrm{lk}_\Delta F$ *over k vanishes in all dimensions except the dimension of* $\mathrm{lk}_\Delta F$*, for each $F \in \Delta$ (including the empty set ϕ).*

For example, simplicial spheres and simplicial balls are Cohen-Macaulay over any field. Constructible simplicial complexes are also Cohen-Macaulay over any field k.

Proposition A.3. *Constructible simplicial complexes are Cohen-Macaulay over any field k.*

Cohen-Macaulayness is known to be a topological property.

Proposition A.4. *If Δ and Δ' are simplicial complexes such that $\|\Delta\|$ is homeomorphic to $\|\Delta'\|$, then Δ is Cohen-Macaulay over k if and only if Δ' is Cohen-Macaulay over k.*

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