DBNS near-factors and 1-overlapped factors

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A pair (A, B) of subsets of a finite group G is *near-factor* if $AB = G\backslash\{g\}$ for some $g \in G$ and $|A|, |B| \geq 2$. Near-factors play important roles in the perfect graph theory and the set packing problems. On the other hand, A pair (A, B) of subsets of a finite group G is 1-*overlapped factor* if $AB = G \cup \{g\}$ for some $g \in G$ and $|A|, |B| \geq 2$. 1-overlapped factors play an important role in the set covering problems. In this talk, we introduce some results on near-factors and 1-overlapped factors.

For integers l and m, let $[l, m]$ denote the set of integers from l by m. Let φ_k be an isomorphism from $\mathbb{Z}$ to $\mathbb{Z}_k$ such that $\varphi_k(i)$ is the residue of i divided by k and ψ_k be the inverse map of φ_k. Let $\rho(\geq 1)$ and $m_1, m_2, \ldots, m_{2\rho}(\geq 2)$, $r, s(\geq 2)$ be integers such that $r = \prod_{i=1}^{\rho} m_{2i-1}$, $s = \prod_{i=1}^{\rho} m_{2i}$ and $n = \prod_{i=1}^{2\rho} m_i$. Let $\mu_j = \prod_{i=1}^{j-1} m_i$ for $2 \leq j \leq 2\rho$ and $\mu_1 = 1$. Define a subset M_i of $\mathbb{N}$ by $[0, m_i - 1]\mu_i$, $A' := M_1 + M_3 + \cdots + M_{2\rho-1}$ and $B' := M_2 + M_4 + \cdots + M_{2\rho}$. It is clear that $(A, B) := (\varphi_{n-1}(A'), \varphi_{n-1}(B'))$ is a 1-overlapped factorization of $\mathbb{Z}_{n-1}$, $(\varphi_{n+1}(A'), \varphi_{n+1}(B'))$ is a near-factorization of $\mathbb{Z}_{n+1}$ and $(\varphi_n(A), \varphi_n(B))$ is a factorization of $\mathbb{Z}_n$. We say that this (A, B) is a *basic DBNS* 1-*overlapped factorization* (resp. to *a basic DBNS near-factorization, a basic DBNS factorization*) of $\mathbb{Z}_n$. For this 1-overlapped factor (resp. to near-factor and factorization), the following three operations construct other 1-overlapped factors (resp. to near-factors and factorizations).

- *Shifting*: Consider $(A + a, B + b)$ for some $a, b \in \mathbb{Z}_n$.
- *Scaling*: Consider $(\lambda A, \lambda B)$ for some $\lambda \in \mathbb{Z}_n^{\times}$.
- *Swapping*: Consider $(-A, B)$.

We say a 1-overlapped factor (resp. to a near-factorization and a factorization) constructed by this method is a *DBNS* 1-*overlapped factor* (resp. to a *DBNS near-factorization*, a *factorization*) of $\mathbb{Z}_n$ and the associated Lehman matrix is a *DBNS Lehman matrix* (resp. to a *DBNS partitionable graph*).

Theorem 1 (D. de Caen, D. A. Gregory, I. G. Hughes and D. L. Kreher 1990)**.** *If (A, B) is a near-factor of G, then*

$$\langle A \rangle = \langle B \rangle = G.$$

Theorem 2. *If (A, B) is a 1-overlapped factor of G, then*

$$\langle A \rangle = \langle B \rangle = G.$$

Theorem 3 (D. de Caen, D. A. Gregory, I. G. Hughes and D. L. Kreher 1990)**.** *For a near-factor* (A, B) *of a finite abelian group* G*, there exist elements* $a, b \in G$ *such that* $(A + a, B + b)$ *is a symmetric near-factor of* G*, and the uncovered element of* $(A + a, B + b)$ *is the identity.*

Theorem 4. *For a* 1*-overlapped factor* (A, B) *of a finite abelian group* G*, there exist elements* $a, b \in G$ *such that* $(A + a, B + b)$ *is a symmetric* 1*-overlapped factor of* G*, and the doubly covered element of* $(A + a, B + b)$ *is the identity.*

Theorem 5 (K. Kashiwabara and T. Sakuma 2006)**.** *A near-factor* (A, B) *of* $\mathbb{Z}_n$ *is a DBNS near-factor if* $\min(|A|, |B|) \leq 8$.

Theorem 6. *A* 1*-overlapped factor* (A, B) *of* $\mathbb{Z}_n$ *is a DBNS* 1*-overlapped factor if* $\min(|A|, |B|) \leq 8$.

A near-factor (1-overlapped factor) is *Krasner* if $0 \leq \psi_n(a) + \psi_n(b) \leq n$ for each $a, b \in \mathbb{Z}_n$.

Theorem 7. *Krasner near-factors and* 1*-overlapped factors of cyclic groups are DBNS.*

A hypergraph is a *clutter* if no two hyperedges have inclusion relations. A clutter $\mathscr{H}$ is called an *ideal clutter* if its *set covering polytope* $\{\vec{x} \in \mathbb{R}^n; M(\mathscr{H})\vec{x} \geq 1, \vec{0} \leq \vec{x} \leq \vec{1}\}$ is an integral polytope. If $\mathscr{H}$ is an ideal clutter, we say that $M(\mathscr{H})$ is an *ideal matrix*. D. R. Fulkerson defined the following two operations, *deletion* and *contraction* of column j.

- deletion: Delete j-th column, and delete i-th row if (i, j)-th entry is 1.
- contraction: Delete j-th column. If there exist dominating rows in resulting matrix, delete them too.

minor of a matrix M if we can obtain M' from M by repeatedly using deletions and contractions. P. Seymour [**?**] proved that each minor of an ideal matrix is also an ideal matrix. A matrix is called a *minimally non-ideal matrix* if it is not an ideal matrix, but its each minor is an ideal matrix. On the other hand, a clutter is *perfect* if its set packing polytope $\{\vec{x} \in \mathbb{R}^n; M(\mathscr{H})\vec{x} \leq 1, \vec{0} \leq \vec{x} \leq \vec{1}\}$ is an integral polytope. Padberg proved the following operation preserves of perfectness of a clutter.

- Delete j-th column. If there exist dominating rows in resulting matrix, delete dominated rows.

A matrix is *minimally imperfect* if it is not perfect, but its each minor is perfect.

Theorem 8 (Grinstead 1982)**.** *A clutter defined by a DBNS near-factor is minimally imperfect if and only if it is a clique clutter of an odd hole or an odd antihole.*

Theorem 9. *A clutter defined by a DBNS* 1*-overlapped factor is minimally non-ideal if and only if it is an edge clutter of an odd cycle or one of exceptional 10 clutters.*