平面分割の数え上げ問題と行列式・パフィアン

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Abstract

平面分割 (plane partitions) の数え上げ問題は MacMahon が研究を始めて以来、古典的な離散数学の問題として研究されてきたが、対称関数・群の表現論・数理物理などの分野にも現れる組合せ論的側面の研究対象でもある。この話の中では、MacMahon に始まる平面分割の母関数の古典論から始めていろいろな対称性を考慮した平面分割の母関数を、対称関数の応用して得る方法について述べ、その表現論や組合せ論との関係を振り返る。さらに、それらの応用として Mills-Robbins-Rumsey によって提出された totally symmetric self-complementary plane partitions や cyclically symmetric transpose-complementary plane partitions など交代符号行列 (alternating sign matrix) との関連を予想される平面分割の数え上げ問題を扱うことを目標にする。それらの母関数として、行列式・パフィアンによる表示や constant term による表示が得られるが、それらの行列式・パフィアンの計算は Plucker 関係式や discrete Hirota equation などの可積分系との深い関連が予想される。また、最近では affine Hecke algebra などの代数的側面との関係も数理物理学者達によって研究されている。

Plan of My Talk

1. Symmetric functions and
2. Plane partitions and symmetries
3. Totally symmetric self-complementary plane partitions

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Given a finite alphabet $\mathbb{A} = \{a_1, \dots, a_n\}$, we define the *Vandermonde matrix*

$$\mathbb{V}(\mathbb{A}) = \left(a_j^{i-1}\right)_{i\geq 1,\, 1\leq j\leq n} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

and the *Vandermonde determinant*

$$\Delta(\mathbb{A}) = \prod_{1\leq i<j\leq n} (a_j - a_i) = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{pmatrix},$$

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Schur functions

Proposition

Let $\lambda \in \mathbb{N}^n$. Then $S_\lambda(\mathbb{A})\,\Delta(\mathbb{A})$ is equal to the minor of index $(J_n(0^n), J_n(\lambda))$ of the Vandermonde matrix $\mathbb{V}(\mathbb{A})$.

Example

If $n = 4$ and $\lambda = (4331)$, then $J_n(\lambda) = (1, 4.5, 7)$ and

$$S_\lambda(\mathbb{A}) = \frac{\det\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1^4 & a_2^4 & a_3^4 & a_4^4 \\ a_1^5 & a_2^5 & a_3^5 & a_4^5 \\ a_1^7 & a_2^7 & a_3^7 & a_4^7 \end{pmatrix}}{\prod_{1\le i<j\le 4}(a_j - a_i)}$$

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Proof.

Put $I = J_n(\lambda) = \{i_1, i_2, \ldots, i_n\}$, and let $\mathbb{S}_I(\mathbb{A})$ denote the submatrix of $\mathbb{S}(\mathbb{A})$ taken on rows i_1, i_2,..., i_n. We consider the product

$$\mathbb{S}_I(\mathbb{A})\,\mathbb{S}(-\mathbb{A})\,\mathbb{V}(\mathbb{A}) = \mathbb{S}_I(\mathbb{A})\left(S^i(a_j - \mathbb{A})\right)_{i\geq 0,\, 1\leq j\leq n} = \mathbb{S}_I(\emptyset)\,\mathbb{V}(\mathbb{A})$$

Note that $S^i(a_j - \mathbb{A})$ vanishes for $i \geq n$ because they are the elementary symmetric functions of alphabets of cardinality $n - 1$.

$n=3,\ I=J_n(\lambda)=\{1,3,4\}$

$$\begin{pmatrix} S^1(\mathbb{A}) & S^0(\mathbb{A}) & 0 & 0 & 0 & 0 & 0 \\ S^4(\mathbb{A}) & S^3(\mathbb{A}) & S^2(\mathbb{A}) & S^1(\mathbb{A}) & S^0(\mathbb{A}) & 0 & 0 \\ S^6(\mathbb{A}) & S^5(\mathbb{A}) & S^4(\mathbb{A}) & S^3(\mathbb{A}) & S^2(\mathbb{A}) & S^1(\mathbb{A}) & S^0(\mathbb{A}) \end{pmatrix}$$

$$\begin{pmatrix} S^0(-\mathbb{A}) & 0 & 0 & 0 & 0 & 0 & 0 \\ S^1(-\mathbb{A}) & S^0(-\mathbb{A}) & 0 & 0 & 0 & 0 & 0 \\ S^2(-\mathbb{A}) & S^1(-\mathbb{A}) & S^0(-\mathbb{A}) & 0 & 0 & 0 & 0 \\ S^3(-\mathbb{A}) & S^2(-\mathbb{A}) & S^1(-\mathbb{A}) & S^0(-\mathbb{A}) & 0 & 0 & 0 \\ S^4(-\mathbb{A}) & S^3(-\mathbb{A}) & S^2(-\mathbb{A}) & S^1(-\mathbb{A}) & S^0(-\mathbb{A}) & 0 & 0 \\ S^5(-\mathbb{A}) & S^4(-\mathbb{A}) & S^3(-\mathbb{A}) & S^2(-\mathbb{A}) & S^1(-\mathbb{A}) & S^0(-\mathbb{A}) & 0 \\ S^6(-\mathbb{A}) & S^5(-\mathbb{A}) & S^4(-\mathbb{A}) & S^3(-\mathbb{A}) & S^2(-\mathbb{A}) & S^1(-\mathbb{A}) & S^0(-\mathbb{A}) \end{pmatrix}$$

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$$\times \begin{pmatrix} S^0(a_1 - \mathbb{A}) & S^0(a_2 - \mathbb{A}) & S^0(a_3 - \mathbb{A}) \\ S^1(a_1 - \mathbb{A}) & S^1(a_2 - \mathbb{A}) & S^1(a_3 - \mathbb{A}) \\ S^2(a_1 - \mathbb{A}) & S^2(a_2 - \mathbb{A}) & S^2(a_3 - \mathbb{A}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$
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The Gessel-Viennot Method

- I. Gessel and G. Viennot, *“Determinants, Paths, and Plane Partitions”*, preprint 1985.

Definition

Let $D = (V, E)$ be an acyclic digraph.

- $\mathscr{P}(u, v)$: the set of all directed paths from u to v.
- An *n-vertex* $\boldsymbol{v} = (v_1, \dots, v_n)$ is an n-tuple of vertices of D.
- An *n-path* from $\boldsymbol{u} = (u_1, \dots, u_n)$ to $\boldsymbol{v} = (v_1, \dots, v_n)$ is an n-tuple $\boldsymbol{P} = (P_1, \dots, P_n)$ such that $P_i \in \mathscr{P}(u_i, v_i)$.
- The n-path $\boldsymbol{P}$ is said to be *non-intersecting* if any two different paths P_i and P_j have no vertex in common.
- $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$ (resp. $\mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v})$) : the set of all (resp. non-intersecting) n-paths from $\boldsymbol{u}$ to $\boldsymbol{v}$
- $\boldsymbol{u}$ is said to be *D-compatible* with $\boldsymbol{v}$ if every path $P \in \mathscr{P}(u_i, v_l)$ intersects with every path $Q \in \mathscr{P}(u_j, v_k)$ whenever $i < j$ and $k < l$.

Non-intersecting lattice paths

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Generating functions

Definition

- We assign a weight x_e of each edge e of D.
- $w(P)$: the product of the weights of its edges for $P \in \mathscr{P}(u, v)$.
- $w(\mathbf{P})$: the product of the weights of its components for $w(\mathbf{P}) \in \mathscr{P}(\mathbf{u}, \mathbf{v})$.
- $\mathrm{GF}[S] = \sum_{\mathbf{P} \in S} w(\mathbf{P})$ for $S \subseteq \mathscr{P}(\mathbf{u}, \mathbf{v})$.
- $h(u, v) = \mathrm{GF}[\mathscr{P}(u, v)]$ for $u, v \in V$.
  $F(\mathbf{u}, \mathbf{v}) = \mathrm{GF}[\mathscr{P}(\mathbf{u}, \mathbf{v})]$ for n-vertices u, v.
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Lidström-Gessel-Viennot Theorem

Lemma (Lidström-Gessel-Viennot)

Let $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ be two n-vertices in an acyclic digraph D. Then

$$\sum_{\pi \in S_n} \operatorname{sgn}\pi \, F_0(\boldsymbol{u}^{\pi}, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \leq i,j \leq n}.$$

In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, then

$$F_0(\boldsymbol{u}, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \leq i,j \leq n}.$$

Proof

From the definition of determinants we have

$$\det[h(u_i, v_j)]_{1\le i,j\le n} = \sum_{\pi\in\mathfrak{S}_n} \operatorname{sgn}(\pi) h(u_1, v_{\pi(1)}) h(u_2, v_{\pi(2)}) \dots h(u_r, v_{\pi(n)}). \tag{1}$$

For $\pi \in \mathfrak{S}_n$, let $P(I, J; \pi)$ denote the set of all the n-paths $\boldsymbol{P} = \{P_1, \dots, P_n\}$ such that each path P_i connects u_i with $v_{\pi(i)}$ for $i = 1, \dots, n$. Let $P^0(I, J; \pi)$ denote the subset of $P(I, J; \pi)$ which consists of all non-intersecting paths $\boldsymbol{P} \in P(I, J; \pi)$. Let us define sets Π and Π^0 of configurations by

$$\Pi = \{(\pi, \boldsymbol{P}) : \pi \in \mathfrak{S}_n \text{ and } \boldsymbol{P} \in \mathscr{P}(I, J; \pi))\},$$
$$\Pi^0 = \left\{(\pi, \boldsymbol{P}) : \pi \in \mathfrak{S}_n \text{ and } \boldsymbol{P} \in \mathscr{P}^0(I, J; \pi)\right\}.$$

Then the right-hand side of (1) is the generating function of configurations $(\pi, \boldsymbol{P}) \in \Pi$ with the weight $w(\pi, \boldsymbol{P}) = \operatorname{sgn}(\pi) w(\boldsymbol{P})$.

Proof

Now we describe an involution on the set $\Pi \setminus \Pi^0$ which reverse the sign of the associated weight. First fix an arbitrary total order on V. Let $C = (\pi, \boldsymbol{P}) \in \Pi \setminus \Pi^0$. Among all vertices that occurs as intersecting points, let v denote the least vertex with respect to the fixed order. Among paths that pass through v, assume that P_i and P_j are the two whose indices i and j are smallest. Let $P_i(\to v)$ (resp. $P_i(v \to)$) denote the subpath of P_i from u_i to v (resp. from v to $v_{\pi(i)}$). Set $C' = (\pi', \boldsymbol{P}')$ to be the configuration in which $P'_k = P_k$ for $k \neq i, j$,

$$P'_i = P_i(\to v)P_j(v \to), \qquad P'_j = P_j(\to v)P_i(v \to),$$

and $\pi' = \pi \circ (i, j)$. It is easy to see that $C' \in \Pi$ and $w(C') = -w(C)$. Thus $C \mapsto C'$ defines a sign reversing involution and, by this involution, one may cancel all of the terms $\{w(C) : C \in \Pi \setminus \Pi^0\}$ and only the terms $\{w(C) : C \in \Pi^0\}$ remains. Since $h(\boldsymbol{u}^{\pi}, \boldsymbol{v}) = h(\boldsymbol{u}, \boldsymbol{v}^{\pi^{-1}})$, we obtain the resulting identity.

Proof

In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, the configurations $C \in \Pi^0$ occur only when $\pi = \mathrm{id}$, and are counted with the weight $w(P)$. This proves the lemma.

Tableaux

Definition

If λ/μ contains no 2×2 sub-diagram and connected (resp. λ/μ contains no two boxes in the same column, resp. no two boxes in the same row), then λ/μ is said to be *ribbon (rim hook)* (resp. *horizontal strip*, resp. *vertical strip*).

Definition

A *tableau* T is a sequence of partitions

$$\mu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)} = \lambda$$

such that each $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip. T is described by numbering each square in $\lambda^{(i)}/\lambda^{(i-1)}$ with number i.

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Example

If $\lambda = (7542)$ and $\mu = (31)$, then the following is a tableau of shape λ/μ.

			1	1	2	3
	1	1	2	2		
1	2	2	3			
2	3					

theorem

$$S_{\lambda/\mu}(\mathbb{A}) = \sum_{T} a^{T}$$

where T runs over all tableaux of shape λ/μ.

Schur functions

Example

If $\lambda = (7542)$ and $\mu = (31)$, then the following is a tableau of shape λ/μ.

			1	1	2	3
	1	1	2	2		
1	2	2	3			
2	3					

theorem

$$S_{\lambda/\mu}(\mathbb{A}) = \sum_{T} a^{T}$$

where T runs over all tableaux of shape λ/μ.

Proof by computation

Use

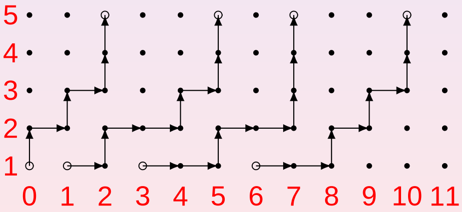
$$S_{\lambda/\mu}\left(\mathbb{A}+\mathbb{B}\right)=\sum_{\nu} S_{\lambda/\nu}\left(\mathbb{A}\right) S_{\nu/\mu}\left(\mathbb{B}\right),$$

and $S_{\lambda/\mu}\left(a\right)=0$ unless λ/μ is a horizontal strip. To prove this, use

$$S_{\lambda/\mu}\left(a\right)=\Lambda_{\lambda'/\mu'}\left(a\right)=\det\left(\Lambda^{\lambda_i'/\mu_j'}\left(a\right)\right),$$

where $\Lambda^{\lambda_i'/\mu_j'}\left(a\right)=0$ unless $\lambda_i'/\mu_j'=0$ or 1.

We consider digraph $G = (V, E)$, where $V = \mathbb{Z}^2 = \{(x, y) | x, y \mathbb{Z}$ and $E = \{(1, 0), (0, 1)$. Take $v_i = (\lambda_i + m - i, n)$ and $u_j = (\mu_j + m - j, 1)$ and asign the weight 1 to each vertical edge, and the weight a_y to to each edge starting at (x, y).



Plane partitions

Definition

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j \geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j \geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the *weight* n.

Example

A plane partition of 14

$$\begin{matrix} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ \vdots & & & & & \\ 0 & 0 & 0 & \ddots & & \end{matrix}$$

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Definition

Let $\pi = (\pi_{ij})_{i,j \geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
1	1		

Shape

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Example of plane partitions

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- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: $\boxed{1}$
- Plane partitions of 2:

$$\begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}$$

- Plane partitions of 3:

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

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Generating Function

Theorem (MacMahon)

The generating function for plane partitions is

$$\sum_{\pi} q^{|\pi|} = \prod_{k=1}^{\infty} (1 - q^k)^{-k},$$

where the sum runs over all (unrestricted) plane partitions.

Example

$$\prod_{k=1}^{\infty} (1 - q^k)^{-k} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \cdots$$

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Ferrers graph

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The *Ferrers graph* $D(\pi)$ of π is the subset of $\mathbb{P}^3$ defined by

$$D(\pi) = \left\{ (i, j, k) : k \leq \pi_{ij} \right\}$$

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Ferrers graph

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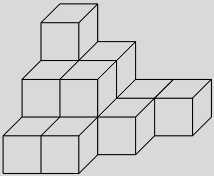


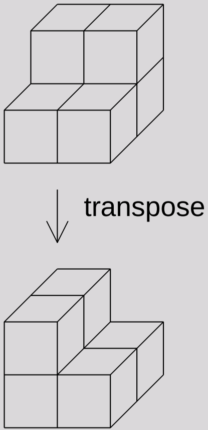
Symmetries of plane paritions

Definition

If $\pi = (\pi_{ij})$ is a plane partition, then the *transpose* π^* of π is defined by $\pi^* = (\pi_{ji})$.

- π is *symmetric* if $\pi = \pi^*$.
- π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$.
- π is called *totally symmetric* if it is cyclically symmetric and symmetric.

Example



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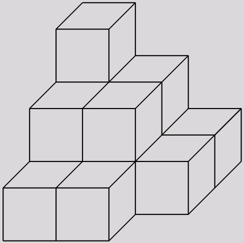
A symmetric PP



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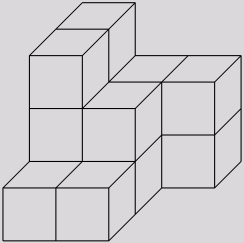

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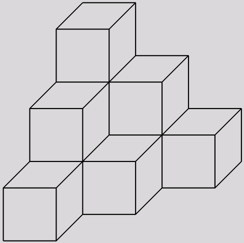

Complement

Definition

Let $\pi = (\pi_{ij})$ be a plane partition contained in the box $B(r, s, t) = [r] \times [s] \times [t]$.

Define the *complement* π^c of π by

$\pi^c = \{ (r+1-i, s+1-j, t+1-k) : (i, j, k) \notin \pi \}$.

- π is said to be *(r, s, t)-self-complementary* if $\pi = \pi^c$. i.e. $(i, j, k) \in \pi \Leftrightarrow (r+1-i, s+1-j, t+1-k) \notin \pi$.

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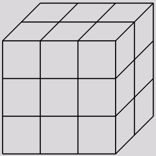

$B(2, 3, 3)$

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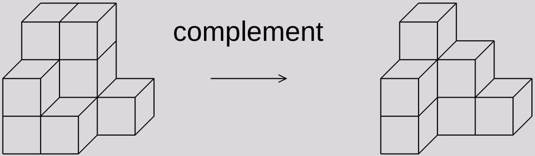

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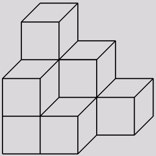

A $(2, 3, 3)$-self-complementary PP

Definition

Let $\pi = (\pi_{ij})$ be a plane partition contained in the box $B(r, r, t)$.
Define the *transpose-complement* π^{tc} of π by
$\pi^{tc} = \{\,(r+1-j, r+1-i, t+1-k)\;:\;(i,j,k) \notin \pi\,\}$.

- π is said to be *complement=transpose* if $\pi = \pi^{tc}$, i.e. $(i,j,k) \in \pi \Leftrightarrow (r+1-j, r+1-i, t+1-k) \notin \pi$.

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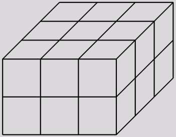

$B(3, 3, 2)$

Transpose-complement

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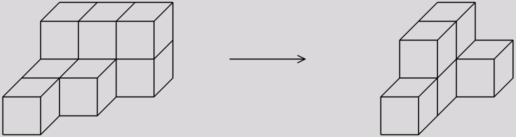

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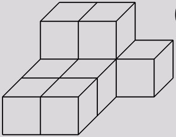

(3, 3, 2)-complement=transpoese

Symmetry classes of plane partitions

Symmetry classes (Stanley)

The transformation ^{c} and the group S_3 generate a group T of order 12. The group T has ten conjugacy classes of subgroups, giving rise to ten enumeration problems.

Table [illegible]

1	[illegible]	[illegible]
2	$B(r,r,t)$	*Symmetric*
[illegible]	[illegible]	*Cyclically symmetric*
4	$B(r,r,r)$	*Totally symmetric*
[illegible]	[illegible]	*Self-complementary*
6	$B(r,r,t)$	*Complement = transpose*
[illegible]	[illegible]	*Symmetric and self-complementary*
8	$B(r,r,r)$	*Cyclically symmetric and complement = transpose*
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Table (R. P. Stanley, "Symmetries of Plane Partitions", *J. Combin. Theory Ser.* **A 43**, 103-113 (1986))

1	$B(r,s,t)$	Any
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