Balanced Nested Designs and Balanced $n$-ary Designs

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Abstract: We introduce here two types of balanced nested designs (BND), which are called symmetric and pair-sum BNDs. In this paper, we give a construction for pair-sum BNDs of BIBDs from nested BIBDs and perpendicular arrays. We also give some direct constructions for pair-sum BNDs of BIBDs, based on the result obtained by Wilson (1972). By use of these constructions, we show some constructions for regular balanced $n$-ary designs.

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To the memory of Professor Sumiyasu Yamamoto

1. Balanced Nested Designs

Let $V$ be a $v$-set and $B$ be a collection of subsets of $V$. The elements of $V$ and $B$ are called points and blocks, respectively. The pair $(V, B)$ with some combinatorial
conditions is called a block design. There are many known types of block designs, for example, a balanced incomplete block design (BIBD), a \( t \)-design, a pairwise balanced design (PBD), an \((r, \lambda)\)-design, a group divisible design (GDD) and so on. We will consider these block designs with further combinatorial conditions and their applications to other block designs.

Suppose that each block \( B \in \mathcal{B} \) of a block design \((\mathcal{V}, \mathcal{B})\) is partitioned into \( g \) subblocks \( B_1, B_2, \ldots, B_g \) (some of them may be empty). Let \( \mathcal{B}_i \) be the collection of \( i \)-th subblocks \( B_i \) for each \( B \in \mathcal{B} \) and let \( \Pi = \{B_1, B_2, \ldots, B_g\} \). If each \((\mathcal{V}, \mathcal{B}_i)\) is also a block design, then the triple \((\mathcal{V}, \mathcal{B}, \Pi)\) is called a nested design. We consider a class of nested designs having one further combinatorial condition between these subdesigns, called balanced nested designs (BND).

We introduce here two types of balanced nested designs. For distinct points \( x \) and \( y \) and distinct subblocks \( B_i \) and \( B_j \) of each \( B \in \mathcal{B} \), we denote the number of blocks \( B \) containing \( x \) in \( B_i \) and \( y \) in \( B_j \) by \( \mu_{ij}(x, y) \). For any distinct subdesigns \((\mathcal{V}, \mathcal{B}_i)\) and \((\mathcal{V}, \mathcal{B}_j)\), if \( \mu_{ij}(x, y) \) is independent of the choice of \( x \) and \( y \), then \((\mathcal{V}, \mathcal{B}, \Pi)\) is called a symmetric BND, and the constant values are denoted by \( \mu_{ij} = \mu_{ij}(x, y) \). If \( \mu_{ij}(x, y) + \mu_{ij}(y, x) \) is independent of the choice of \( x \) and \( y \), then \((\mathcal{V}, \mathcal{B}, \Pi)\) is called a pair-sum BND, where \( \mu_{ij}(x, y) + \mu_{ij}(y, x) \) are denoted by \( \nu_{ij} \). It is clear that \( \mu_{ij} = \mu_{ji} \) in a symmetric BND and \( \nu_{ij} = \nu_{ji} \) in a pair-sum BND for every \( i \) and \( j \) \((i \neq j, 1 \leq i, j \leq g)\). A symmetric BND satisfies stronger conditions than a pair-sum type, and it is also a pair-sum BND with \( \nu_{ij} = 2\mu_{ij} \).

As the first stage of our research on BNDs, we consider a pair-sum BND such that \((\mathcal{V}, \mathcal{B})\) is a BIBD. A balanced incomplete block design BIBD\((v, k, \lambda)\) is a pair \((\mathcal{V}, \mathcal{B})\) satisfying the following conditions:

1. every point occurs in precisely \( r \) blocks,

2. each block contains precisely \( k \) points,
every pair of distinct points occurs in precisely $\lambda$ blocks.

An $(r, \lambda)$-design is a pair $(\mathcal{V}, \mathcal{B})$ satisfying the conditions (1) and (3), and a 1-design is a pair satisfying the conditions (1) and (2).

Kuriki and Fuji-Hara (1994) considered a symmetric BND $(\mathcal{V}, \mathcal{B}, \Pi)$ such that $(\mathcal{V}, \mathcal{B})$ is an $(r, \lambda)$-design and that each $(\mathcal{V}, \mathcal{B}_i)$ is also an $(r_i, \lambda_i)$-design. They called this design an $(r, \lambda)$-design with mutually balanced nested subdesigns and gave some constructions for such $(r, \lambda)$-designs. Note that such an $(r, \lambda)$-design is equivalent to some statistical array (called a balanced array). From now on we use the terminology “balanced nested designs” including the above two types instead of “mutually balanced nested subdesigns”.

In this paper, we consider a pair-sum BND $(\mathcal{V}, \mathcal{B}, \Pi)$ such that the size of each $i$-th subblock of $\mathcal{B}_i$ is equal to $k_i$ and that $(\mathcal{V}, \mathcal{B}_i)$ is a BIBD($v, k_i, \lambda_i$) if $k_i \geq 2$ and a 1-design if $k_i = 1$ for $1 \leq i \leq g$. Let $\mathcal{B}_{ij}$ be the collection of the blocks $B_i \cup B_j$ for the $i$-th subblock $B_i$ and the $j$-th subblock $B_j$ of each $B \in \mathcal{B}$. Then $(\mathcal{V}, \mathcal{B}_{ij})$ is a BIBD($v, k_i+k_j, \lambda_{ij}$) such that $\lambda_{ij} = \lambda_i + \lambda_j + \nu_{ij}$ for every $i$ and $j$ ($i \neq j, 1 \leq i, j \leq g$), and $(\mathcal{V}, \mathcal{B})$ is also a BIBD($v, k, \lambda$) such that

$$k = \sum_{i=1}^{g} k_i \quad \text{and} \quad \lambda = \sum_{i=1}^{g} \lambda_i + \sum_{1 \leq i < j \leq g} \nu_{ij},$$

where $\lambda_i = 0$ if $k_i = 1$. Such a triple $(\mathcal{V}, \mathcal{B}, \Pi)$ is called a pair-sum BND of BIBDs.

We give in Sections 2 and 3 some constructions for such pair-sum BNDs of BIBDs. In Section 4, we construct balanced $n$-ary designs using these constructions in Sections 2 and 3.

**Example 1.1.** Let $\mathcal{V} = \{0, 1, 2, 3, 4, 5, 6\}$ and $\mathcal{B}$ be a collection of 7 blocks:

$$\{0, 1, 3, 6\}, \{1, 2, 4, 0\}, \{2, 3, 5, 1\}, \{3, 4, 6, 2\}, \{4, 5, 0, 3\}, \{5, 6, 1, 4\}, \{6, 0, 2, 5\}.$$
The pair \((V, B)\) is a BIBD\((7, 4, 2)\). We partition each of these blocks into two subblocks as follows:

\[
\begin{align*}
\{0, 1, 3\} \cup \{6\}, & \quad \{1, 2, 4\} \cup \{0\}, & \quad \{2, 3, 5\} \cup \{1\}, & \quad \{3, 4, 6\} \cup \{2\}, \\
\{4, 5, 0\} \cup \{3\}, & \quad \{5, 6, 1\} \cup \{4\}, & \quad \{6, 0, 2\} \cup \{5\}.
\end{align*}
\]

The triple \((V, B, \Pi)\) is a pair-sum BND of BIBDs with the parameters \(v = 7\), \(g = 2\), \(k_1 = 3\), \(k_2 = 1\), \(\lambda_1 = 1\), \(\lambda_2 = 0\), and \(\nu_{12} = 1\), where \(\Pi = \{B_1, B_2\}\) and \(B_1\) and \(B_2\) are collections of the first and the second subblocks of each block of \(B\).

\section{2. Pair-sum BNDs of BIBDs from nested BIBDs and perpendicular arrays}

In order to describe our first construction for pair-sum BNDs of BIBDs, we need the definitions of a nested BIBD and a perpendicular array. Let \((V, B)\) be a BIBD\((v, k, \lambda)\). Suppose that each block of \(B\) is partitioned into \(k/k'\) subblocks with \(k'\) points each. We denote the collection of all subblocks by \(B'\). If \((V, B')\) is a BIBD\((v, k', \lambda')\), then the triple \((V, B, B')\) is called a \textit{nested BIBD} and denoted by \text{NBIBD}\((v; k, \lambda; k', \lambda')\). Obviously,

\[
vr = bk = b'k', \quad \lambda(v - 1) = r(k - 1) \quad \text{and} \quad \lambda'(v - 1) = r(k' - 1),
\]

where \(r\) is the number of blocks of \(B\) containing each point of \(V\), \(b\) and \(b'\) are the numbers of blocks and subblocks of \(B\) and \(B'\), respectively. The notion of a nested BIBD was first introduced by Preece (1967) and a list of nested BIBDs for small values of the parameters was given there. Note that there is another definition for nested BIBDs (e.g., Federer (1972)). Kageyama and Miao (1997) have recently generalized the concept of a nested BIBD in order to unify these different nested BIBDs due to Preece (1967) and Federer (1972).
A perpendicular array \( PA(g, s, \lambda) \) is a \( \lambda \left( \frac{s}{2} \right) \times g \) matrix \( A \) with entries from \( S = \{1, 2, \ldots, s\} \) such that each row has \( g \) distinct entries of \( S \) and that each two columned submatrix \( A_0 \) of \( A \) contains each unordered pair of two distinct entries of \( S \) as a row of \( A_0 \) exactly \( \lambda \) times, where \( \left( \frac{s}{2} \right) \) denotes the binomial coefficient. If each column has each entry of \( S \lambda(s-1)/2 \) times, the perpendicular array is said to be regular. Note that a pair-sum BND of BIBDs with \( k_1 = k_2 = \cdots = k_g = 1 \) is equivalent to a regular perpendicular array.

We can construct a pair-sum BND of BIBDs by means of a nested BIBD and a regular perpendicular array.

**Theorem 2.1.** If there exist an NBIBD \( (v; k, \lambda; k', \lambda') \) and a regular PA \( (g, s, \lambda'') \) such that \( s = k/k' \), then there exists a pair-sum BND \( (V, B, \Pi) \) of BIBDs with the parameters

\[
k_i = k', \quad \lambda_i = \frac{\lambda\lambda''(s-1)}{2} \quad \text{and} \quad \nu_{ij} = (\lambda - \lambda')\lambda'',
\]

where \( |V| = v \) and \( |\Pi| = g \).

**Proof.** Let \( (V, B', B'') \) be an NBIBD \( (v; k, \lambda; k', \lambda') \) and let \( A = (a_{ui}) \) be a regular PA \( (g, s, \lambda'') \) such that \( s = k/k' \). For each block \( B' \) of \( B' \), we construct \( \lambda'' \left( \frac{s}{2} \right) \) blocks such that the \( i \)-th subblock of the \( u \)-th block is the \( a_{ui} \)-th subblock of \( B' \) for \( i = 1, 2, \ldots, g \) and \( u = 1, 2, \cdots, \lambda'' \left( \frac{s}{2} \right) \). Let \( B \) and \( B_i \) be the collections of these \( b'\lambda'' \left( \frac{s}{2} \right) \) blocks and their \( i \)-th subblocks, respectively, and \( \Pi = \{B_1, B_2, \cdots, B_g\} \), where \( b' \) is the number of blocks of \( B' \). We show that the triple \( (V, B, \Pi) \) is a pair-sum BND of BIBDs with the parameters given in (2.1). Since \( (V, B'') \) is a BIBD \( (v, k', \lambda') \) and the perpendicular array \( A \) is regular, \( (V, B_i) \) is a BIBD \( (v, k', \lambda\lambda''(s-1)/2) \) for \( i = 1, 2, \cdots, g \). Also since \( (V, B', B'') \) is an NBIBD \( (v; k, \lambda; k', \lambda') \) and \( A \) is a PA \( (g, s, \lambda'') \), the number \( \mu_{ij}(x, y) + \mu_{ij}(y, x) \) is equal to \( (\lambda - \lambda')\lambda'' \), which is independent of the choice of the distinct points \( x \) and \( y \). \( \square \)
A large number of constructions for nested BIBDs and regular perpendicular arrays can be found in Morgan (1996) and Bierbrauer (1996). The following propositions were given by Jimbo and Kuriki (1983) and Rao (1961), respectively.

**Proposition 2.2.** Let $q$ be a prime or a prime power. Then, for any positive integers $d$ and $d'$ such that $d > d'$, there exists an NBIBD$(q^N; q^d, \lambda; q^{d'}, \lambda')$, where $N = d + d'$, $\lambda = \phi(N - 2, d - 2, q)$ and $\lambda' = \phi(N - 2, d' - 2, q)$.

Here
\[
\phi(N - 1, d - 1, q) = \frac{(q^N - 1)(q^{N-1} - 1) \cdots (q^{N-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q - 1)}.
\]

**Proposition 2.3.** Let $q$ be an odd prime or an odd prime power. Then there exists a regular PA$(q, q, 1)$.

Applying Propositions 2.2 and 2.3 to Theorem 2.1, we have:

**Corollary 2.4.** Let $q$ be an odd prime or an odd prime power. Then, for any positive integers $d$ and $d'$ such that $d > d'$, there exists a pair-sum BND $(V, B, \Pi)$ of BIBDs with the parameters $k_i = q^{d'}$, $\lambda_i = (q^{d-d'} - 1)\phi(N - 2, d' - 2, q)/2$ and $\nu_{ij} = \phi(N - 2, d - 2, q) - \phi(N - 2, d' - 2, q)$, where $|V| = q^N$, $|\Pi| = q^{d-d'}$ and $N = d + d'$.

3. Pair-sum BNDs of BIBDs from finite fields

There are numerous direct constructions for various types of block designs. The most commonly used direct construction technique is the method of differences developed by Bose (1939). In this section, we will use this method to provide some series of pair-sum BNDs of BIBDs from finite fields. Let $q$ be a prime or a prime power and let $V = GF(q)$, a finite field of order $q$. For any subsets $W$ and $W'$ of $V$, let $W + W'$, $W - W'$ and $W \circ W'$ be multisets $\{w + w'|w \in W, w' \in W\}$,
\(\{w - w' | w \in \mathcal{W}, w' \in \mathcal{W}'\}\) and \(\{ww' | w \in \mathcal{W}, w' \in \mathcal{W}'\}\), respectively. For any nonnegative integer \(\lambda\), \(\lambda \times \mathcal{W}\) denotes the multiset containing every element of \(\mathcal{W}\) exactly \(\lambda\) times. For brevity, \(\{w\} + \mathcal{W}'\) and \(\{w\} \circ \mathcal{W}'\) are denoted by \(w + \mathcal{W}'\) and \(w\mathcal{W}'\), respectively. For an integer \(h\) satisfying \(h|q - 1\), we define \(H^h_u\) as the set \(\{x^e | e \equiv u (\text{mod } h)\}\), where \(x\) is a primitive element of \(\mathcal{V}\). Clearly \(H^h_u\) is a subgroup of \(\mathcal{V}^*\), which is denoted by \(H^h\), where \(\mathcal{V}^*\) denotes the multiplicative group over the set of nonzero elements of \(\mathcal{V}\). Select an element \(c_u\) from each \(H^h_u\) and call the set \(C_h = \{c_0, c_1, \cdots, c_{h-1}\}\) the system of distinct representatives for the cosets modulo \(H^h\). Then \(\mathcal{V}^* = H^h \circ C_h\).

For an \(\ell\)-subset \(L\) of \(C_h\), let \(B = L \circ H^h\), i.e., \(B\) is a union of \(\ell\) cosets of \(H^h_0, H^h_1, \cdots, H^h_{h-1}\). Let \(\mathcal{B}\) be a collection of blocks \(y + cB\) for \(y \in \mathcal{V}\) and \(c \in C_h\). In the case when \(h\) is even and \((q - 1)/h(=f, \text{say})\) is odd, let \(\mathcal{B}'\) be a collection of blocks \(y + cB\) for \(y \in \mathcal{V}\) and \(c \in C_{h/2}\). Then Wilson (1972) proved that \((\mathcal{V}, \mathcal{B})\) is a BIBD\((q, f\ell, \ell(f\ell - 1))\) and \((\mathcal{V}, \mathcal{B}')\) is a BIBD\((q, f\ell, \ell(f\ell - 1)/2)\).

**Proposition 3.1.** Let \(q = hf + 1\) be a prime or a prime power. For any positive integer \(\ell\) such that \(\ell \leq h\), there exists a BIBD\((q, f\ell, \ell(f\ell - 1))\). Moreover, in the case when \(h\) is even and \(f\) is odd, there exists a BIBD\((q, f\ell, \ell(f\ell - 1)/2)\).

By Proposition 3.1, we can show the following theorem:

**Theorem 3.2.** Let \(q = hf + 1\) be a prime or a prime power. For any positive integers \(\ell_1, \ell_2, \cdots, \ell_g\) such that \(\sum_{i=1}^{g} \ell_i \leq h\), there exists a pair-sum BND \((\mathcal{V}, \mathcal{B}, \Pi)\) of BIBDs with the parameters

\[
k_i = f\ell_i, \quad \lambda_i = \ell_i(f\ell_i - 1) \quad \text{and} \quad \nu_{ij} = 2f\ell_i\ell_j.
\]  

Moreover, in the case when \(h\) is even and \(f\) is odd, there exists a pair-sum BND \((\mathcal{V}, \mathcal{B}', \Pi')\) of BIBDs with the parameters

\[
k_i = f\ell_i, \quad \lambda_i = \ell_i(f\ell_i - 1)/2 \quad \text{and} \quad \nu_{ij} = f\ell_i\ell_j,
\]
where $|\mathcal{V}| = q$ and $|\Pi| = |\Pi'| = g$.

**Proof.** For any mutually disjoint subsets $L_1, L_2, \ldots, L_g$ of $C_h$ such that $|L_i| = \ell_i$, let $B_i = L_i \circ H^h$ and $B = B_1 \cup B_2 \cup \cdots \cup B_g$. Let $\mathcal{B}$ and $\mathcal{B}_i$ be collections of blocks $y + cB$ and $i$-th subblocks $y + cB_i$ for $y \in \mathcal{V}$ and $c \in C_h$, respectively, and $\Pi = \{B_1, B_2, \ldots, B_g\}$. In the case when $h$ is even and $f$ is odd, let $\mathcal{B}'$ and $\mathcal{B}'_i$ be collections of blocks $y + cB$ and $i$-th subblocks $y + cB_i$ for $y \in \mathcal{V}$ and $c \in C_{h/2}$, and $\Pi' = \{\mathcal{B}'_1, \mathcal{B}'_2, \ldots, \mathcal{B}'_g\}$. Kuriki and Fuji-Hara (1994) have shown that $(\mathcal{V}, \mathcal{B}, \Pi)$ is a symmetric BND such that $\mu_{ij} = f\ell_i\ell_j$. By this fact and Proposition 3.1, $(\mathcal{V}, \mathcal{B}, \Pi)$ is a pair-sum BND of BIBDs with the parameters given in (3.1). Now we will show that $(\mathcal{V}, \mathcal{B}', \Pi')$ is a pair-sum BND of BIBDs with (3.2). By Proposition 3.1, $(\mathcal{V}, \mathcal{B}')$ is a BIBD($q, f\ell_i, \ell_i(f\ell_i - 1)/2$) for $1 \leq i \leq g$. For any two elements $x^{e_i}$ of $L_i$ and $x^{e_j}$ of $L_j$, the multiset $x^{e_i}H^h - x^{e_j}H^h$ consists of $f$ cosets modulo $H^h$, where $x$ is a primitive element of $\mathcal{V}$. So the multiset $L_i \circ H^h - L_j \circ H^h$ consists of $f\ell_i\ell_j$ cosets modulo $H^h$. It is easy to show that $C_{h/2} \cup (-1)C_{h/2}$ is a system of distinct representatives for the cosets modulo $H^h$. Then we have

$$\bigcup_{c \in C_{h/2}} \{c(L_i \circ H^h) - c(L_j \circ H^h)\} \supseteq \{c(L_i \circ H^h) - c(L_i \circ H^h)\}$$

$$= \bigcup_{c \in C_{h/2}} \{c, -c\} \circ (L_i \circ H^h - L_j \circ H^h) = \bigcup_{c \in C_h} c(L_i \circ H^h - L_j \circ H^h) = f\ell_i\ell_j \times \mathcal{V}^*.$$

Hence, for distinct points $x^e$ and $x^{e'}$ of $\mathcal{V}$, the number $\mu_{ij}(x^e, x^{e'}) + \mu_{ij}(x^{e'}, x^e)$ is equal to $f\ell_i\ell_j$, which is independent of the choice of $x^e$ and $x^{e'}$. This completes the proof. \hfill \Box

Replacing the block $B = B_1 \cup B_2 \cup \cdots \cup B_g$ in the proof of Theorem 3.2 with $B = B_1 \cup B_2 \cup \cdots \cup B_g \cup \{0\}$, which contains the zero element of $\mathcal{V}$, we can show the following theorem in a similar way to Theorem 3.2.

**Theorem 3.3.** Let $q = hf + 1$ be a prime or a prime power. For any positive integers $\ell_1, \ell_2, \ldots, \ell_g$ such that $\sum_{i=1}^g \ell_i \leq h$, there exists a pair-sum BND $(\mathcal{V}, \mathcal{B}, \Pi)$
of BIBDs with the parameters (3.1) and $k_{g+1} = 1$, $\lambda_{g+1} = 0$ and $\nu_{ig+1} = 2\ell_i$ for $1 \leq i \leq g$. Moreover, in the case when $h$ is even and $f$ is odd, there exists a pair-sum BND $(\mathcal{V}, \mathcal{B}', \Pi')$ of BIBDs with the parameters (3.2) and $k_{g+1} = 1$, $\lambda_{g+1} = 0$ and $\nu_{ig+1} = \ell_i$ for $1 \leq i \leq g$, where $|\mathcal{V}| = q$ and $|\Pi| = |\Pi'| = g + 1$.

If, instead of using $\{0\}$ as a distinct subblock of size 1, we use $B_g \cup \{0\}$, say, as a subblock of size $f\ell_g + 1$, we can get another theorem in a similar way.

**Theorem 3.4.** Let $q = hf + 1$ be a prime or a prime power. For any positive integers $\ell_1, \ell_2, \cdots, \ell_g$ such that $\sum_{i=1}^g \ell_i \leq h$, there exists a pair-sum BND $(\mathcal{V}, \mathcal{B}, \Pi)$ of BIBDs with the parameters (3.1) and $k_g = f\ell_g + 1$, $\lambda_g = \ell_g(f\ell_g + 1)$ and $\nu_g = 2\ell_i(f\ell_g + 1)$ for $1 \leq i \leq g - 1$. Moreover, in the case when $h$ is even and $f$ is odd, there exists a pair-sum BND $(\mathcal{V}, \mathcal{B}', \Pi')$ of BIBDs with the parameters (3.2) and $k_g = f\ell_g + 1$, $\lambda_g = \ell_g(f\ell_g + 1)/2$ and $\nu_g = \ell_i(f\ell_g + 1)$ for $1 \leq i \leq g - 1$, where $|\mathcal{V}| = q$ and $|\Pi| = |\Pi'| = g$.

### 4. Balanced $n$-ary Designs

In this section, we give some series of balanced $n$-ary designs by use of these constructions for pair-sum BNDs of BIBDs obtained in the preceding sections. Let $(\mathcal{V}, \mathcal{B})$ be a pair such that $\mathcal{B}$ is a collection of multi-subsets of $\mathcal{V}$. A balanced $n$-ary design is a pair $(\mathcal{V}, \mathcal{B})$ satisfying the following conditions:

1. **(A1)** every block contains precisely $K$ points,

2. **(A2)** every point occurs 0, 1, $\cdots$, or $n-1$ times in any block,

3. **(A3)** every pair of distinct points occurs precisely $\Lambda$ times.

Note that, for example, a pair $\{x, y\}$ is counted twice in a block $\{x, x, y\}$ and once in a block $\{x, y, z\}$. Such a balanced $n$-ary design is denoted by $BnD(\mathcal{V}, K, \Lambda)$,
where $V$ is the number of points of $V$. Let $\rho^*_x$ be the number of blocks in which the point $x$ occurs exactly $i$ times. If $\rho^*_x$ is independent of the choice of $x$, then the balanced $n$-ary design is said to be regular and $\rho^*_x$ is denoted by $\rho_i$. In this case, every point occurs the same number $\sum_{i=1}^{n-1} i \rho_i$ times. Balanced $n$-ary designs were first introduced by Tocher (1952) and capital letters have been used for the parameters of the design, traditionally. Many authors have studied balanced $n$-ary designs and we especially refer to Billington (1984,1989), which are excellent survey papers.

Consider the $V \times B^*$ (multi) incidence matrix $M = (m_{xu})$ of a pair $(V, B)$ such that a point $x$ of $V$ occurs $m_{xu}$ times in the $u$-th block of $B$, where $B^*$ is the number of blocks of $B$. We rewrite the conditions (A1),(A2),(A3) for $(V, B)$ to be a balanced $n$-ary design as follows:

\begin{itemize}
  \item [(B1)] $\sum_{x \in V} m_{xu} = K$ for every $u = 1, 2, \cdots, B^*$,
  \item [(B2)] $0 \leq m_{xu} \leq n - 1$ for every point $x$ of $V$ and every $u = 1, 2, \cdots, B^*$,
  \item [(B3)] $\sum_{u=1}^{B^*} m_{xu}m_{x'u} = \Lambda$ for every pair of distinct points $x$ and $x'$ of $V$.
\end{itemize}

Now we consider a relationship between a BND and a balanced $n$-ary design. Let $(V, B, \Pi)$ be a BND, where $\Pi = \{B_1, B_2, \cdots, B_g\}$. The size of the $i$-th subblock of the $u$-th block of $B$ is denoted by $k_{iu}$ for $1 \leq i \leq g$ and $1 \leq u \leq b$, where $b$ is the number of blocks of $B$. For distinct points $x$ and $y$, $\lambda_i(x, y)$ denotes the number of $i$-th subblocks containing $x$ and $y$. Let

$$K_u = \sum_{i=1}^{g} i k_{iu} \quad \text{and} \quad \Lambda(x, y) = \sum_{i=1}^{g} i^2 \lambda_i(x, y) + \sum_{1 \leq i < j \leq g} ij \{\mu_{ij}(x, y) + \mu_{ij}(y, x)\}.$$ 

Then a BND satisfying some conditions is equivalent to a balanced $n$-ary design.

**Theorem 4.1.** The existence of a BND $(V, B, \Pi)$ satisfying the following conditions:
(1) $K_u$ is a constant for every $u = 1, 2, \ldots, b$.

(2) for every pair of distinct points $x$ and $y$ of $V$, $\Lambda(x, y)$ is independent of the choice of $x$ and $y$.

is equivalent to the existence of a $BnD(V, K, \Lambda)$ such that

$$V = v, \quad K = K_u, \quad n = g + 1 \quad \text{and} \quad \Lambda = \Lambda(x, y),$$

where $|V| = v$, $|B| = b$ and $|\Pi| = g$.

**Proof.** Let $(V, B, \Pi)$ be a BND satisfying the conditions (1) and (2) given in this theorem. We define a $v \times b$ matrix $M = (m_{xu})$ as

$$m_{xu} = \begin{cases} 
    i, & \text{if a point } x \text{ of } V \text{ occurs in the } i\text{-th subblock} \\
    0, & \text{of the } u\text{-th block of } B,
\end{cases}$$

We show that the matrix $M$ is the incidence matrix of a $BnD(V, K, \Lambda)$ with (4.1). Obviously, from the definition of $M$, we have $V = v$ and $n = g + 1$. Since the BND $(V, B, \Pi)$ satisfies the conditions (1) and (2), we have

$$\sum_{x \in V} m_{xu} = \sum_{i=1}^g ik_{iu} = K_u,$$

which is a constant for every $u = 1, 2, \ldots, b$, and for every pair of distinct points $x$ and $y$,

$$\sum_{u=1}^b m_{xu}m_{yu} = \sum_{i=1}^g i^2 \lambda_i(x, y) + \sum_{1 \leq i < j \leq g} ij \{\mu_{ij}(x, y) + \mu_{ij}(y, x)\} = \Lambda(x, y),$$

which is independent of the choice of $x$ and $y$.

Conversely, let $M$ be the incidence matrix of a $BnD(V, K, \Lambda)$. We give a correspondence between points of a $v$-set $V$ and rows of $M$, and between blocks of a collection $B$ and columns of $M$. Each block of $B$ consists of points corresponding to nonzero entries of $M$. For each block $B$ of $B$, we partition $B$ into $n - 1$ subblocks $B_1, B_2, \ldots, B_{n-1}$ such that $B_i$ consists of points with the entry $i$. Let $B_i$ be a collection of the $i$-th subblocks $B_i$ of each $B$ and $\Pi = \{B_1, B_2, \ldots, B_{n-1}\}$. It is easy to show that the triple $(V, B, \Pi)$ is a BND satisfying the conditions (1) and (2).
Since a pair-sum BND of BIBDs satisfies the conditions (1) and (2) in Theorem 4.1, we can construct a balanced $n$-ary design from a pair-sum BND of BIBDs.

**Theorem 4.2.** If there exists a pair-sum BND $(V, \mathcal{B}, \Pi)$ of BIBDs, then there exists a regular balanced $n$-ary design $BnD(V, K, \Lambda)$ with the parameters

$$V = v, \quad K = \sum_{i=1}^{g} ik_i, \quad n = g + 1 \quad \text{and} \quad \Lambda = \sum_{i=1}^{g} i^2 \lambda_i + \sum_{1 \leq i < j \leq g} ij \nu_{ij},$$

where $\lambda_i = 0$ if $k_i = 1$, $|V| = v$ and $|\Pi| = g$.

Note that if we change the order of the subblocks of each block of a pair-sum BND of BIBDs, we may have a regular balanced $n$-ary design with different values of the parameters.

Theorem 4.2 and some constructions for pair-sum BNDs of BIBDs in Sections 2 and 3 yield some series of regular balanced $n$-ary designs. Combining Theorem 2.1, Corollary 2.4 and Theorem 4.2, we obtain:

**Corollary 4.3.** If there exist an NBIBD $(v; k, \lambda; k', \lambda')$ and a regular PA$(g, s, \lambda'')$ such that $s = k/k'$, then there exists a regular $BnD(V, K, \Lambda)$ with the parameters

$$V = v, \quad K = \frac{1}{2}g(g + 1)k', \quad n = g + 1$$

and

$$\Lambda = \frac{1}{12}g(g + 1)(2g + 1)\lambda'\lambda''(s - 1) + \frac{1}{24}g(g + 1)(g - 1)(3g + 2)(\lambda - \lambda')\lambda''.$$

**Corollary 4.4.** Let $q$ be an odd prime or an odd prime power. Then, for any positive integers $d$ and $d'$ such that $d > d'$, there exists a regular $BnD(V, K, \Lambda)$ with the parameters

$$V = q^N, \quad K = \frac{1}{2}q^d(q^{d-d'} + 1), \quad n = q^{d-d'} + 1.$$
\( \Lambda = \frac{1}{24} q^{d-d'}(q^{d-d'}+1)(q^{d-d'}-1)\{ (3q^{d-d'}+2)\phi(N-2,d-2,q)+q^{d-d'}\phi(N-2,d'-2,q) \} \),

where \( N = d + d' \).

Combining Theorems 3.2, 3.3, 3.4 and 4.2, we obtain immediately the following corollaries:

**Corollary 4.5.** Let \( q = hf + 1 \) be a prime or a prime power. For any positive integers \( \ell_1, \ell_2, \cdots, \ell_g \) such that \( \sum_{i=1}^{g} \ell_i \leq h \), there exists a regular \( B_{nD}(V, K, \Lambda) \) with the parameters

\[
V = q, \quad K = f \sum_{i=1}^{g} i\ell_i, \quad n = g + 1 \quad \text{and} \quad \Lambda = \sum_{i=1}^{g} \ell_i (f\ell_i - 1) + 2f \sum_{1 \leq i < j \leq g} ij\ell_i\ell_j.
\]

Moreover, in the case when \( h \) is even and \( f \) is odd, there exists a regular \( B_{nD}(V, K, \Lambda) \) with the parameters

\[
V = q, \quad K = f \sum_{i=1}^{g} i\ell_i, \quad n = g + 1 \quad \text{and} \quad \Lambda = \frac{1}{2} \sum_{i=1}^{g} \ell_i (f\ell_i - 1) + f \sum_{1 \leq i < j \leq g} ij\ell_i\ell_j.
\]

**Corollary 4.6.** Let \( q = hf + 1 \) be a prime or a prime power. For any positive integers \( \ell_1, \ell_2, \cdots, \ell_g \) such that \( \sum_{i=1}^{g} \ell_i \leq h \), there exists a regular \( B_{nD}(V, K, \Lambda) \) with the parameters

\[
V = q, \quad K = f \sum_{i=1}^{g} i\ell_i + g + 1, \quad n = g + 2
\]

and

\[
\Lambda = \sum_{i=1}^{g} \ell_i (f\ell_i - 1) + 2f \sum_{1 \leq i < j \leq g} ij\ell_i\ell_j + 2(g+1) \sum_{i=1}^{g} i\ell_i.
\]

Moreover, in the case when \( h \) is even and \( f \) is odd, there exists a regular \( B_{nD}(V, K, \Lambda) \) with the parameters

\[
V = q, \quad K = f \sum_{i=1}^{g} i\ell_i + g + 1, \quad n = g + 2
\]
\[ \Lambda = \frac{1}{2} \sum_{i=1}^{g} i^{2} \ell_{i}(f\ell_{i} - 1) + f \sum_{1 \leq i < j \leq g} ij \ell_{i}\ell_{j} + (g + 1) \sum_{i=1}^{g} i \ell_{i}. \]

**Corollary 4.7.** Let \( q = hf + 1 \) be a prime or a prime power. For any positive integers \( \ell_{1}, \ell_{2}, \ldots, \ell_{g} \) such that \( \sum_{i=1}^{g} \ell_{i} \leq h \), there exists a regular \( B_{n}D(V, K, \Lambda) \) with the parameters

\[ V = q, \quad K = f \sum_{i=1}^{g-1} i \ell_{i} + g(f\ell_{g} + 1), \quad n = g + 1 \]

and

\[ \Lambda = \sum_{i=1}^{g-1} i^{2} \ell_{i}(f\ell_{i} - 1) + g^{2}\ell_{g}(f\ell_{g} + 1) + 2f \sum_{1 \leq i < j \leq g-1} ij \ell_{i}\ell_{j} + 2g(f\ell_{g} + 1) \sum_{i=1}^{g-1} i \ell_{i}. \]

Moreover, in the case when \( h \) is even and \( f \) is odd, there exists a regular \( B_{n}D(V, K, \Lambda) \) with the parameters

\[ V = q, \quad K = f \sum_{i=1}^{g-1} i \ell_{i} + g(f\ell_{g} + 1), \quad n = g + 1 \]

and

\[ \Lambda = \frac{1}{2} \sum_{i=1}^{g} i^{2} \ell_{i}(f\ell_{i} - 1) + \frac{1}{2} g^{2}\ell_{g}(f\ell_{g} + 1) + f \sum_{1 \leq i < j \leq g-1} ij \ell_{i}\ell_{j} + g(f\ell_{g} + 1) \sum_{i=1}^{g-1} i \ell_{i}. \]

**References**


