Balanced arrays of strength two and nested \((r,\lambda)\)-designs

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Abstract: An \((r,\lambda)\)-design with mutually balanced nested subdesigns (for brevity, \((r,\lambda)\)-design with MBN) is introduced firstly in this paper. It is shown that an \((r,\lambda)\)-design with MBN is equivalent to a balanced array of strength 2 with \(s\) symbols. By the use of a nested design and an orthogonal array, a construction of an \((r,\lambda)\)-design with MBN is given. A direct construction of such an \((r,\lambda)\)-design, based on the result obtained by Wilson (1972), is also given. By these constructions, new balanced arrays with \(s \geq 3\) are presented.
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Abbreviated title: Balanced arrays and nested $(r,\lambda)$-designs
1. Introduction

A balanced array was first introduced by Chakravarti (1956,1961) in connection with some class of statistical designs. Some constructions of balanced arrays have been studied by several authors (e.g., Kageyama (1975), Rafter and Seiden (1974), Shirakura (1977), Srivastava (1972) and Srivastava and Chopra (1973)). Our concern in this paper is to construct balanced arrays of strength 2 with \( s \geq 3 \) symbols.

First of all we begin with the definition of a balanced array.

Let \( S \) be \( \{0, 1, \ldots, s-1\} \), called symbols. A balanced array of strength 2, denoted by \( BA(m, n, s, 2) \), is an \( m \times n \) matrix \( T \) whose entries are from \( S \), satisfying the following conditions:

(i) in any two rowed submatrix \( T_0 \) of \( T \), a vector \([i, j]\) occurs exactly \( \mu_{ij} \) times as columns in \( T_0 \) for any \( i, j \in S \),

(ii) \( \mu_{ij} = \mu_{ji} \) for any \( i, j \in S \).

The \( \mu_{ij} \)'s are called indices. If \( \mu_{ij} = \mu \) for every \( i, j \in S \), then the array is called an orthogonal array with the index \( \mu \), and is denoted by \( OA(m, n, s, 2) \), where \( n = \mu s^2 \).

Example 1.1. A 6 \( \times \) 20 matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 \\
0 & 2 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 2 & 1 & 0 & 0 \\
2 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

is a \( BA(6, 20, 3, 2) \) with indices \( \mu_{00} = 4, \mu_{01} = \mu_{02} = 3 \) and \( \mu_{11} = \mu_{12} = \mu_{22} = 1 \).

In order to construct such balanced arrays, we introduce in Section 2 an \((r, \lambda)\)-design satisfying some conditions. And we show that it is equivalent to a balanced array of strength 2 with \( s \) symbols. In Section 3, we give some constructions of
such \((r, \lambda)\)-designs. By these constructions, we present new balanced arrays with \(s\) symbols.

2. \((r, \lambda)\)-designs with mutually balanced nested subdesigns

An \((r, \lambda)\)-design is a pair \((V, B)\), where \(V\) is a \(v\)-set (called points) and \(B\) is a collection of subsets of \(V\) (called blocks), satisfying the following conditions:

(i) every point occurs in precisely \(r\) blocks,

(ii) every pair of distinct points occurs in precisely \(\lambda\) blocks.

If each block contains \(k\) points, then it is called a balanced incomplete block design, and is denoted by \((v, k, \lambda)\)-BIBD.

It is well-known that the existence of an \((r, \lambda)\)-design \((V, B)\) is equivalent to the existence of a \(BA(v, b, 2, 2)\) with indices \(\mu_{11} = \lambda, \mu_{01} = r - \lambda\) and \(\mu_{00} = b - 2r + \lambda\), where \(b\) is the number of blocks of \(B\). In the case of more than 2 symbols, Dey, Kulshreshtha and Saha (1972) have shown a construction of balanced arrays with 3 symbols from BIBD’s. Kageyama (1975) has improved the result for \(s \geq 3\).

Now we introduce an \((r, \lambda)\)-design with mutually balanced nested subdesigns and show that it is equivalent to a balanced array of strength 2 with \(s\) symbols. Let \((V, B)\) be an \((r, \lambda)\)-design. Suppose that each block \(B \in B\) is partitioned into \(g\) subblocks \(B_1, B_2, \ldots, B_g\) (some of them may be empty). We denote the collection of the \(i\)-th subblocks \(B_i\)’s for each \(B \in B\) by \(B_i\) and \(\Pi = \{B_1, B_2, \ldots, B_g\}\). An \((r, \lambda)\)-design with mutually balanced nested subdesigns (for brevity, \((r, \lambda)\)-design with MBN) is a triple \((V, B, \Pi)\) satisfying the following conditions:

(i) \((V, B_i)\) is an \((r_i, \lambda_i)\)-design for \(i = 1, 2, \ldots, g\),
(ii) for distinct points $x$ and $y$ of $V$, the number of blocks $B$ of $B$ containing $x$ in the $i$-th and $y$ in the $j$-th subblocks of $B$ is $\lambda_{i,j}$ which is independent of the $x$ and $y$ chosen.

Note that $\lambda_{i,j} = \lambda_{j,i}$ and $\lambda_i = \lambda_{i,i}$. It is easily seen that

$$r = \sum_{i=1}^{g} r_i \quad \text{and} \quad \lambda = \sum_{i=1}^{g} \lambda_i + 2 \sum_{1 \leq i < j \leq g} \lambda_{i,j}.$$  

**Theorem 2.1.** The existence of an $(r,\lambda)$-design $(V,B,\Pi)$ with mutually balanced nested subdesigns is equivalent to the existence of a $BA(v,b,s,2)$ with indices

$$\mu_{ij} = \begin{cases} 
\lambda_{i,j}, & \text{if } i \neq j, i, j \neq 0, \\
\lambda_i, & \text{if } i = j \neq 0, \\
r_i - \sum_{u=1}^{s-1} \lambda_{i,u}, & \text{if } i \neq 0 \text{ and } j = 0, \\
b - 2r + \lambda, & \text{if } i = j = 0,
\end{cases}$$

(2.1)

where $v = |V|$, $b$ is the number of blocks of $B$ and $s = |\Pi| + 1$.

**Proof.** Let $(V,B,\Pi)$ be an $(r,\lambda)$-design with MBN. Then we define a $v \times b$ matrix $T = [t_{x,u}]$ as

$$t_{x,u} = \begin{cases} 
i, & \text{if a point } x \text{ of } V \text{ occurs in the } i\text{-th subblock} \\
0, & \text{otherwise.}
\end{cases}$$

Let $T_0$ be any two rowed submatrix of $T$. It is immediately seen, from the definition of an $(r,\lambda)$-design with MBN, that $\mu_{ij} = \lambda_{i,j}$ for $i \neq j, i, j \neq 0$; and $\mu_{ii} = \lambda_i$ for $i \neq 0$. Since every point of $V$ occurs in precisely $r_i$ blocks in $B_i$, each of vectors $[0,i]$ and $[i,0]$ for $i \neq 0$ occurs exactly $r_i - \sum_{u=1}^{s-1} \lambda_{i,u}$ times as columns in $T_0$. So the vector $[0,0]$ occurs

$$b - 2 \sum_{1 \leq i < j \leq s-1} \lambda_{i,j} - \sum_{i=1}^{s-1} \lambda_i - 2 \sum_{i=1}^{s-1} \{r_i - \sum_{u=1}^{s-1} \lambda_{i,u}\}$$

$$= b - 2 \sum_{i=1}^{s-1} r_i + \sum_{i=1}^{s-1} \lambda_i + 2 \sum_{1 \leq i < j \leq s-1} \lambda_{i,j}$$

$$= b - 2r + \lambda$$
times. Hence $T$ is a $BA(v, b, s, 2)$ with indices given in (2.1).

Conversely, let $T$ be a $BA(v, b, s, 2)$ with indices given in (2.1). We give a correspondence between points of a $v$-set $V$ and rows of $T$, and between blocks of a collection $\mathcal{B}$ and columns of $T$. Each block of $\mathcal{B}$ consists of points of $V$ corresponding to nonzero entries of $T$. $T$ is also a balanced array with 2 symbols, the pair $(V, \mathcal{B})$ therefore is an $(r, \lambda)$-design. For each $B \in \mathcal{B}$, we partition $B$ into $s - 1$ subblocks $B_1, B_2, \ldots, B_{s-1}$ such that $B_i$ consists of points with the entry $i$. Let $\mathcal{B}_i$ be a collection of the $i$-th subblocks $B_i$ of $B \in \mathcal{B}$ and $\Pi = \{B_1, B_2, \ldots, B_{s-1}\}$. Since $T$ is a $BA(v, b, s, 2)$ with indices given in (2.1), the number $r_i$ of blocks of $\mathcal{B}_i$ containing any point is $\sum_{u=0}^{s-1} \mu_{iu}$, and the number $\lambda_i$ of blocks of $\mathcal{B}_i$ containing any pair of distinct points is $\mu_{ii}$. For distinct points $x$ and $y$ of $V$, the number $\lambda_{ij}$ of blocks $B$ of $\mathcal{B}$ containing $x$ in the $i$-th and $y$ in the $j$-th subblocks of $B$ is $\mu_{ij}$ which is independent of the $x$ and $y$ chosen. Hence the triple $(V, \mathcal{B}, \Pi)$ is an $(r, \lambda)$-design with MBN. \quad \square

**Example 2.2.** Let $V = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{B}$ be a collection of 20 blocks:

$$
\{1, 2, 3\}, \{1, 2, 5\}, \{1, 3, 6\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 4, 5\}, \{3, 4, 6\}, \{4, 5, 6\};
\{1, 2, 4\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 5, 6\}, \{3, 4, 5\}, \text{ twice.}
$$

The pair $(V, \mathcal{B})$ is an $(r, \lambda)$-design with $r = 10$ and $\lambda = 4$. We partition each of these blocks into two subblocks as follows:

$$
\begin{align*}
\{1, 3\} \cup \{2\}, & \quad \{1, 2, 5\} \cup \phi, \quad \{6\} \cup \{1, 3\}, \quad \{5\} \cup \{1, 6\}, \quad \phi \cup \{2, 3, 4\}, \\
\{2, 4\} \cup \{5\}, & \quad \{3, 4\} \cup \{6\}, \quad \{6\} \cup \{4, 5\}, \quad \{2\} \cup \{1, 4\}, \quad \{4\} \cup \{1, 2\}, \\
\{1\} \cup \{3, 5\}, & \quad \{3\} \cup \{1, 5\}, \quad \{1, 4, 6\} \cup \phi, \quad \{1\} \cup \{4, 6\}, \quad \{2, 3, 6\} \cup \phi, \\
\{2\} \cup \{3, 6\}, & \quad \{5, 6\} \cup \{2\}, \quad \phi \cup \{2, 5, 6\}, \quad \{3, 5\} \cup \{4\}, \quad \{4, 5\} \cup \{3\}.
\end{align*}
$$

The triple $(V, \mathcal{B}, \Pi)$ is an $(r, \lambda)$-design with MBN having parameters $r_1 = r_2 = 5$ and $\lambda_1 = \lambda_2 = \lambda_{1,2} = 1$, where $\Pi = \{\mathcal{B}_1, \mathcal{B}_2\}$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ are collections of the first and the second subblocks of each block of $\mathcal{B}$. A $6 \times 20$ matrix $T$, as defined in the proof of Theorem 2.1 for the $(r, \lambda)$-design $(V, \mathcal{B}, \Pi)$ with MBN, is a $BA(6, 20, 3, 2)$ with indices $\mu_{00} = 4$, $\mu_{01} = \mu_{02} = 3$ and $\mu_{11} = \mu_{12} = \mu_{22} = 1$. After some rearrangement of columns, this balanced array is seen to be the same as the one in Example 1.1.
3. Constructions of \((r,\lambda)\)-designs with mutually balanced nested subdesigns

Let \((V,\mathcal{B})\) be an \((r,\lambda)\)-design. Suppose that each block of \(\mathcal{B}\) is partitioned into \(g\) subblocks some of which may be empty. We denote the collection of all subblocks by \(\mathcal{B}'\). A nested \((r,\lambda)\)-design is a triple \((V,\mathcal{B},\mathcal{B}')\) such that every pair of distinct points of \(V\) occurs in precisely \(\lambda'\) blocks in \(\mathcal{B}'\). If each block contains \(k\) points and each subblock contains \(k'\) points, then it is called a nested BIBD, and is denoted by \(\text{NBIBD}(v;k,\lambda;k',\lambda')\). Obviously,

\[
vr = bk = b'k', \quad \lambda(v - 1) = r(k - 1) \quad \text{and} \quad \lambda'(v - 1) = r(k' - 1),
\]

where \(v = |V|\) and \(b\) and \(b'\) are the numbers of blocks of \(\mathcal{B}\) and \(\mathcal{B}'\), respectively.

The nested BIBD was introduced by Preece (1967) and a list of nested BIBD’s for small values of parameters was given. Note that there is a different definition for nested designs (see, Colbourn and Colbourn (1983) and Longyear (1981)).

Using a nested \((r',\lambda')\)-design and an orthogonal array, we construct an \((r,\lambda)\)-design with MBN.

**Theorem 3.1.** If there exist a nested \((r',\lambda')\)-design \((V,\mathcal{B}',\mathcal{B}'')\) and an \(\text{OA}(g + 1, g, 2)\) \(A = [a_{i,u}]\) is standardized as follows:

\[
a_{g+1,u+u'} = u, \quad \text{for any} \ 0 \leq u \leq g - 1 \quad \text{and} \quad 1 \leq u' \leq g,
\]

then there exists an \((r,\lambda)\)-design \((V,\mathcal{B},\Pi)\) (\(|\Pi| = g\)) with mutually balanced nested subdesigns having parameters

\[
r_i = (g - 1)r', \quad \lambda_i = (g - 1)\lambda'' \quad \text{and} \quad \lambda_{i,j} = \lambda' - \lambda'', \quad (3.1)
\]

where \(g\) is the number of subblocks in each block of \(\mathcal{B}'\) and \(\lambda''\) is the number of blocks of \(\mathcal{B}''\) containing any pair of distinct points of \(V\).

**Proof.** Without loss of generality, we may assume that an \(\text{OA}(g + 1, g^2, g, 2)\) \(A = [a_{i,u}]\) is standardized as follows:
and 
\[ a_{i,0} = 0, \quad a_{i,1} = 1, \quad \text{for any } 1 \leq i \leq g \text{ and } 1 \leq u \leq g. \]

From this assumption, it is seen that, in the columns of a submatrix \( \tilde{A} = [a_{i,u}] \) 
(1 \leq i \leq g, 1 \leq u \leq g(g-1)) of \( A \), entries of each column are all distinct. For each block \( B' \) of \( B^r \), we construct \( g(g-1) \) blocks such that the \( i \)-th subblock of the \( u \)-th block is the \((a_{i,u} + 1)\)-th subblock of \( B' \) for 1 \leq i \leq g and 1 \leq u \leq g(g-1). \)

Let \( B \) and \( B_i \) be collections of these \( g(g-1)b' \) blocks and the \( i \)-th subblocks, respectively, and \( \Pi = \{B_1, B_2, \ldots, B_g\} \), where \( b' \) is the number of blocks of \( B^r \). Obviously, \((V,B)\) is an \((r,\lambda)\)-design. Since \((V,B')\) is an \((r',\lambda'')\)-design and \( A \) is an \( OA(g+1, g^2, g, 2) \), the number \( r_i \) of blocks of \( B_i \) containing any point is \((g - 1)r'\), and the number \( \lambda_i \) of blocks of \( B_i \) containing any pair of distinct points is \((g - 1)\lambda''\). Since \((V,B',\Pi)\) is an \((r',\lambda')\)-design, for distinct points \( x \) and \( y \) of \( V \), the number \( \lambda_{i,j} \) of blocks \( B \) of \( B \) containing \( x \) in the \( i \)-th and \( y \) in the \( j \)-th subblocks of \( B \) is \( \lambda' - \lambda'' \) which is independent of the \( x \) and \( y \) chosen. Hence the triple \((V,B,\Pi)\) is an \((r,\lambda)\)-design with \( MBN \) having parameters given in (3.1). \( \square \)

Combining Theorems 2.1 and 3.1, we obtain immediately the following corollary:

**Corollary 3.2.** If there exist a nested \((r,\lambda)\)-design \((V,B,B')\) and an \( OA(s, (s - 1)^2, s - 1, 2) \), then there exists a \( BA(v, (s - 1)(s - 2)b, s, 2) \) with indices

\[ \mu_{ij} = \begin{cases} 
\lambda' - \lambda', & \text{if } i \neq j, \ i, j \neq 0, \\
(s - 2)\lambda', & \text{if } i = j \neq 0, \\
(s - 2)(r - \lambda), & \text{if } i \neq 0 \text{ and } j = 0, \\
(s - 1)(s - 2)(b - 2r + \lambda), & \text{if } i = j = 0, 
\end{cases} \]

where \( s - 1 \) is the number of subblocks in each block of \( B \), \( v = |V| \), \( b \) is the number of blocks of \( B \) and \( \lambda' \) is the number of blocks of \( B' \) containing any pair of distinct points of \( V \).

Let \((V,B)\) be an \((r,\lambda)\)-design. We partition each block \( B \in B \) into two subblocks \( B \) and \( \phi \). Then a triple \((V,B,B')\) is a nested \((r,\lambda)\)-design. From Corollary 3.2, we
construct a $BA(v, 2b, 3, 2)$ with indices $\mu_{12} = 0$, $\mu_{11} = \mu_{22} = \lambda$, $\mu_{01} = \mu_{02} = r - \lambda$ and $\mu_{00} = 2(b - 2r + \lambda)$. This is the case of Dey, Kulshreshtha and Saha (1972).

If we partition each block $B$ into $B$, $\phi$, $\phi, \ldots$, then we have the case of Kageyama (1975).

Several constructions of nested BIBD’s are shown by Homel and Robinson (1975) and Jimbo and Kuriki (1983). For example, Jimbo and Kuriki have shown:

**Theorem 3.3.** Let $q$ be a prime power. Then, for all positive integers $d$ and $d'$ such that $d > d'$, there exists a NBIBD($q^N; q^d, \lambda; q^{d'}, \lambda'$), where $N = d + d'$, \( \lambda = \phi(N - 2, d - 2, q) \) and \( \lambda' = \phi(N - 2, d' - 2, q) \).

Here

\[
\phi(N - 1, d - 1, q) = \frac{(q^N - 1)(q^{N-1} - 1) \cdots (q^{N-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q - 1)}.
\]

We apply Theorem 3.3 to Corollary 3.2. Since it is well-known that there exists an $OA(s, (s - 1)^2, s - 1, 2)$ for a prime power $s - 1 = q^{d-d'}$, we have:

**Corollary 3.4.** Let $q$ be a prime power. Then, for all positive integers $d$ and $d'$ such that $d > d'$, there exists a $BA(q^N, q^d(q^{d-d'} - 1)\phi(N - 1, d' - 1, q), q^{d-d'} + 1, 2)$ with indices

\[
\mu_{ij} = \begin{cases} 
q^d/\Phi, & \text{if } i \neq j, i, j \neq 0, \\
\Phi, & \text{if } i = j \neq 0, \\
q^d\Phi, & \text{if } i \neq 0 \text{ and } j = 0, \\
q^{d-d'}(q^N - q^d - 1)\Phi, & \text{if } i = j = 0,
\end{cases}
\]

where $N = d + d'$ and $\Phi = \phi(N - 2, d' - 2, q)$.

We give a direct construction of $(r, \lambda)$-designs with MBN. Let $v$ be a prime power and let $V = GF(v)$, a finite field of order $v$. For any subsets $W$ and $W'$ of $V$, let $W + W'$, $W - W'$ and $W \circ W'$ be multisets $\{w + w'|w \in W, w' \in W\}$, $\{w - w'|w \in W, w' \in W\}$ and $\{ww'|w \in W, w' \in W\}$, respectively. And, for any nonnegative integer $\lambda$, $\lambda \times W$ denotes a multiset containing every element of $W \lambda$ times. For
brevity, \( \{w\} + W' \) and \( \{w\} \circ W' \) are denoted by \( w + W' \) and \( wW' \), respectively. For an integer \( p \) satisfying \( p|v-1 \), we define \( H^p_u \) as a set \( \{x^e | e \equiv u \mod p \} \), where \( x \) is a primitive element of \( V \). Clearly \( H^p_0 \) is a subgroup of \( V \), which is denoted by \( H^p \). We select an element \( c_u \) from each \( H^p_u \) and call elements of the set \( C_p = \{c_0, c_1, \ldots, c_{p-1}\} \) representatives for the cosets modulo \( H^p \). Then \( V - \{0\} = H^p \circ C_p \).

For an \( \ell \)-subset \( L \) of \( C_p \), let \( B = L \circ H^p \), i.e., \( B \) is a union of \( \ell \) cosets of \( H^p_0, H^p_1, \ldots, H^p_{p-1} \). And let \( B \) be a collection of blocks \( y + cB \) for \( y \in V \) and \( c \in C_p \). Then Wilson (1972) has shown that the pair \( (V, B) \) is a \( (v, f\ell, \ell(f\ell - 1)) \)-BIBD, where \( v = pf + 1 \).

**Lemma 3.5.** Let \( v = pf + 1 \) be a prime power. For all positive integer \( \ell \) such that \( \ell \leq p \), there exists a \( (v, f\ell, \ell(f\ell - 1)) \)-BIBD.

By Lemma 3.5, we can show the following theorem:

**Theorem 3.6.** Let \( v = pf + 1 \) be a prime power. For all positive integers \( \ell_1, \ell_2, \ldots, \ell_g \) such that \( \sum_{i=1}^g \ell_i \leq p \), there exists an \( (r, \lambda) \)-design, \( r = pf\ell \) and \( \lambda = \ell(f\ell - 1) \), with mutually balanced nested subdesigns having parameters

\[
 r_i = pf\ell_i, \quad \lambda_i = \ell_i(f\ell_i - 1) \quad \text{and} \quad \lambda_{i,j} = f\ell_i\ell_j, \quad (3.2)
\]

where \( \ell = \sum_{i=1}^g \ell_i \).

**Proof.** For any mutually disjoint subsets \( L_1, L_2, \ldots, L_g \) of \( C_p \), \( \ell_i = |L_i| \), let \( B_i = L_i \circ H^p \) and \( B = B_1 \cup B_2 \cup \cdots \cup B_g \). And let \( B \) and \( B_i \) be collections of blocks \( y + cB \) and the \( i \)-th subblocks \( y + cB_i \) for \( y \in V \) and \( c \in C_p \), respectively, and \( \Pi = \{B_1, B_2, \ldots, B_g\} \). Now we show that the triple \( (V, B, \Pi) \) is an \( (r, \lambda) \)-design with MBN having parameters given in (3.2). By Wilson’s construction, \( (V, B) \) is an \( (r, \lambda) \)-design and \( (V, B_i) \) is an \( (r_i, \lambda_i) \)-design for \( i = 1, 2, \ldots, g \), where \( r_i = pf\ell_i \) and \( \lambda_i = \ell_i(f\ell_i - 1) \). For any two elements \( x^{e_i} \) of \( L_i \) and \( x^{e_j} \) of \( L_j \), a multiset \( (x^{e_i}H^p) - (x^{e_j}H^p) \) consists of \( f \) cosets modulo \( H^p \), where \( x \) is a primitive element.
of \( V \). So a multiset \((L_i \circ H^p) - (L_j \circ H^p)\) consists of \( f\ell_i\ell_j \) cosets modulo \( H^p \). Then we have
\[
\bigcup_{c \in C_p} \{c(L_i \circ H^p) - c(L_j \circ H^p)\} = \bigcup_{c \in C_p} c\{(L_i \circ H^p) - (L_j \circ H^p)\} = (f\ell_i\ell_j) \times (V - \{0\}).
\]
Hence, for distinct points \( x^e \) and \( x^{e'} \) of \( V \), the number \( \lambda_{i,j} \) of blocks \( \mathcal{B}' \) of \( \mathcal{B} \) containing \( x^e \) in the \( i \)-th and \( x^{e'} \) in the \( j \)-th subblocks of \( \mathcal{B}' \) is \( f\ell_i\ell_j \) which is independent of the \( x^e \) and \( x^{e'} \) chosen. This completes the proof. \( \square \)

Note that the design \((V, \mathcal{B})\) is a \((v, f\ell, \ell(f\ell - 1))\)-BIBD and the \( i \)-th subdesign \((V, \mathcal{B}_i)\) is a \((v, f\ell_i, \ell_i(f\ell_i - 1))\)-BIBD.

Combining Theorems 2.1 and 3.6, we obtain:

**Corollary 3.7.** Let \( v = pf + 1 \) be a prime power. For all positive integers \( \ell_1, \ell_2, \ldots, \ell_{s-1} \) such that \( \sum_{i=1}^{s-1} \ell_i \leq p \), there exists a \( BA(v, pv, s, 2) \) with indices
\[
\mu_{ij} = \begin{cases} 
  f\ell_i\ell_j, & \text{if } i \neq j, \ i, j \neq 0, \\
  \ell_i(f\ell_i - 1), & \text{if } i = j \neq 0, \\
  \ell_i\{f(p - \ell) + 1\}, & \text{if } i \neq 0 \text{ and } j = 0, \\
  (p - \ell)\{f(p - \ell) + 1\}, & \text{if } i = j = 0,
\end{cases}
\]

where \( \ell = \sum_{i=1}^{s-1} \ell_i \).

Chakravarty and Dey (1976) have given, for a prime power \( 4t + 1 \), a \( BA(4t + 1, 8t + 2, 3, 2) \) with indices \( \mu_{12} = 2t, \mu_{11} = \mu_{22} = 2t - 1, \mu_{01} = \mu_{02} = 1 \) and \( \mu_{00} = 0 \). This construction is a special case of Corollary 3.7 with \( p = 2, f = 2t, s = 3 \) and \( \ell_1 = \ell_2 = 1 \).
References


