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Evaluating approximations of the semidefinite cone with trace normalized distance*

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Abstract

We evaluate the dual cone of the set of diagonally dominant matrices (resp., scaled diagonally dominant matrices), namely $\mathcal{DD}_n^*$ (resp., $\mathcal{SDD}_n^*$), as an approximation of the semidefinite cone. Using the measure proposed by Blekherman et al. (2020) called norm normalized distance, we prove that the norm normalized distance between a set $\mathcal{S}$ and the semidefinite cone has the same value whenever $\mathcal{SDD}_n^* \subseteq \mathcal{S} \subseteq \mathcal{DD}_n^*$. This implies that the norm normalized distance is not a sufficient measure to evaluate these approximations. As a new measure to compensate for the weakness of that distance, we propose a new distance, called trace normalized distance. We prove that the trace normalized distance between $\mathcal{DD}_n^*$ and $\mathcal{S}_+^n$ has a different value from the one between $\mathcal{SDD}_n^*$ and $\mathcal{S}_+^n$, and give the exact values of these distances.

Key words: Semidefinite optimization problem; Diagonally dominant matrix; Scaled diagonally dominant matrix.

1 Introduction

Semidefinite optimization problems (SDPs) can provide powerful convex relaxations for combinatorial and nonconvex optimization problems ([7], [8], [13], [15], etc.). An SDP can be written in the standard form:

$$\begin{aligned} \min\ & \langle C, X\rangle \\ \text{s.t. } & \langle A_j, X\rangle = b_j, j = 1, 2, \ldots, m, \\ & X \in \mathcal{S}_+^n, \end{aligned}$$

where $C \in \mathbb{S}^n$, $A_j \in \mathbb{S}^n$, $b_j \in \mathbb{R}$ $(j = 1, 2, \ldots, m)$, and $\langle A, B\rangle := \sum_{i,j=1}^n A_{i,j}B_{i,j}$ is the inner product over $\mathbb{S}^n$. The space of symmetric matrices is denoted as $\mathbb{S}^n := \{X \in \mathbb{R}^{n\times n} \mid X_{i,j} = X_{j,i}\ (1 \le i < j \le n)\}$ and the semidefinite cone is defined as $\mathcal{S}_+^n := \{X \in \mathbb{S}^n \mid d^T X d \ge 0 \text{ for any } d \in \mathbb{R}^n\}$.

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SDPs are theoretically attractive because they can be solved in polynomial time to any desired precision. However, it is difficult to solve large-scale SDPs even using the state-of-the-art solvers, such as Mosek [10]. One technique to overcome this deficiency is to relax the semidefinite constraint and solve the resulting easily handled problem using, e.g., linear programming (LP) or second order cone programming (SOCP). A typical relaxation technique is to replace the constraint $X \in \mathcal{S}_+^n$ by a relaxed constraint $X \in \mathcal{S}$, where $\mathcal{S}$ is a subset of $\mathbb{S}^n$ containing $\mathcal{S}_+^n$, i.e., $\mathcal{S}_+^n \subseteq \mathcal{S} \subseteq \mathcal{S}^n$. If $X \in \mathcal{S}$ is represented by linear constraints, the resulting problem becomes an LP and if $X \in \mathcal{S}$ is represented by second-order constraints, the resulting problem becomes an SOCP. Such a set $\mathcal{S}$ is called an outer approximation of $\mathcal{S}_+^n$. There are two kinds of approximation, i.e., inner approximation and outer approximation, and an inner approximation (an outer approximation) of $\mathcal{S}_+^n$ can be obtained by constructing the dual cone of an outer approximation (an inner approximation). We focus on the outer approximations of $\mathcal{S}_+^n$ and refer to them as approximations of $\mathcal{S}_+^n$ throughout this paper.

Several sets have been used as approximations of the semidefinite cone, including the k-PSD closure, namely $\mathcal{S}^{n,k}$ ([5]) and the dual cone of the set of diagonally dominant matrices (resp., scaled diagonally dominant matrices), namely $\mathcal{DD}_n^*$ (resp., $\mathcal{SDD}_n^*$) ([2]). Multiple experimental results have shown the efficiency of cutting-plane methods using these approximations ([2], [4], [14]). Although the inclusive relationship of these approximations has been given (see, e.g. [2], [4]), theoretical analyses of how well these sets approximate the semidefinite cone have been limited.

Bertsimas and Cory-Wright [4] evaluated $\mathcal{DD}_n^*$ and $\mathcal{SDD}_n^*$ as approximations of the semidefinite cone by comparing lower bounds of the minimum eigenvalues of matrices from these two sets. Blekherman et al. [5] evaluated $\mathcal{S}^{n,k}$ as an approximation of the semidefinite cone by using an evaluation method called norm normalized distance. The norm normalized distance between a given approximation $\mathcal{S} \subseteq \mathbb{S}^n$ and $\mathcal{S}_+^n$ is the maximum distance from a matrix $X \in \mathcal{S}$ to $\mathcal{S}_+^n$, where the Frobenius norm of the matrix X is assumed to be one. They obtained several upper bounds and lower bounds of the norm normalized distance between $\mathcal{S}^{n,k}$ and $\mathcal{S}_+^n$.

In this paper, we first show that the norm normalized distance between a set $\mathcal{S}$ and $\mathcal{S}_+^n$ has the same value whenever $\mathcal{SDD}_n^* \subseteq \mathcal{S} \subseteq \mathcal{DD}_n^*$. This implies that the norm normalized distance is not a sufficient measure to evaluate these approximations. As a new measure to compensate for the weakness of that distance, we introduce a new distance, called trace normalized distance. We prove that the trace normalized distance between $\mathcal{DD}_n^*$ and $\mathcal{S}_+^n$ has a different value from the one between $\mathcal{SDD}_n^*$ and $\mathcal{S}_+^n$, and give the exact values of these distances.

The organization of this paper is as follows. Section 2 introduces approximations of the semidefinite cone $\mathcal{S}^{n,k}$, $\mathcal{DD}_n^*$ and $\mathcal{SDD}_n^*$ and their inclusive relationship. In Section 3, the norm normalized distance proposed by Blekherman et al. [5] is used to evaluate $\mathcal{DD}_n^*$ and $\mathcal{SDD}_n^*$. In Section 4, the trace normalized distance is proposed and used to evaluate $\mathcal{DD}_n^*$ and $\mathcal{SDD}_n^*$. We conclude our work in Section 5.

2 Approximations of the semidefinite cone

We consider the following three sets as approximations of the semidefinite cone. Let k and n be positive integers satisfying $2 \leq k \leq n$ and

$$\mathcal{S}^{n,k} := \{X \in \mathbb{S}^n \mid \text{ All } k \times k \text{ submatrices of } X \text{ are positive semidefinite}\}. \tag{1}$$

$$\mathcal{DD}_n^* := \{X \in \mathbb{S}^n \mid X_{i,i} + X_{j,j} \pm 2X_{i,j} \geq 0 \ (1 \leq i \leq j \leq n)\}. \tag{2}$$

$$\mathcal{SDD}_n^* := \{X \in \mathbb{S}^n \mid X_{i,i}X_{j,j} \geq X_{i,j}^2 \ (1 \leq i < j \leq n), X_{i,i} \geq 0 \ (i = 1, \dots, n)\}. \tag{3}$$

$\mathcal{S}^{n,k}$ is called the *k-PSD closure*, whose properties are discussed in [5]. It is obvious from the definition (1) that $\mathcal{S}^n_+ = \mathcal{S}^{n,n} \subseteq \mathcal{S}^{n,k_1} \subseteq \mathcal{S}^{n,k_2}$ when $n \geq k_1 \geq k_2 \geq 2$.

$\mathcal{DD}^*_n$ (resp. $\mathcal{SDD}^*_n$) is the dual cone of the set of diagonally dominant matrices (resp. scaled diagonally dominant matrices). These duality relationships imply that $\mathcal{SDD}^*_n$ is a subset of $\mathcal{DD}^*_n$. It is worth noting that $\mathcal{DD}^*_n$ and $\mathcal{SDD}^*_n$ are used as approximations of the semidefinite cone in cutting-plane methods ([1],[2],[4],[14]) and facial reduction methods [11].

According to the definitions (1) and (3), it is clear that $\mathcal{S}^{n,2} = \mathcal{SDD}^*_n$. The relation among $\mathcal{S}^{n,k}$, $\mathcal{DD}^*_n$ and $\mathcal{SDD}^*_n$ can be concluded to be

$$\mathcal{S}^n_+ = \mathcal{S}^{n,n} \subseteq \cdots \subseteq \mathcal{S}^{n,2} = \mathcal{SDD}^*_n \subseteq \mathcal{DD}^*_n. \tag{4}$$

3 The norm normalized distance between the semidefinite cone and its approximation

Blekherman et al. [5] proposed a method of evaluating approximations of the semidefinite cone, which is based on the maximum distance from a matrix in a given approximation $\mathcal{S}^n_+ \subseteq \mathcal{S} \subseteq \mathbb{S}^n$ to $\mathcal{S}^n_+$. A feature of their method is that the distance is evaluated under the constraint that the value of the Frobenius norm is one, and the norm normalized distance between a set $\mathcal{S}$ and $\mathcal{S}^n_+$ is defined as

$$\overline{\mathrm{dist}}_F(\mathcal{S}, \mathcal{S}^n_+) := \sup_{X \in \mathcal{S}, \|X\|_F = 1} \|X - \mathrm{P}_{\mathcal{S}^n_+}(X)\|_F, \tag{5}$$

where $\mathrm{P}_{\mathcal{S}^n_+}(X) := \mathrm{argmin}_{Y \in \mathcal{S}^n_+} \|X - Y\|_F$ is the metric projection of X on $\mathcal{S}^n_+$.

In [5], the authors showed that $\overline{\mathrm{dist}}_F(\mathcal{S}^{n,k}, \mathcal{S}^n_+) \leq \frac{n-k}{n+k-2}$. Through a similar discussion, we can prove the following theorem:

Theorem 3.1. *For $n \geq 4$,*

$$\overline{\mathrm{dist}}_F(\mathcal{DD}^*_n, \mathcal{S}^n_+) = \overline{\mathrm{dist}}_F(\mathcal{SDD}^*_n, \mathcal{S}^n_+) = \frac{n-2}{n}.$$

The proof of this theorem is provided in Appendix A.

Theorem 3.1 shows unfortunately that the norm normalized distance (5) gives the same value $\overline{\mathrm{dist}}_F(\mathcal{S}, \mathcal{S}^n_+) = \frac{n-2}{n}$ for any approximation $\mathcal{S} \subseteq \mathbb{S}^n$ whenever it satisfies $\mathcal{SDD}^*_n \subseteq \mathcal{S} \subseteq \mathcal{DD}^*_n$. In the next section, we introduce a new distance, called the trace normalized distance. We show that the new distance between $\mathcal{SDD}^*_n$ and $\mathcal{S}^n_+$ has a different value from the one between $\mathcal{DD}^*_n$ and $\mathcal{S}^n_+$.

4 The trace normalized distance between the semidefinite cone and its approximation

We define the trace normalized distance between a set $\mathcal{S}$ and $\mathcal{S}^n_+$ as

$$\overline{\mathrm{dist}}_T(\mathcal{S}, \mathcal{S}^n_+) := \sup_{X \in \mathcal{S}, \mathrm{Tr}(X) = 1} \|X - \mathrm{P}_{\mathcal{S}^n_+}(X)\|_F, \tag{6}$$

where $\mathrm{Tr}(X) := \sum_{i=1}^n X_{i,i}$ is the trace of matrix $X \in \mathbb{S}^n$.

As shown in the sections below, $\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n)$ and $\overline{\mathrm{dist}}_T(\mathcal{DD}_n^*, \mathcal{S}_+^n)$ are different, i.e., $\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n) = \frac{n-2}{n}$ (Theorem 4.1) and $\overline{\mathrm{dist}}_T(\mathcal{DD}_n^*, \mathcal{S}_+^n) = \frac{\sqrt{n}-1}{2}$ (Theorem 4.4).

4.1 The trace normalized distance between $\mathcal{SDD}_n^*$ and $\mathcal{S}_+^n$

Theorem 4.1. *For all $n \geq 2$,*

$$\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n) = \frac{n-2}{n}.$$

To prove this theorem, we introduce Lemmas 4.2 and 4.3. Lemma 4.2 gives a lower bound of $\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n)$ and Lemma 4.3 gives an upper bound of $\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n)$. In these lemmas, we assume that $n \geq 3$. If $n = 2$, we can easily see that $\overline{\mathrm{dist}}_T(\mathcal{SDD}_2^*, \mathcal{S}_+^2) = \frac{n-2}{n} = 0$.

Lemmas 1 and 2 are based on the results shown in the proofs of Theorems 3 and 1 in [5].

Lemma 4.2. *For all $n \geq 3$,*

$$\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n) \geq \frac{n-2}{n}.$$

Proof. Let $I \in \mathbb{S}^n$ be the identity matrix and $e := (1, \ldots, 1)^T \in \mathbb{R}^n$. Given scalars $a, b \geq 0$, we define a matrix,

$$G(a, b, n) := (a + b)I - aee^T. \tag{7}$$

If $G(a, b, n) \in \mathcal{SDD}_n^* \setminus \mathcal{S}_+^n$ and $\mathrm{Tr}(G(a, b, n)) = 1$, then by definition (6), $\|G(a, b, n) - \mathrm{P}_{\mathcal{S}_+^n}(G(a, b, n))\|_F$ gives a lower bound of $\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n)$. To find a tighter lower bound, we consider the following problem (8) on the parameters a and b:

$$\begin{aligned}
\max_{a,b\geq 0} \quad & \|G(a, b, n) - \mathrm{P}_{\mathcal{S}_+^n}(G(a, b, n))\|_F & \text{(8a)}\\
\text{s.t.} \quad & G(a, b, n) \notin \mathcal{S}_+^n, & \text{(8b)}\\
& G(a, b, n) \in \mathcal{SDD}_n^*, & \text{(8c)}\\
& \mathrm{Tr}(G(a, b, n)) = 1. & \text{(8d)}
\end{aligned}$$

Problem (8) can be equivalently written as problem (9):

$$\begin{aligned}
\max_{a,b\geq 0} \quad & (n-1)a - b & \text{(9a)}\\
\text{s.t.} \quad & b < (n-1)a, & \text{(9b)}\\
& b \geq a, & \text{(9c)}\\
& nb = 1. & \text{(9d)}
\end{aligned}$$

To prove the equivalence between (8) and (9), we first show that the constraints (8b) and (9b) are equivalent. Proposition 4 in [5] ensures that the eigenvalues of $G(a, b, n)$ are $a+b$ with multiplicity $n-1$ and $b - (n-1)a$ with multiplicity 1. Note that $a, b \geq 0$, hence

$$G(a, b, n) \notin \mathcal{S}_+^n \text{ if and only if } b < (n-1)a. \tag{10}$$

Next, we verify that (8c) and (9c) are equivalent. It follows from definition (3) that $G(a,b,n) \in \mathcal{SDD}_n^*$ if and only if all the 2×2 submatrices of $G(a,b,n)$ are positive semidefinite. It is obvious from (7) that any 2×2 submatrix of $G(a,b,n)$ is $G(a,b,2)$. (10) ensures that $G(a,b,2) \in \mathcal{S}_+^2$ if and only if $b \geq a$ and we can conclude that

$$G(a,b,n) \in \mathcal{SDD}_n^* \text{ if and only if } b \geq a. \tag{11}$$

The equivalence between (8d) and (9d) comes from the fact that the definition (7) implies that

$$\mathrm{Tr}(G(a,b,n)) = nb. \tag{12}$$

We finally show that the objective functions (8a) and (9a) are equivalent. Since (8b) implies that $G(a,b,n) \notin \mathcal{S}_+^n$, it is apparent from (10) that $b-(n-1)a < 0$. Then, $b-(n-1)a$ is the only negative eigenvalue of $G(a,b,n)$, and hence,

$$\|G(a,b,n) - \mathrm{P}_{\mathcal{S}_+^n}(G(a,b,n))\|_F = (n-1)a - b. \tag{13}$$

By (10), (11), (12) and (13), one can see that problems (8) and (9) are equivalent. The optimal solution of problem (9) is $\bar{a} = \bar{b} = \frac{1}{n}$, and hence, we have

$$\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n) \geq \|G(\bar{a},\bar{b},n) - \mathrm{P}_{\mathcal{S}_+^n}(G(\bar{a},\bar{b},n))\|_F = \frac{n-2}{n}.$$

□

□

Lemma 4.3. *For all $n \geq 3$,*

$$\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n) \leq \frac{n-2}{n}.$$

Proof. If a scalar U satisfies $\|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F \leq U$ for every $X \in \mathcal{SDD}_n^*$ with $\mathrm{Tr}(X) = 1$, then U is an upper bound of $\overline{\mathrm{dist}}_T(\mathcal{SDD}_n^*, \mathcal{S}_+^n)$. Below, we find such a scalar U.

Let X be a matrix in $\mathcal{SDD}_n^*$ satisfying $\mathrm{Tr}(X) = 1$. We construct a matrix $\tilde{X} \in \mathcal{S}_+^n$ and a scalar $\tilde{\alpha} \geq 0$ so that $\|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F \leq \|X - \tilde{\alpha}\tilde{X}\|_F$.

Define a matrix $X^{(i,j)} \in \mathbb{S}^n$ for every $1 \leq i < j \leq n$:

$$X_{p,q}^{(i,j)} := \begin{cases} X_{i,i} & (\text{if } p = q = i), \\ X_{j,j} & (\text{if } p = q = j), \\ X_{i,j} & (\text{if } (p,q) \in \{(i,j),(j,i)\}), \\ 0 & (\text{otherwise}). \end{cases} \tag{14}$$

Let $\tilde{X} = \frac{1}{C_n^2}\sum_{1\leq i<j\leq n} X^{(i,j)}$. Then (14) implies that

$$\begin{aligned} \tilde{X}_{i,i} =& \frac{C_n^2 - C_{n-1}^2}{C_n^2} X_{i,i} = \frac{2}{n} X_{i,i} \ (i = 1,\dots,n), \\ \tilde{X}_{i,j} =& \frac{1}{C_n^2} X_{i,j} = \frac{2}{n(n-1)} X_{i,j} \ (1 \leq i < j \leq n). \end{aligned}$$

By (3), we know that $X^{(i,j)} \in \mathcal{S}^n_+$ for all $1 \le i < j \le n$ and hence $\tilde{X} \in \mathcal{S}^n_+$.

Let α be a scalar satisfying $\alpha \ge 0$. Then,

$$
\begin{aligned}
\|X - \alpha\tilde{X}\| &= \sqrt{\sum_{i=1}^{n}(X-\alpha\tilde{X})^2_{i,i} + \sum_{i\neq j}(X-\alpha\tilde{X})^2_{i,j}} \\
&= \sqrt{\sum_{i=1}^{n}(1-\frac{2\alpha}{n})^2 X^2_{i,i} + \sum_{i\neq j}(1-\frac{2\alpha}{n(n-1)})^2 X^2_{i,j}} \\
&\le \sqrt{(1-\frac{2\alpha}{n})^2 \sum_{i=1}^{n} X^2_{i,i} + (1-\frac{2\alpha}{n(n-1)})^2 \sum_{i\neq j} X_{i,i}X_{j,j}} \\
&= \sqrt{(1-\frac{2\alpha}{n})^2 \sum_{i=1}^{n} X^2_{i,i} + (1-\frac{2\alpha}{n(n-1)})^2 (\mathrm{Tr}(X)^2 - \sum_{i=1}^{n} X^2_{i,i})} \\
&= \sqrt{\left((1-\frac{2\alpha}{n})^2 - (1-\frac{2\alpha}{n(n-1)})^2\right) \sum_{i=1}^{n} X^2_{i,i} + (1-\frac{2\alpha}{n(n-1)})^2 \mathrm{Tr}(X)^2} \\
&= \sqrt{\left(1-\frac{4\alpha}{n} + \frac{4\alpha^2}{n^2} - (1-\frac{4\alpha}{n(n-1)} + \frac{4\alpha^2}{n^2(n-1)^2})\right) \sum_{i=1}^{n} X^2_{i,i} + (1-\frac{2\alpha}{n(n-1)})^2 \mathrm{Tr}(X)^2} \\
&= \sqrt{\left(\frac{4\alpha^2(n-2)}{n(n-1)^2} - \frac{4\alpha(n-2)}{n(n-1)}\right) \sum_{i=1}^{n} X^2_{i,i} + (1-\frac{2\alpha}{n(n-1)})^2 \mathrm{Tr}(X)^2}. \qquad (15)
\end{aligned}
$$

Note that $\mathrm{Tr}(X) = 1$ and $\tilde{\alpha} := n-1 \ge 0$ satisfies that $\frac{4\tilde{\alpha}^2(n-2)}{n(n-1)^2} - \frac{4\tilde{\alpha}(n-2)}{n(n-1)} = 0$. By substituting $\tilde{\alpha}$ into (15), we have

$$\|X - \tilde{\alpha}\tilde{X}\|_F \le \sqrt{(1-\frac{2\tilde{\alpha}}{n(n-1)})^2} = \frac{n-2}{n}.$$

Since $\tilde{\alpha} \ge 0$ and $\tilde{X} \in \mathcal{S}^n_+$, by letting $U = \frac{n-2}{n}$, we can see that

$$\|X - \mathrm{P}_{\mathcal{S}^n_+}(X)\|_F \le \|X - \tilde{\alpha}\tilde{X}\|_F \le U = \frac{n-2}{n},$$

and hence,

$$\overline{\mathrm{dist}}_T(\mathcal{SDD}^*_n, \mathcal{S}^n_+) = \sup_{X \in \mathcal{SDD}^*_n, \mathrm{Tr}(X)=1} \|X - \mathrm{P}_{\mathcal{S}^n_+}(X)\|_F \le U = \frac{n-2}{n}.$$

□

□

4.2 The trace normalized distance between $\mathcal{DD}^*_n$ and $\mathcal{S}^n_+$

In this section, we prove the following theorem:

Theorem 4.4. *For all $n \geq 2$,*

$$\overline{\mathrm{dist}}_T(\mathcal{DD}_n^*, \mathcal{S}_+^n) = \frac{\sqrt{n}-1}{2}.$$

The idea for proving Theorem 4.4 is as follows. Define

$$\mathcal{DDT}_n^* := \mathcal{DD}_n^* \cap \{X \in \mathbb{S}^n \mid \mathrm{Tr}(X) = 1\}. \tag{16}$$

Definition (6) ensures that

$$\overline{\mathrm{dist}}_T(\mathcal{DD}_n^*, \mathcal{S}_+^n) = \max_{X \in \mathcal{DDT}_n^*} \|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F.$$

Note that $\|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F$ is continuous and convex on $\mathbb{S}^n$ and $\mathcal{DDT}_n^*$ is closed, bounded, and convex. The Bauer maximum principle [3] states that any continuous convex function defined on a compact convex set in $\mathbb{R}^n$ attains its maximum at some extreme point of the set. As its corollary, we have the following result:

Corollary 4.5. $\max_{X \in \mathcal{DDT}_n^*} \|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F$ *attains its maximum at some extreme point of $\mathcal{DDT}_n^*$.*

In Proposition 4.6, we show that every extreme point of $\mathcal{DDT}_n^*$ has a special structure. By using this special structure, Lemma 4.7 shows that the distance between each extreme point of $\mathcal{DDT}_n^*$ and $\mathcal{S}_+^n$ is the same. The exact value of this distance is also given in Lemma 4.7. Theorem 4.4 can be obtained as a direct result of Corollary 4.5 and Lemma 4.7.

Proposition 4.6. *For $n \geq 2$, let X be an extreme point of $\mathcal{DDT}_n^*$. There exists an integer q satisfying $1 \leq q \leq n$ such that*

$$X_{i,j} = \begin{cases} 1 & (\text{ if } i = j = q), \\ \frac{1}{2} \text{ or } -\frac{1}{2} & (\text{ if either } i = q \text{ or } j = q), \\ 0 & (\text{ otherwise}). \end{cases} \tag{17}$$

Proof. Let $X \in \mathcal{DDT}_n^*$. By (2) and (16), we see that for every $i = 1, \ldots, n$,

$$X_{i,i} \geq 0,$$

and for every $1 \leq i < j \leq n$,

$$X_{i,i} + X_{j,j} + 2X_{i,j} \geq 0, \tag{18}$$

$$X_{i,i} + X_{j,j} - 2X_{i,j} \geq 0. \tag{19}$$

Thus, the set $\mathcal{DDT}_n^*$ can be written as

$$\begin{aligned} \mathcal{DDT}_n^* = \{X \in \mathbb{S}^n \mid & \mathrm{Tr}(X) = 1, \\ & X_{i,i} \geq 0 \ (i = 1, \ldots, n), & (20) \\ & X_{i,i} + X_{j,j} + 2X_{i,j} \geq 0 \ (1 \leq i < j \leq n), & (21) \\ & X_{i,i} + X_{j,j} - 2X_{i,j} \geq 0 \ (1 \leq i < j \leq n)\}. & (22) \end{aligned}$$

Let $\bar{X}$ be an extreme point of $\mathcal{DDT}_n^*$ and let $N(X)$ be the number of linearly independent inequalities in (20), (21) and (22) that are active (i.e., the equalities hold) at $X \in \mathcal{DDT}_n^*$. By the characterization of extreme points of a polyhedron (see, e.g., Theorem 5.7, [12]), we know that

$$N(\bar{X}) = \frac{n(n+1)}{2} - 1. \tag{23}$$

We prove that $\bar{X}$ satisfies (17) by observing the active inequalities at $\bar{X}$.

It follows from $\text{Tr}(\bar{X}) = 1$ that $\bar{X}$ has at least one nonzero diagonal element. This implies that the number of active inequalities in (20) at $\bar{X}$ is at most $n-1$. Suppose that $n-k$ inequalities in (20) are active at $\bar{X}$, where k is an integer and $1 \le k \le n$. Below, we show that $k \ne n$ by contradiction.

Assume that $k = n$. Then we have $\bar{X}_{i,i} > 0$ for each $1 \le i \le n$. At most one of (18) and (19) can be active at $\bar{X}$ for each $1 \le i < j \le n$. In fact, suppose that (18) and (19) are active simultaneously for some $1 \le i < j \le n$:

$$\bar{X}_{i,i} + \bar{X}_{j,j} + 2\bar{X}_{i,j} = 0,\ \bar{X}_{i,i} + \bar{X}_{j,j} - 2\bar{X}_{i,j} = 0.$$

Then $\bar{X}_{i,j} = \bar{X}_{i,i} + \bar{X}_{j,j} = 0$ and since $\bar{X}_{i,i}, \bar{X}_{j,j} \ge 0$, we obtain $\bar{X}_{i,i} = \bar{X}_{j,j} = 0$, which is a contradiction to the assumption $\bar{X}_{i,i}, \bar{X}_{j,j} > 0$. This implies that $N(\bar{X})$ is at most $\frac{n(n-1)}{2}$, that is strictly less than the number $\frac{n(n+1)}{2} - 1$ in (23). This contradiction implies that $k \ne n$.

Since we have shown that $1 \le k \le n-1$, there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix $X^* := P\bar{X}P^T$ satisfies

$$\begin{aligned} X^*_{i,i} &= 0 \quad (1 \le i \le n-k), \\ X^*_{i,i} &> 0 \quad (n-k+1 \le i \le n). \end{aligned} \tag{24}$$

Note that $X^* \in \mathcal{DDT}^*_n$ and $N(X^*) = \frac{n(n+1)}{2} - 1$. Below, we show that X^* satisfies (17) by observing the active inequalities at X^* instead of $\bar{X}$.

Next, we show that $k = 1$; i.e., exactly $n-1$ inequalities in (20) are active at X^*. It follows from (21), (22) and (24) that $X^*_{i,j} = 0$ for each $1 \le i < j \le n-k$. This implies that all inequalities (18) and (19) with $1 \le i < j \le n-k$ at X^* are active. For each pair of (i,j) where $X_{j,j} > 0$ and $1 \le i < j$, one can show again by contradiction that at most one of (18) and (19) can be active at X^*. Consider the case when the number of active inequalities at X^* attains its maximum; i.e., exactly one of (18) and (19) is active at X^* for each pair of (i,j) where $n-k+1 \le j \le n$ and $1 \le i < j$. The following system,

$$\begin{cases} 0 = X^*_{i,i} \quad (1 \le i \le n-k), \\ 0 = X^*_{i,i} + X^*_{j,j} + 2X^*_{i,j} \quad (1 \le i < j \le n-k), \\ 0 = X^*_{i,i} + X^*_{j,j} - 2X^*_{i,j} \quad (1 \le i < j \le n-k), \\ \text{either (18) or (19) is active at } X^* \quad (n-k+1 \le j \le n,\ 1 \le i < j) \end{cases}$$

includes exactly $(n-k) + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} - k$ linearly independent active inequalities. This implies that $N(X^*) \le \frac{n(n+1)}{2} - k$. By (23), we know that $k = 1$ and the number $\frac{n(n+1)}{2} - 1$ in (23) is attained only if the number of active inequalities in (21) and (22) attains its maximum.

The fact $k = 1$ implies that $X^*_{n,n} = 1$ and $X^*_{i,i} = 0$ for each $1 \le i \le n-1$, and hence $X^*_{i,j} = 0$ for each $1 \le i < j \le n-1$. Since the number of active inequalities in (21) and (22) attains its maximum, we know that either (18) or (19) is active at X^* for each (i,j) satisfying $j = n$ and $1 \le i < j$, which implies that $X^*_{i,n} \in \{\frac{1}{2}, -\frac{1}{2}\}$ for each $1 \le i < n$.

Finally, by applying the permutation $\bar{X} = P^T X^* P$, we know that there exists an integer q satisfying $1 \le q \le n$ for which $\bar{X}$ satisfies (17).

□

□

Lemma 4.7. *For $n \geq 2$, let X be an extreme point of $\mathcal{DDT}_n^*$. There exist scalars $\alpha_1, \ldots, \alpha_{n-1} \in \{\frac{1}{2}, -\frac{1}{2}\}$ such that the following matrix,*

$$X^* := \begin{pmatrix} 0 & & & a_1 \\ & \ddots & & \vdots \\ & & 0 & a_{n-1} \\ a_1 & \ldots & a_{n-1} & 1 \end{pmatrix} \tag{25}$$

satisfies

$$\|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F = \|X^* - \mathrm{P}_{\mathcal{S}_+^n}(X^*)\|_F = \frac{\sqrt{n}-1}{2}.$$

Proof. Let X be an extreme point of $\mathcal{DDT}_n^*$. By Proposition 4.6, there exists an integer q such that $1 \leq q \leq n$ for which X satisfies (17). Note that X only has one nonzero diagonal element $X_{q,q} = 1$. Let $P \in \mathbb{R}^{n \times n}$ be a permutation matrix such that $(PXP^T)_{n,n} = 1$. It is easy to see that there exist scalars $\alpha_1, \ldots, \alpha_{n-1} \in \{\frac{1}{2}, -\frac{1}{2}\}$ such that the matrix X^* defined in (25) satisfies $X^* = PXP^T$. Since the permutation matrix P is orthogonal, we see that $Y \in \mathcal{S}_+^n$ if and only if $PYP^T \in \mathcal{S}_+^n$ for any $Y \in \mathbb{S}^n$. This fact implies that

$$\begin{aligned} \|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F &= \inf_{Y \in \mathcal{S}_+^n} \|X - Y\|_F \\ &= \inf_{Y \in \mathcal{S}_+^n} \|PXP^T - PYP^T\|_F \\ &= \inf_{PYP^T \in \mathcal{S}_+^n} \|X^* - PYP^T\|_F \\ &= \|X^* - \mathrm{P}_{\mathcal{S}_+^n}(X^*)\|_F. \end{aligned} \tag{26}$$

By solving the eigenvalue equation $0 = |\lambda I - X^*|$ with respect to the scalar λ, we obtain that:

1. **If** $n = 2$**,** the eigenvalues of X^* are $\frac{1+\sqrt{n}}{2}$ with multiplicity 1 and $\frac{1-\sqrt{n}}{2}$ with multiplicity 1.
2. **If** $n \geq 3$**,** the eigenvalues of X^* are $\frac{1+\sqrt{n}}{2}$ with multiplicity 1, $\frac{1-\sqrt{n}}{2}$ with multiplicity 1 and 0 with multiplicity $n-2$.

From these observations, for every $n \geq 2$, X^* has only one negative eigenvalue $\lambda_{\min} := \frac{1-\sqrt{n}}{2}$ and hence,

$$\|X^* - \mathrm{P}_{\mathcal{S}_+^n}(X^*)\|_F = \sqrt{\lambda_{\min}^2} = \frac{\sqrt{n}-1}{2}. \tag{27}$$

We can conclude from (26) and (27) that

$$\|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F = \|X^* - \mathrm{P}_{\mathcal{S}_+^n}(X^*)\|_F = \frac{\sqrt{n}-1}{2}.$$

□

□

5 Concluding remarks

In this paper, we first showed that the norm normalized distance $\overline{\text{dist}}_F(\mathcal{S}, \mathcal{S}^n_+)$ has the same value whenever $\mathcal{SDD}^*_n \subseteq \mathcal{S} \subseteq \mathcal{DD}^*_n$, since $\overline{\text{dist}}_F(\mathcal{DD}^*_n, \mathcal{S}^n_+) = \overline{\text{dist}}_F(\mathcal{SDD}^*_n, \mathcal{S}^n_+)$ holds. This implies that the norm normalized distance is not a sufficient measure to evaluate these approximations. As a new measure to compensate for the weakness of that distance, we proposed a new distance, the trace normalized distance $\overline{\text{dist}}_T(\mathcal{S}, \mathcal{S}^n_+)$. Using this new measure, we proved that $\overline{\text{dist}}_T(\mathcal{DD}^*_n, \mathcal{S}^n_+)$ and $\overline{\text{dist}}_T(\mathcal{SDD}^*_n, \mathcal{S}^n_+)$ are different, i.e., $\overline{\text{dist}}_T(\mathcal{DD}^*_n, \mathcal{S}^n_+) = \frac{\sqrt{n-1}}{2}$ and $\overline{\text{dist}}_T(\mathcal{SDD}^*_n, \mathcal{S}^n_+) = \frac{n-2}{n}$.

In [14], the authors proposed a class of polyhedral approximations of the semidefinite cone, denoted as $\mathcal{SDB}^*_n$. The experimental results on cutting-plane methods, where $\mathcal{SDB}^*_n$ is used as the approximation of $\mathcal{S}^n_+$, for solving SDPs are promising. It is an interesting but also challenging issue to analyze the value of $\overline{\text{dist}}_T(\mathcal{SDB}^*_n, \mathcal{S}^n_+)$.

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A Proof of Theorem 3.1

Proof. We prove the following inequalities:

$$\frac{n-2}{n} \leq \overline{\mathrm{dist}}_F(\mathcal{SDD}_n^*, \mathcal{S}_+^n) \leq \overline{\mathrm{dist}}_F(\mathcal{DD}_n^*, \mathcal{S}_+^n) \leq \frac{n-2}{n}. \tag{28}$$

The relation $\mathcal{S}^{n,2} = \mathcal{SDD}_n^*$ in (4) implies that $\overline{\mathrm{dist}}_F(\mathcal{SDD}_n^*, \mathcal{S}_+^n) = \overline{\mathrm{dist}}_F(\mathcal{S}^{n,2}, \mathcal{S}_+^n)$. By Theorem 3 in [5], we know that $\overline{\mathrm{dist}}_F(\mathcal{S}^{n,k}, \mathcal{S}_+^n) \geq \frac{n-k}{\sqrt{(k-1)^2 n + n(n-1)}}$ and hence,

$$\overline{\mathrm{dist}}_F(\mathcal{SDD}_n^*, \mathcal{S}_+^n) = \overline{\mathrm{dist}}_F(\mathcal{S}^{n,2}, \mathcal{S}_+^n) \geq \frac{n-2}{\sqrt{(2-1)^2 n + n(n-1)}} = \frac{n-2}{n}. \tag{29}$$

The relation $\mathcal{SDD}_n^* \subseteq \mathcal{DD}_n^*$ in (4) ensures that

$$\overline{\mathrm{dist}}_F(\mathcal{SDD}_n^*, \mathcal{S}_+^n) \leq \overline{\mathrm{dist}}_F(\mathcal{DD}_n^*, \mathcal{S}_+^n). \tag{30}$$

Next, we prove that $\overline{\mathrm{dist}}_F(\mathcal{DD}_n^*, \mathcal{S}_+^n) \leq \frac{n-2}{n}$ with the following idea. If a scalar U satisfies $\|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F \leq U$ for every $X \in \mathcal{DD}_n^*$ with $\|X\|_F = 1$, then U is an upper bound of $\overline{\mathrm{dist}}_F(\mathcal{DD}_n^*, \mathcal{S}_+^n)$. We can find such a scalar U by constructing a matrix $\tilde{X} \in \mathcal{S}_+^n$ and a scalar $\tilde{\alpha} \geq 0$ for every $X \in \mathcal{DD}_n^*$ with $\|X\|_F = 1$ so that $\|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F \leq \|X - \tilde{\alpha}\tilde{X}\|_F$.

Let X be a matrix in $\mathcal{DD}_n^*$ satisfying $\|X\|_F = 1$. Define a matrix $X^{(i,j)} \in \mathbb{S}^n$ for every $1 \leq i < j \leq n$:

$$X_{p,q}^{(i,j)} := \begin{cases} \frac{X_{i,i}+X_{j,j}}{2} & (\text{if } p = q \in \{i,j\}), \\ X_{i,j} & (\text{if } (p,q) \in \{(i,j),(j,i)\}), \\ 0 & (\text{otherwise}). \end{cases} \tag{31}$$

Let $\bar{X} := \frac{1}{C_n^2}\sum_{1\le i<j\le n} X^{(i,j)}$. By definitions (2) and (31), one can verify that $X^{(i,j)} \in \mathcal{S}_+^n$ for all $1 \le i < j \le n$, and hence $\bar{X} \in \mathcal{S}_+^n$. Let α be a scalar satisfying $\alpha \ge \frac{2n(n-1)}{3n-4} > 0$. For all $1 \le i < j \le n$, we can obtain from (31) that $\bar{X}_{i,j} = \bar{X}_{j,i} = \frac{1}{C_n^2} X_{i,j}$ and hence,

$$\sum_{i\neq j}(X_{i,j} - \alpha\bar{X}_{i,j})^2 = \sum_{i\neq j}(1 - \frac{\alpha}{C_n^2})^2 X_{i,j}^2. \tag{32}$$

For all $i = 1, \ldots, n$, (31) implies that $\bar{X}_{i,i} = \frac{1}{C_n^2}\left(\frac{n-2}{2}X_{i,i} + \frac{1}{2}\mathrm{Tr}(X)\right)$ and hence,

$$\begin{aligned}
\sum_{i=1}^n (X_{i,i} - \alpha\bar{X}_{i,i})^2 =& \sum_{i=1}^n ((1 - \frac{\alpha(n-2)}{2C_n^2})X_{i,i} - \frac{\alpha}{2C_n^2}\mathrm{Tr}(X))^2 \\
=& \sum_{i=1}^n ((1 - \frac{\alpha(n-2)}{2C_n^2})^2 X_{i,i}^2 - 2(1 - \frac{\alpha(n-2)}{2C_n^2})X_{i,i}\frac{\alpha}{2C_n^2}\mathrm{Tr}(X) \\
&+ \frac{\alpha^2}{4(C_n^2)^2}\mathrm{Tr}(X)^2) \\
=& (1 - \frac{\alpha(n-2)}{2C_n^2})^2 \sum_{i=1}^n X_{i,i}^2 - (\frac{\alpha}{C_n^2} - \frac{\alpha^2(n-2)}{2(C_n^2)^2})\mathrm{Tr}(X)\sum_{i=1}^n X_{i,i} \\
&+ \frac{\alpha^2 n}{4(C_n^2)^2}\mathrm{Tr}(X)^2 \\
=& (1 - \frac{\alpha(n-2)}{2C_n^2})^2 \sum_{i=1}^n X_{i,i}^2 + (\frac{\alpha^2(3n-4)}{4(C_n^2)^2} - \frac{\alpha}{C_n^2})\mathrm{Tr}(X)^2.
\end{aligned} \tag{33}$$

The assumption $\alpha \ge \frac{2n(n-1)}{3n-4}$ ensures that $\frac{\alpha^2(3n-4)}{4(C_n^2)^2} - \frac{\alpha}{C_n^2} \ge 0$. One can verify that $\mathrm{Tr}(X)^2 \le n\sum_{i=1}^n X_{i,i}^2$ by using Cauchy-Schwarz inequality. Then, it follows from (33) that

$$\begin{aligned}
\sum_{i=1}^n (X_{i,i} - \alpha\bar{X}_{i,i})^2 \le& (1 - \frac{\alpha(n-2)}{2C_n^2})^2 \sum_{i=1}^n X_{i,i}^2 + (\frac{\alpha^2(3n-4)}{4(C_n^2)^2} - \frac{\alpha}{C_n^2})n\sum_{i=1}^n X_{i,i}^2 \\
=& \left((1 - \frac{\alpha(n-2)}{2C_n^2})^2 + \frac{\alpha^2 n(3n-4)}{4(C_n^2)^2} - \frac{\alpha n}{C_n^2}\right)\sum_{i=1}^n X_{i,i}^2 \\
=& \left(1 - \frac{2\alpha(n-2)}{2C_n^2} + \frac{\alpha^2(n-2)^2}{4(C_n^2)^2} + \frac{\alpha^2 n(3n-4)}{4(C_n^2)^2} - \frac{\alpha n}{C_n^2}\right)\sum_{i=1}^n X_{i,i}^2 \\
=& \left(1 - \frac{\alpha(n-2+n)}{C_n^2} + \frac{\alpha^2(n^2-4n+4+3n^2-4n)}{4(C_n^2)^2}\right)\sum_{i=1}^n X_{i,i}^2 \\
=& \left(1 - \frac{2\alpha(n-1)}{C_n^2} + \frac{4\alpha^2(n-1)^2}{4(C_n^2)^2}\right)\sum_{i=1}^n X_{i,i}^2 \\
=& \left(1 - \frac{\alpha(n-1)}{C_n^2}\right)^2 \sum_{i=1}^n X_{i,i}^2.
\end{aligned} \tag{34}$$

Combining (32) and (34) gives

$$\|X - \alpha\bar{X}\|_F \le \sqrt{\sum_{i\neq j}(1 - \frac{\alpha}{C_n^2})^2 X_{i,j}^2 + \left(1 - \frac{\alpha(n-1)}{C_n^2}\right)^2 \sum_{i=1}^n X_{i,i}^2}. \tag{35}$$

Note that $\bar{\alpha} := n-1$ satisfies $\bar{\alpha} \geq \frac{2n(n-1)}{3n-4}$ when $n \geq 4$, and the coefficients in (35) satisfy

$$1 - \frac{\bar{\alpha}}{C_n^2} = -(1 - \frac{\bar{\alpha}(n-1)}{C_n^2}) = \frac{n-2}{n}.$$

Since $\|X\|_F = 1$, by substituting $\bar{\alpha}$ into (35), we have

$$\begin{aligned}\|X - \bar{\alpha}\bar{X}\|_F \leq& \sqrt{\left(\frac{n-2}{n}\right)^2 \sum_{i \neq j} X_{i,j}^2 + \left(\frac{n-2}{n}\right)^2 \sum_{i=1}^{n} X_{i,i}^2} \\ =& \frac{n-2}{n}\|X\|_F^2 \\ =& \frac{n-2}{n}.\end{aligned}$$

Because $\bar{X} \in \mathcal{S}_+^n$ and $\bar{\alpha} \geq 0$, by letting $U = \frac{n-2}{n}$, we have

$$\|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F \leq \|X - \bar{\alpha}\bar{X}\|_F \leq U = \frac{n-2}{n}$$

and hence,

$$\overline{\mathrm{dist}}_F(\mathcal{DD}_n^*, \mathcal{S}_+^n) = \sup_{X \in \mathcal{DD}_n^*, \|X\|_F = 1} \|X - \mathrm{P}_{\mathcal{S}_+^n}(X)\|_F \leq U = \frac{n-2}{n}. \tag{36}$$

(29), (30) and (36) imply that (28) holds, which proves this theorem.

□

□